# *Foundations of Modern Macroeconomics*

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## Chapter 17: Intergenerational Economics, II

# Aims of this lecture

- Study second “work-horse” model of overlapping generations based on discrete time. Motivation for doing this:
  - key model in modern macroeconomics and public finance theory
  - better captures life-cycle behaviour
  - chain of bequests easier to study
  - natural extension to Computable General Equilibrium (CGE) policy models (e.g. Auerbach & Kotlikoff)

- Apply model to various issues:
  - funded vs. unfunded pensions
  - pension reform
  - ageing and the macroeconomy
- Study various extensions
  - growth and human capital
  - public investment
  - intergenerational accounting

# The Diamond-Samuelson model

## Households

- live two periods: "youth" (superscript $Y$) and "old age" (superscript $O$)
- consume in both periods
- work only during youth
- unlinked with past or future generations (no bequests)
- save during youth to finance old-age consumption (life-cycle saving)
- utility function of young agent at time $t$:

$$\Lambda_t^Y \equiv U(C_t^Y) + \left(\frac{1}{1+\rho}\right) U(C_{t+1}^O)$$

  - $U(\cdot)$ is felicity function (Inada-style conditions)

- $\rho > 0$ captures time preference

- budget identities:

$$\begin{aligned} C_t^Y + S_t &= W_t \\ C_{t+1}^O &= (1 + r_{t+1}) S_t \end{aligned}$$

  - $S_t$ is saving
  - $W_t$ is wage income (exogenous labour supply)
  - $r_{t+1}$ is real interest rate

- consolidated (lifetime) budget constraint:

$$W_t = C_t^Y + \frac{C_{t+1}^O}{1 + r_{t+1}}$$

- utility maximization yields consumption Euler equation:

$$\frac{U'(C^O_{t+1})}{U'(C^Y_t)} = \frac{1+\rho}{1+r_{t+1}}$$

- savings function:

$$S_t = S(W_t, r_{t+1})$$

  - $0 < S_W < 1$: both goods are normal
  - $S_r$ ambiguous (offsetting income and substitution effects)
  - if intertemporal substitution elasticity is high ($\sigma > 1$) then $S_r > 0$ (and vice versa)

# Firms

- perfect competition, CRTS technology $Y_t = F(K_t, L_t)$, Inada conditions
- hire $L_t$ from young (at wage $W_t$) and $K_t$ from old (at rental rate $r_t + \delta$):

$$\begin{aligned} W_t &= F_L(K_t, L_t) \\ r_t + \delta &= F_K(K_t, L_t) \end{aligned}$$

- interest rate facing young depends on future (aggregate) capital-labour ratio: $r_{t+1} + \delta = F_K(K_{t+1}, L_{t+1})$
- intensive-form expressions:

$$\begin{aligned} y_t &= f(k_t) \\ W_t &= f(k_t) - k_t f'(k_t) \\ r_{t+1} + \delta &= f'(k_{t+1}) \end{aligned}$$

where $y_t \equiv Y_t/L_t$ and $k_t \equiv K_t/L_t$

# Aggregate market equilibrium

- resource constraint:

$$Y_t + (1-\delta)K_t = K_{t+1} + C_t, \tag{A}$$

where $C_t$ is *aggregate* consumption:

$$C_t \equiv L_{t-1}C_t^O + L_t C_t^Y$$

- consumption by the old:

$$L_{t-1}C_t^O = (r_t + \delta)K_t + (1-\delta)K_t$$

- consumption by the young:

$$L_t C_t^Y = W_t L_t - S_t L_t$$

- Hence, aggregate output is:

$$\begin{aligned} C_t &= (r_t + \delta)K_t + (1-\delta)K_t + W_t L_t - S_t L_t \\ &= Y_t + (1-\delta)K_t - S_t L_t \end{aligned} \qquad \text{(B)}$$

- Comparing (A) and (B) yields:

$$S_t L_t = K_{t+1}$$

saving by the young determines the future capital stock

- Population growth:

$$L_t = L_0(1+n)^t, \quad n > -1$$

- Intensive-form expression:

$$S\left(W_t, r_{t+1}\right) = (1+n)\, k_{t+1}$$

# Dynamics and stability

- Model can be expressed in single nonlinear difference equation:

$$(1+n)k_{t+1} = S\Big[\underbrace{f(k_t) - k_t f'(k_t)}_{W_t}, \underbrace{f'(k_{t+1}) - \delta}_{r_{t+1}}\Big]$$

- Slope of fundamental difference equation:

$$\frac{dk_{t+1}}{dk_t} = \frac{-S_W k_t f''(k_t)}{1 + n - S_r f''(k_{t+1})}$$

  - stability condition is $\left|\frac{dk_{t+1}}{dk_t}\right| < 1$
  - numerator is positive (because $0 < S_W < 1$ and $f''(\cdot) < 0$)
  - denominator is ambiguous (because $S_r$ is)

- For expository purposes focus on *unit-elastic* case:

$$y_t = k_t^{1-\epsilon_L} \qquad \text{so that} \quad W_t = \epsilon_L k_t^{1-\epsilon_L}$$

$$U(x) = \log x \quad \text{so that} \quad S_t = W_t/(2+\rho)$$

- Fundamental difference equation for unit-elastic model:

$$k_{t+1} = g(k_t) \equiv \left(\frac{\epsilon_L}{(1+n)(2+\rho)}\right) k_t^{1-\epsilon_L}$$

  - **Figure 17.1** shows the phase diagram
  - Steady-state equilibrium is unique and stable

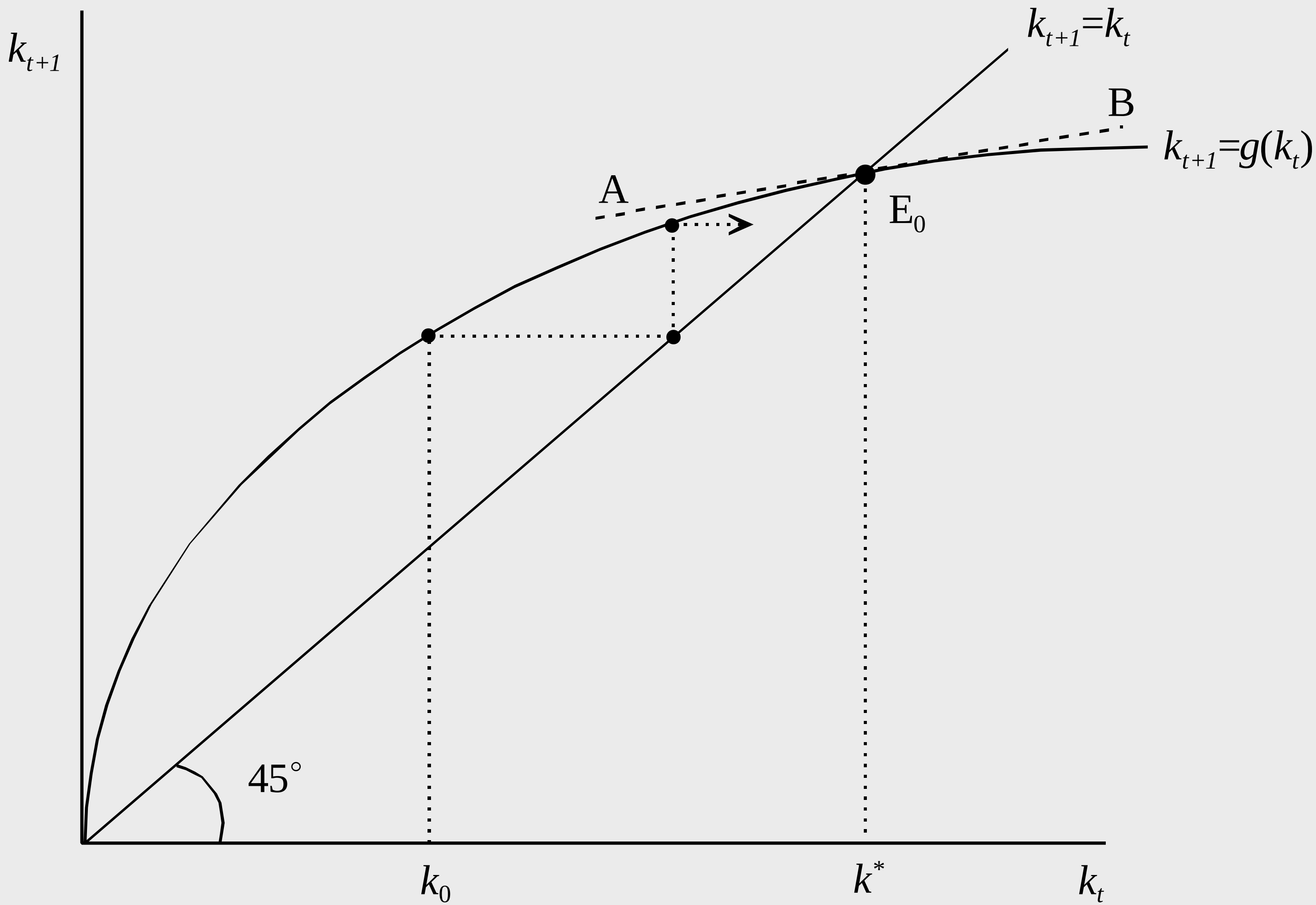


Figure 17.1: The Unit-Elastic Diamond-Samuelson Model

# Efficiency

- Ignoring transitional dynamics, what would an *optimal steady-state* look like?
- Optimal steady-state is such that the lifetime utility of a "representative" young agent is maximized subject to the resource constraint:

$$\max_{\{C^Y, C^O, k\}} \quad \Lambda^Y \equiv U(C^Y) + \left(\frac{1}{1+\rho}\right) U(C^O)$$

subject to: $$f(k) - (n+\delta)k = C^Y + \frac{C^O}{1+n}$$

- The first-order conditions give rise to two types of golden rules:
  - FONC #1, biological-interest-rate *consumption* golden-rule:

$$\frac{U'(C^O)}{U'(C^Y)} = \frac{1+\rho}{1+n}$$

  - FONC #2, *production* golden-rule:

$$f'(k) = n + \delta$$

  - even if one is violated the other must still hold

- In decentralized setting, $r = f'(k) - \delta$ so production rule calls for $r = n$. If $r < n$ there is overaccumulation (dynamic inefficiency). This is quite possible in the unit-elastic model.

# Applications of the model

- Old-age pensions
  - fully-funded versus pay-as-you-go (PAYG) pensions
  - reforming the pension system: transitional problems
- Ageing of the population

# Old-age pensions

- to study a pension system we must add government taxes and transfers to the model
- budget identities:

$$\begin{aligned} C_t^Y + S_t &= W_t - T_t \\ C_{t+1}^O &= (1 + r_{t+1})S_t + Z_{t+1} \end{aligned}$$

  - $T_t$ is tax levied on the young
  - $Z_t$ is transfer provided to the old

- consolidated lifetime budget constraint:

$$W_t - T_t + \frac{Z_{t+1}}{1 + r_{t+1}} = C_t^Y + \frac{C_{t+1}^O}{1 + r_{t+1}}$$

- financing method of the government distinguishes two prototypical systems:
  - Fully-funded system:

$$Z_{t+1} = (1 + r_{t+1})T_t$$

    Contribution $T_t$ earns market interest rate ($r_{t+1}$)

  - PAYG system:

$$L_{t-1}Z_t = L_tT_t \quad \Leftrightarrow \quad Z_t = (1 + n)T_t$$

    Contribution $T_t$ earns the right to receive $(1 + n)T_{t+1}$ when old, where $n$ is the biological interest rate

## Fully-funded pensions

- striking neutrality property
- recall that lifetime budget constraint is:

$$W_t - T_t + \frac{Z_{t+1}}{1+r_{t+1}} = C_t^Y + \frac{C_{t+1}^O}{1+r_{t+1}}$$

- recall that under fully-funded system we have:

$$Z_{t+1} = (1+r_{t+1})T_t$$

- so $T_t$ and $Z_{t+1}$ drop out of the lifetime budget constraint:

$$W_t = C_t^Y + \frac{C_{t+1}^O}{1+r_{t+1}}$$

- economies with or without fully-funded system are identical!

- *Intuition*: household only worries about its total saving $S_t + T_t = S(W_t, r_{t+1})$. Part of this is carried out by the government but it carries the same rate of return.

- Proviso: system should not be "too severe" ($T_t < S(W_t, r_{t+1})$). Otherwise households are forced to save too much by the pension system.

# PAYG pensions

- features transfer from young to old in each period
- We look at *defined-contribution* system: $T_t = T$ for all $t$ so that $Z_{t+1} = (1+n)T$
- household lifetime budget constraint becomes:

$$\hat{W}_t \equiv W_t - \left(\frac{r_{t+1} - n}{1 + r_{t+1}}\right) T = C_t^Y + \frac{C_{t+1}^O}{1 + r_{t+1}}$$

- ceteris paribus factor prices, the PAYG system expands (contracts) the household's resources if the market interest rate, $r_{t+1}$, falls short of (exceeds) the biological interest rate ($n$)

- For logarithmic felicity the savings function becomes:

$$S(W_t, r_{t+1}, T) \equiv \left(\frac{1}{2+\rho}\right) W_t - \left[1 - \left(\frac{1+\rho}{2+\rho}\right)\left(\frac{r_{t+1}-n}{1+r_{t+1}}\right)\right] T$$

with $0 < S_W < 1$, $S_r > 0$, $-1 < S_T < 0$ (if $r_{t+1} > n$), and $S_T < -1$ (if $r_{t+1} < n$).

- capital accumulation:

$$S(W_t, r_{t+1}, T) = (1+n)\, k_{t+1}$$

- Factor rewards under Cobb-Douglas technology:

$$W_t \equiv W(k_t) = \epsilon_L k_t^{1-\epsilon_L}$$

$$r_{t+1} \equiv r(k_{t+1}) = (1-\epsilon_L) k_{t+1}^{-\epsilon_L} - \delta$$

- Fundamental difference equation is illustrated in **Figure 17.2**
  - two equilibria: unstable on (at D) and stable one (at $E_0$)
  - introduction of PAYG system is windfall gain to the then old but leads to crowding out of capital (see path A to C to $E_0$). In the long run, wages fall and the interest rate rises.

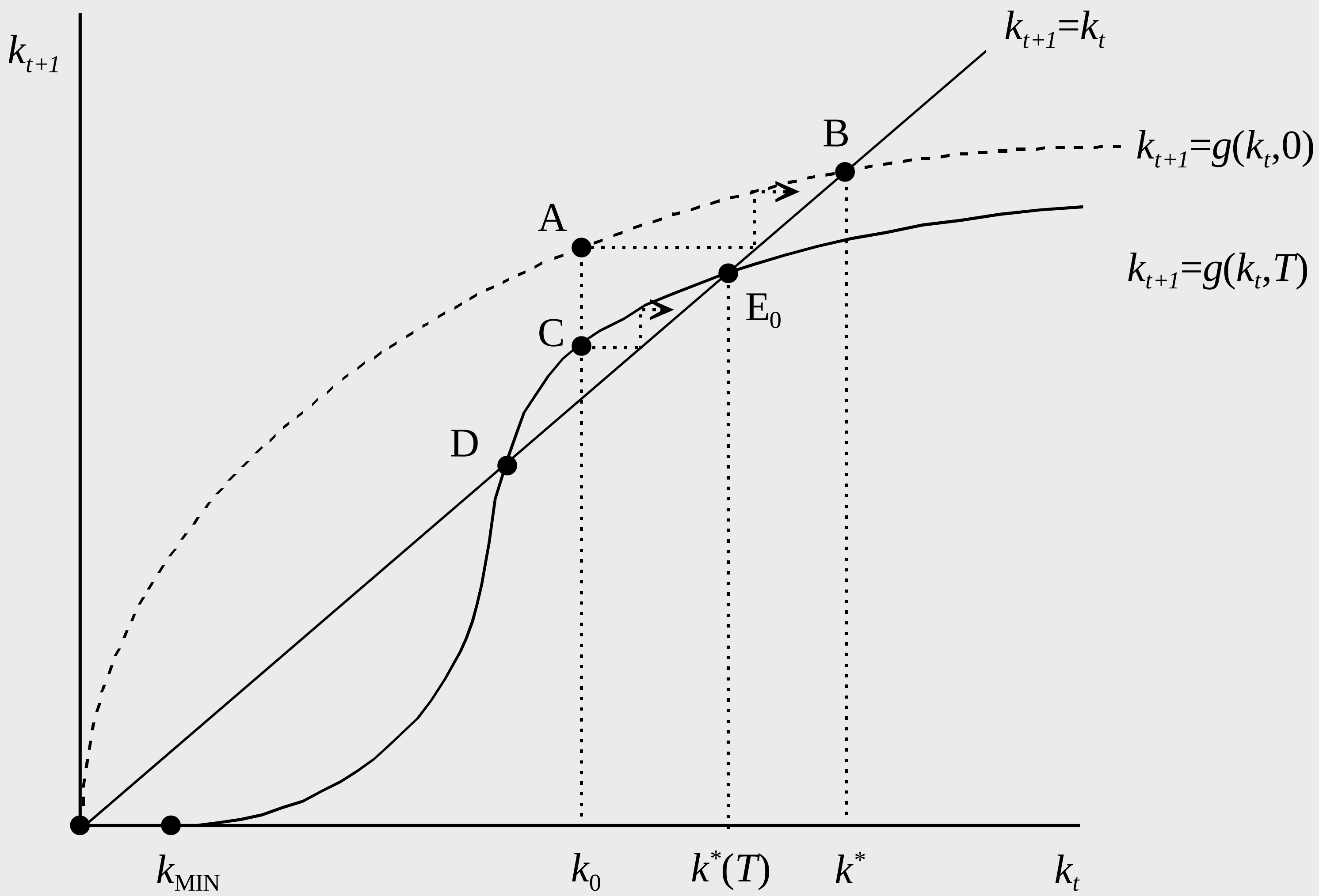


Figure 17.2: PAYG Pensions in the Unit-Elastic Model

# Digression: welfare effect of PAYG system

- Ignoring transitional dynamics, what is the effect on welfare if $T$ is changed marginally?
- Two useful tools:
  - Indirect utility function
  - Factor price frontier
- Indirect utility function is defined as follows:

$$\bar{\Lambda}^Y(W, r, T) \equiv \max_{\{C^Y, C^O\}} \left\{ \Lambda^Y(C^Y, C^O) \text{ subject to } \hat{W} = C^Y + \frac{C^O}{1+r} \right\},$$

with:

$$\hat{W} = W - \left(\frac{r-n}{1+r}\right) T$$

- Key properties of the IUF:

$$\frac{\partial \bar{\Lambda}^Y}{\partial W} = \frac{\partial \Lambda^Y}{\partial C^Y} > 0$$

$$\frac{\partial \bar{\Lambda}^Y}{\partial r} = \left(\frac{S}{1+r}\right)\frac{\partial \Lambda^Y}{\partial C^Y} > 0$$

$$\frac{\partial \bar{\Lambda}^Y}{\partial T} = -\left(\frac{r-n}{1+r}\right)\frac{\partial \Lambda^Y}{\partial C^Y} \gtreqless 0$$

- An increase in $T$ has three effects:
  - wage effect: $W\downarrow$ which is bad for welfare
  - interest rate effect: $r\uparrow$ which is good for welfare
  - direct effect depending on sign of $r-n$

- Factor price frontier is defined as follows:

$$W_t = \phi(r_t)$$

- Key property of FPF:

$$\frac{dW_t}{dr_t} \equiv \phi'(r_t) = -k_t$$

- Welfare effect of marginal change in $T$:

$$\frac{d\bar{\Lambda}^Y}{dT} = \frac{\partial\bar{\Lambda}^Y}{\partial W}\frac{dW}{dT} + \frac{\partial\bar{\Lambda}^Y}{\partial r}\frac{dr}{dT} + \frac{\partial\bar{\Lambda}^Y}{\partial T}$$
$$= \frac{\partial\Lambda^Y}{\partial C^Y}\left[\frac{dW}{dT} + \left(\frac{S}{1+r}\right)\frac{dr}{dT} - \left(\frac{r-n}{1+r}\right)\right]$$
$$= -\left(\frac{r-n}{1+r}\right)\left(\frac{\partial\Lambda^Y}{\partial C^Y}\right)\left[1 + k\left(\frac{dr}{dT}\right)\right] \propto \operatorname{sgn}(n-r)$$

- There is thus an intimate link between the welfare effect and dynamic (in)efficiency:
  - if $r = n$ then $\frac{d\bar{\Lambda}^Y}{dT} = 0$ (no first-order welfare effects despite capital crowding out)
  - if economy is initially dynamically inefficient ($r < n$) then $\frac{d\bar{\Lambda}^Y}{dT} > 0$ (yield on PAYG pension is higher than market interest rate *and* capital crowding out is a good thing)

# From PAYG to funded system

- Ignoring transitional dynamics is not a good idea: there may be non-trivial welfare costs due to transition from one to another equilibrium
- In a dynamically inefficient economy (with $r < n$ initially) an *increase* in $T$ moves the economy in the direction of the golden-rule equilibrium *and* improves welfare for all generations during transition. Optimal to expand and not to abolish the system.
- In a dynamically efficient economy (with $r > n$ initially) an *decrease* in $T$ moves the economy in the direction of the golden-rule equilibrium *but* during transition it improves welfare for some generations (e.g. those born in the steady-state) and deteriorates it for other generations (e.g. the currently old). How do we evaluate the desirability?
  - postulate social welfare function, weighing all generations
  - adopt the Pareto criterion

- In a dynamically efficient economy it is impossible to move from a PAYG to a funded system in a Pareto-improving manner: a cut in $T$ makes the old worse off and there is no way to compensate them without making some future generation worse off.

# PAYG pensions and induced retirement

- Martin Feldstein: PAYG system not only affects the household's savings decision but also its retirement decision
  - Labour supply is endogenous during youth
  - The pension contribution rate is potentially distorting (proportional to labour income)
  - *Intragenerational* fairness: pension is proportional to contribution during youth (the lazy get less than the diligent)

- Preview of some key results:
  - pension contribution acts like an employment *subsidy* if the so-called *Aaron condition* holds
  - the general model displays a continuum of perfect foresight equilibria (Cobb-Douglas case has unique perfect foresight equilibrium)
  - if economy is in golden-rule equilibrium ($r = n$) then the contribution rate is non-distorting at the margin
  - Pareto-improving transition from PAYG to fully-funded system *may* now be possible

# The D-S model with endogenous labour supply

## Households

- Retired in old-age but endogenous labour supply during youth (early retirement)
- utility function of agent $i$:

$$\Lambda_t^{Y,i} \equiv \Lambda^Y(C_t^{Y,i}, C_{t+1}^{O,i}, 1 - N_t^i)$$

- budget identities:

$$\begin{aligned} C_t^{Y,i} + S_t^i &= W_t N_t^i - T_t^i \\ C_{t+1}^{O,i} &= (1 + r_{t+1}) S_t^i + Z_{t+1}^i \end{aligned}$$

- pension contribution proportional to wage income:

$$T_t^i = t_L W_t N_t^i$$

where $t_L$ is the statutory tax rate ($0 < t_L < 1$)

- pension received during old age:

$$Z_{t+1}^i = \underbrace{\left( t_L W_{t+1} \sum_{j=1}^{L_{t+1}} N_{t+1}^j \right)}_{\text{(a)}} \left( \underbrace{\frac{N_t^i}{\sum_{j=1}^{L_t} N_t^j}}_{\text{(b)}} \right)$$

  – Term (a): pension contributions of the future young generation (to be disbursed to the then old)

  – Term (b): share of pension revenue received by household $i$ (intragenerational fairness)

- consolidated (lifetime) budget constraint:

$$\left(1 - t_{L,t}^{E}\right) W_t N_t^i = C_t^{Y,i} + \frac{C_{t+1}^{O,i}}{1 + r_{t+1}}$$

$$t_{L,t}^{E} \equiv t_L \left[1 - \left(\frac{W_{t+1}}{W_t}\right) \left(\frac{\sum_{j=1}^{L_{t+1}} N_{t+1}^j}{\sum_{j=1}^{L_t} N_t^j}\right) \left(\frac{1}{1 + r_{t+1}}\right)\right]$$

  - agent has perfect foresight regarding labour supply of the future young
  - effective tax rate, $t_{L,t}^{E}$, different from the statutory tax rate, $t_L$

- Household chooses $C_t^{Y,i}$, $C_{t+1}^{O,i}$, and $N_t^i$ in order to maximize lifetime utility subject to the lifetime budget constraint. First-order conditions:

$$\frac{\partial \Lambda^Y}{\partial C_{t+1}^{O,i}} = \left( \frac{1}{1+r_{t+1}} \right) \left( \frac{\partial \Lambda^Y}{\partial C_t^{Y,i}} \right)$$

$$\left[ -\frac{\partial \Lambda^Y}{\partial N_t^i} = \right] \frac{\partial \Lambda^Y}{\partial (1 - N_t^i)} = (1 - t_{L,t}^E) W_t \left( \frac{\partial \Lambda^Y}{\partial C_t^{Y,i}} \right)$$

  – MRS between future and present consumption is equated to the relative price of future consumption

  – MRS between leisure and consumption (during youth) is equated to the after-effective-tax wage rate

  – it is not $t_L$ but $t_{L,t}^E$ which exerts a potentially distorting effect on labour supply

- Symmetric solution as all agents are identical: $N_t^i = N_t$. With constant population growth $L_{t+1} = (1+n)L_t$ and $t_{L,t}^E$ simplifies to:

$$t_{L,t}^E \equiv t_L \left[1 - \left(\frac{W_{t+1}}{W_t}\right)\left(\frac{N_{t+1}}{N_t}\right)\left(\frac{1+n}{1+r_{t+1}}\right)\right]$$

- $t_{L,t}^E$ is negative if the *Aaron condition* holds, i.e. if the combined effect of growth in wage income and in the population exceeds the interest rate:

$$t_{L,t}^E < 0 \quad \Leftrightarrow \quad \left(1 + \frac{\Delta W_{t+1} N_{t+1}}{W_t N_t}\right)(1+n) > 1 + r_{t+1}$$

  - growth in wage income widens the revenue obtained per young household
  - population growth increases the number of young households and thus widens the total revenue

- effect of $t_L$ on labour supply is ambiguous for two reasons:
  - depends on Aaron condition (is $t^E_{L,t}$ negative of positive?)
  - depends on income versus substitution effect

# The macroeconomy

- Relation between household saving and the capital-labour ratio:

$$S_t = (1+n)N_{t+1}k_{t+1}$$

where $k_t \equiv K_t / (L_t N_t)$.

- Labour supply and the savings function:

$$\begin{aligned} N_t &= N\left(W_t(1-t^E_{L,t}), r_{t+1}\right) \\ S[\cdot] &\equiv \frac{C^O\left(W_t(1-t^E_{L,t}), r_{t+1}\right) - (1+n)t_L W_{t+1} N_{t+1}}{1+r_{t+1}} \end{aligned}$$

- Fundamental difference equation:

$$S\left[W_t(1-t^E_{L,t}),r_{t+1},t_L W_{t+1} N\left(W_{t+1}(1-t^E_{L,t+1}),r_{t+2}\right)\right]$$
$$=(1+n)N\left(W_{t+1}(1-t^E_{L,t+1}),r_{t+2}\right)k_{t+1}$$

  - (bad) $W_t = W(k_t)$ and $r_t = r(k_t)$ so expression contains $k_t$, $k_{t+1}$, *and* $k_{t+2}$ via the factor prices alone!
  - (worse) $t^E_{L,t+1}$ depends on $N_{t+2}$ which itself depends on $k_{t+2}$, $k_{t+3}$, and $t^E_{L,t+2}$ (infinite regress)
  - (disaster) FDE depends on the entire sequence of capital stocks $\{k_{t+\tau}\}_{\tau=0}^{\infty}$ so there is a continuum of perfect foresight equilibria
  - (but) if the utility function is Cobb-Douglas, then labour supply is constant and the perfect foresight equilibrium is unique (case discussed below)

## Steady-state welfare effect

- Despite non-uniqueness of transition path, the steady state equilibrium is unique, so we can study its welfare properties
- The indirect utility function is now:

$$\bar{\Lambda}^Y(W, r, t_L) \equiv \max_{\{C^Y, C^O, N\}} \Lambda^Y(C^Y, C^O, 1-N)$$

subject to: $$WN\left[1 - t_L\left(\frac{r-n}{1+r}\right)\right] = C^Y + \frac{C^O}{1+r}$$

- The welfare effect of a marginal change in the statutory tax is:

$$\frac{d\Lambda^Y}{dt_L} = -N\left(\frac{r-n}{1+r}\right)\left(\frac{\partial \Lambda^Y}{\partial C^Y}\right)\left[W + (1-t_L)k\left(\frac{dr}{dt_L}\right)\right]$$

  - no first-order welfare effect if $r = n$ (golden-rule equilibrium)
  - if $r \neq n$ then welfare effect is ambiguous because $\frac{dr}{dt_L}$ is ambiguous

## Cobb-Douglas preferences

- Assume that the utility function is now:

$$\Lambda_t^Y \equiv \log C_t^Y + \lambda_C \log[1 - N_t] + \left(\frac{1}{1+\rho}\right) \log C_t^O$$

where $\lambda_C \geq 0$ regulates the strength of the labour supply effect.

- Optimal household decision rules:

$$\begin{aligned} C_t^Y &= \left(\frac{1+\rho}{2+\rho+\lambda_C(1+\rho)}\right) W_t^N \\ C_{t+1}^O &= \left(\frac{1+r_{t+1}}{2+\rho+\lambda_C(1+\rho)}\right) W_t^N \\ N_t &= \frac{2+\rho}{2+\rho+\lambda_C(1+\rho)} \end{aligned}$$

with:

$$W_t^N \equiv W_t(1 - t_{L,t}^E) \equiv W_t \left[1 - t_L \left(1 - \left(\frac{W_{t+1}}{W_t}\right)\left(\frac{1+n}{1+r_{t+1}}\right)\right)\right]$$

– labour supply is constant (IE and SE offset each other)

– consumption during youth depends on the future interest rate via the effective tax rate

- Fundamental difference equation is now:

$$(1+n)k_{t+1} = \frac{W(k_t)(1-t_L)}{2+\rho} - \left(\frac{1+\rho}{2+\rho}\right)\left(\frac{t_L\,(1+n)\,W\,(k_{t+1})}{1+r(k_{t+1})}\right)$$

– first-order difference equation in the capital-labour ratio so the transition path is determinate

– assuming stability, there is a unique perfect foresight equilibrium adjustment path

– an increase in $t_L$ leads to crowding out of the steady-state capital stock (just as when lump-sum taxes are used)

- Unlike the lump-sum case, the increase in $t_L$ causes a distortion in the labour supply decision (provided $r \neq n$)
  - recall that the deadweight loss of the distorting tax hinges on the elasticity of the *compensated* labour supply curve (which is positive) not of the *uncompensated* labour supply curve (which is zero for CD preferences).
  - (weak) implication for pension reform: provided lump-sum contributions can be used during transition, a gradual move from PAYG to a funded system is possible

# Digression on deadweight loss of taxation

- Deadweight loss of a distorting tax: the loss in welfare due to the use of a distorting rather than a non-distorting tax.
- In the context of our model, the DWL of the pension tax $t_L$ can be illustrated with **Figure 17.3**
- Assumptions: $(W, r)$ held constant and $r > n$ (dynamic efficiency)
- Model solved in two steps to develop diagrammatic approach
- We define lifetime income as:

$$X \equiv WN\left[1 - t_L\left(\frac{r-n}{1+r}\right)\right] \equiv WN(1 - t_L^E)$$

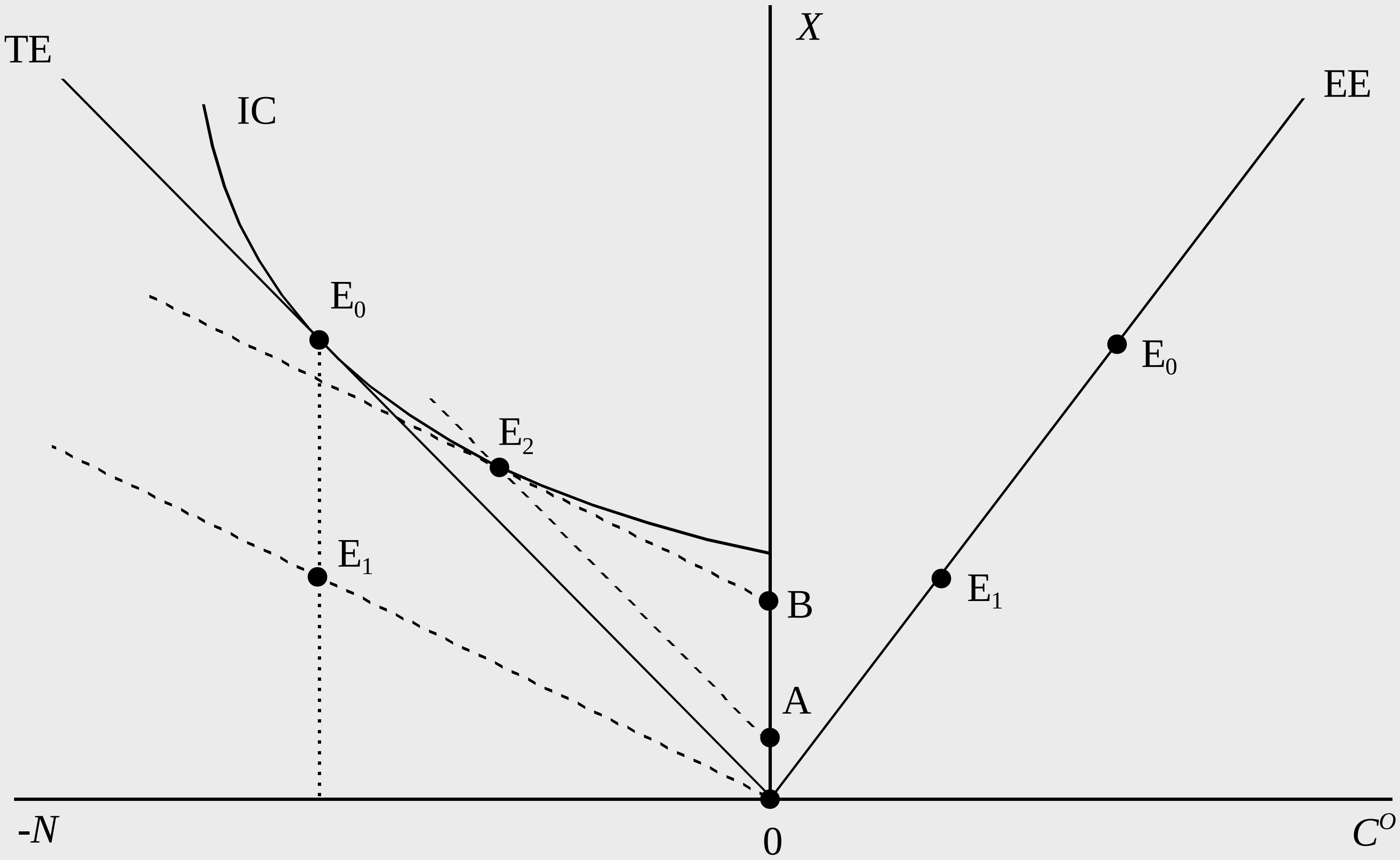


Figure 17.3: Deadweight Loss of Taxation

- *Stage 1*: household chooses $C^Y$ and $C^O$ to maximize:

$$\log C^Y + \left(\frac{1}{1+\rho}\right) \log C^O,$$

subject to the constraint:

$$C^Y + \frac{C^O}{1+r} = X$$

This yields:

$$C^Y = \left(\frac{1+\rho}{2+\rho}\right) X, \qquad C^O = \left(\frac{1+r}{2+\rho}\right) X$$

The second expression is plotted in the right-hand panel of Figure 17.3.

- By substituting the solutions for $C^Y$ and $C^O$ into the utility function we find:

$$\Lambda^Y \equiv \left(\frac{2+\rho}{1+\rho}\right) \log X + \lambda_C \log[1 - N_t] + \text{constant}$$

- *Stage 2*: the household chooses $X$ and $N$ to maximize $\Lambda^Y$ subject to the constraint $X = WN(1-t_L^E)$. The resulting expressions are:

$$N = \frac{2+\rho}{2+\rho+\lambda_C(1+\rho)}$$

$$X = \frac{(2+\rho)\,W(1-t_L^E)}{2+\rho+\lambda_C(1+\rho)}$$

The maximization problem is shown in the left-hand panel of Figure 17.3: IC is the indifference curve and TE is the constraint. Both are downward sloping because $-N$ is measured on the horizontal axis!

- The optimal solution for $t_L^E = 0$ is given by point E$_0$ in both panels. Now consider what happens if $t_L^E$ is increased:
  - right-hand panel: no effect on EE curve ($r$ is constant)
  - left-hand panel: TE rotates counterclockwise. New equilibrium at E$_1$ (directly below E$_0$)
  - decomposition of total effect: SE: move from E$_0$ to E$_2$; IE move from E$_2$ to E$_1$
- On the vertical axis:
  - 0B is the income one would have to give the household to restore it to its initial indifference curve IC (hypothetical transfer $Z_0$)
  - AB represents the tax revenue collected from the agent (i.e. $t_L^E W N$)
  - 0B minus AB is the dead-weight loss of the tax.
- If lump-sum tax were used then the slope of TE would not change and the DWL would be zero (hypothetical transfer equal to tax revenue).

# Macroeconomic effects of ageing

- The old-age *dependency ratio* is the number of retired people divided by the working-age population
- In the models studied so far, the old-age dependency ratio is assumed to be constant: $\frac{L_{t-1}}{L_t} = \frac{1}{1+n}$.
- As the data in **Table 17.1** show, this is rather unrealistic:
  - in the OECD and the US the population is ageing: proportion of young falls whilst proportion of old rises
  - NOTE: demographic predictions are notoriously unreliable!!!

## Table 17.1. Age composition of the population

| | 1950 | 1990 | 2025 |
|---|---|---|---|
| *World* | | | |
| 0-19 | 44.1 | 41.7 | 32.8 |
| 20-65 | 50.8 | 52.1 | 57.5 |
| 65+ | 5.1 | 6.2 | 9.7 |
| *OECD* | | | |
| 0-19 | 35.0 | 27.2 | 24.8 |
| 20-64 | 56.7 | 59.9 | 56.6 |
| 65+ | 8.3 | 12.8 | 18.6 |
| *United States* | | | |
| 0-19 | 33.9 | 28.9 | 26.8 |
| 20-65 | 57.9 | 58.9 | 56.0 |
| 65+ | 8.1 | 12.2 | 17.2 |

- In the absence of immigration, there are two causes for ageing:
  - decrease in fertility
  - decrease in mortality
- We can study the first effect with D-S model: focus on interaction with pension system

## Revised model

- Population:

$$L_t = (1 + n_t) L_{t-1}$$

with $n_t$ variable

- Saving-capital link:

$$S(W_t, r_{t+1}, n_{t+1}, T) = (1 + n_{t+1})k_{t+1}$$

  - $S_n < 0$: as $n_{t+1}$ decreases, the future pension decreases ($Z_{t+1} = (1 + n_{t+1})\,T$), and saving increases.
  - LHS: a reduction in $n_{t+1}$ allows for a higher capital-labour ratio for a given level of saving

- A permanent decrease in the fertility rate increases the long-run capital stock. The transition path is shown in **Figure 17.4**. Economy-wide asset ownership rises because the proportion of old increases.

- Qualitatively the same conclusion as Auerbach & Kotlikoff reach on basis of detailed CGE model!

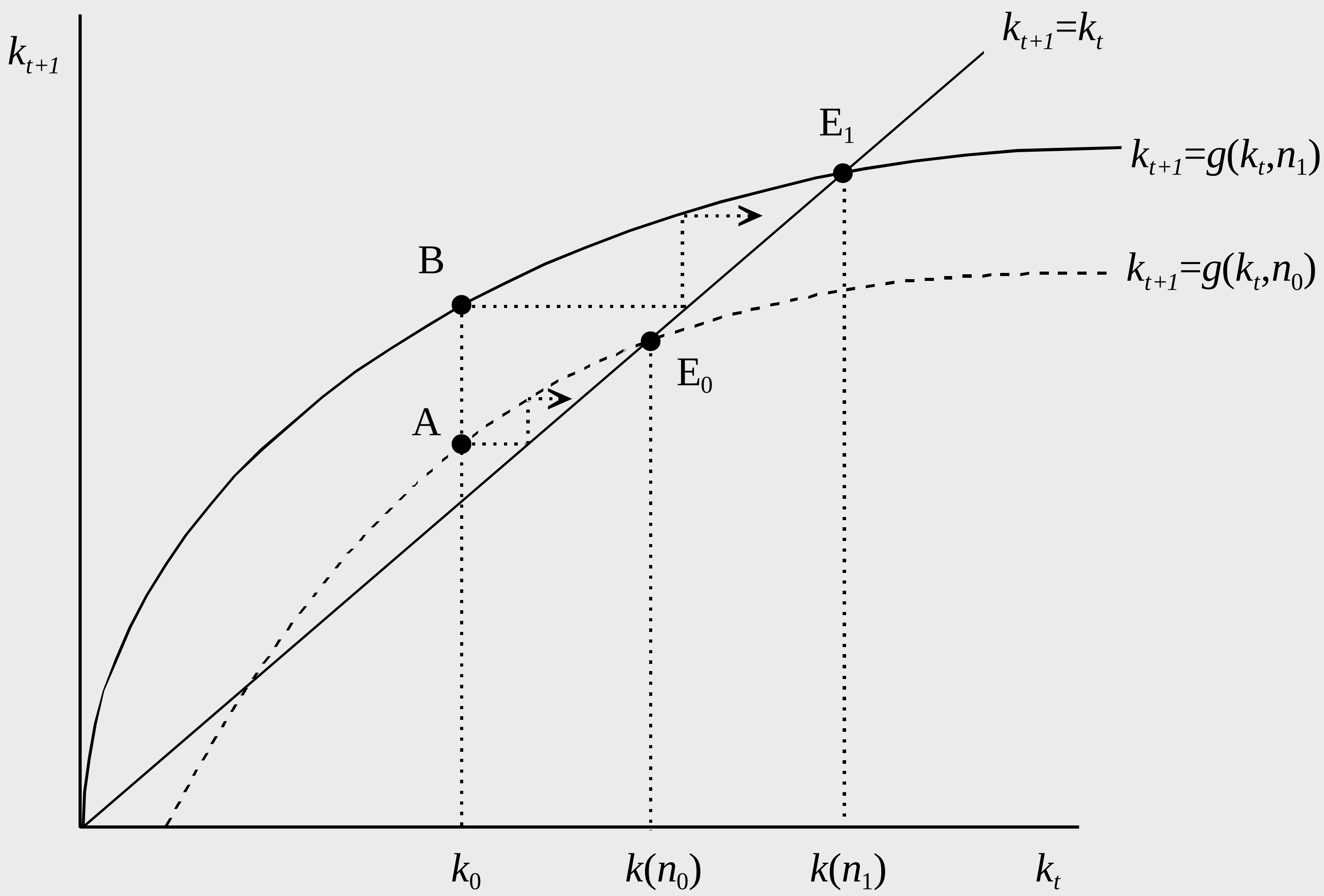


Figure 17.4: The Effects of Ageing

# Extensions

- Human capital accumulation
  - automatic knowledge transfer and endogenous growth
  - a family externality and the benefit of a mandatory education system
- Public investment
  - macroeconomic effects
  - some modified golden rules
- Intergenerational accounting

# Human capital accumulation

- Human capital creation may be an important engine of growth in the economy
- We study an OLG version of the Lucas-Uzawa model (also studied in Chapter 14) proposed by Azariadis-Drazen

# Households

- work full time during second period of life
- divide time between training and working during first period
- lifetime utility:

$$\Lambda_t^{Y,i} \equiv \Lambda^Y(C_t^{Y,i}, C_{t+1}^{O,i})$$

  - no direct utility attached to leisure and to training (knowledge not value *per se*)

- budget identities:

$$
\begin{aligned}
C_t^{Y,i} + S_t^i &= W_t H_t^i N_t^i \\
C_{t+1}^{O,i} &= (1 + r_{t+1}) S_t^i + W_{t+1} H_{t+1}^i
\end{aligned}
$$

  - $W_t$ is the wage rate per efficiency unit of labour
  - $H_t^i$ is the level of human capital of worker $i$ at time $t$
  - $N_t^i$ is the amount of time spent working (rather than training) during youth
  - $C_t^{Y,i}$, $C_{t+1}^{O,i}$, and $S_t^i$ have their usual meaning

- time constraint during youth:

$$
E_t^i = 1 - N_t^i \geq 0
$$

  - time endowment is unity
  - $E_t^i$ is time spent on training during youth

- Training technology:

$$H_{t+1}^{i} = G(E_{t}^{i})H_{t}^{i}$$

  - positive but non-increasing returns to training ($G' > 0 \geq G''$)
  - no knowledge depreciation ($G(0) = 1$)

- Household optimization in two steps:
  - choose training level to maximize lifetime income
  - choose consumption and savings (subject to lifetime income)

## Step 1: Training decision

- Household chooses $E_t^i$ such that lifetime income is maximized:

$$I_t^i(E_t^i) \equiv H_t^i \left[ W_t(1 - E_t^i) + \frac{W_{t+1} G(E_t^i)}{1 + r_{t+1}} \right]$$

- first-order (Kuhn-Tucker) condition:

$$\begin{aligned} \frac{dI_t^i}{dE_t^i} &= H_t^i \left[ -W_t + \frac{W_{t+1} G'(E_t^i)}{1 + r_{t+1}} \right] \leq 0, \\ E_t^i &\geq 0, \qquad E_t^i \left( \frac{dI_t^i}{dE_t^i} \right) = 0 \end{aligned}$$

- Two possible solutions:
  - no-training solution:

$$G'(0) < \frac{W_t(1+r_{t+1})}{W_{t+1}} \qquad \Rightarrow \qquad E_t^i = 0$$

Corner solution because training technology not productive enough!

  - training solution:

$$E_t^i > 0 \qquad \Rightarrow \qquad 1 + r_{t+1} = \left(\frac{W_{t+1}}{W_t}\right) G'(E_t^i)$$

Invest in human capital until its yield equals the yield on financial assets.

# Step 2: Consumption-saving decisions

- Household chooses $C_t^{Y,i}$, $C_{t+1}^{O,i}$ and $S_t^i$ in order to maximize lifetime utility subject to the lifetime budget constraint:

$$C_t^{Y,i} + \frac{C_{t+1}^{O,i}}{1 + r_{t+1}} = I_t^i,$$

where $I_t^i$ is now maximized lifetime income (see Step 1).

- Key expression is the savings function:

$$S_t^i = S\left(r_{t+1}, (1 - E_t^i) W_t H_t^i, W_{t+1} H_{t+1}^i\right)$$

# Further elements of the model

- Initial condition for household $i$:

$$H_t^i = H_t$$

  Household "inherits" average level of human capital in the economy (*osmotic* human capital transfer across generations)

- Model is symmetric so index $i$ can be dropped
- Constant population ($L_t = L_{t-1} = 1$)
- Total labour supply in efficiency units is $N_t = (1 - E_t)H_t + H_t$

# Summary of the model

$$N_{t+1}k_{t+1} = S(r_{t+1},(1-E_t)W_tH_t,W_{t+1}H_{t+1}) \quad \text{(a)}$$

$$r_{t+1}+\delta = f'(k_{t+1}) \quad \text{(b)}$$

$$W_t = f(k_t)-k_tf'(k_t) \quad \text{(c)}$$

$$N_t = (2-E_t)H_t \quad \text{(d)}$$

$$1+r_{t+1} = \left(\frac{W_{t+1}}{W_t}\right)G'(E_t) \quad \text{(e)}$$

$$H_{t+1} = G(E_t)H_t \quad \text{(f)}$$

- Model displays endogenous growth in the steady state. This is illustrated in **Figure 17.5** for the unit-elastic case with technology $y_t = k_t^{1-\epsilon_L}$ and utility $\Lambda_t^Y = \log C_t^Y + (1/(1+\rho)) \log C_{t+1}^O$.
  - PB is portfolio balance line: $(k, E)$ combinations for which yields on physical and human capital equalize
  - SI is the savings-investment line: $(k, E)$ combinations for which savings equals investment
  - equilibrium at $E_0$
  - growth rate is $\gamma \equiv G(E) - 1$ is depicted in bottom panel
- Engine of growth in the model is the training technology $H_{t+1}^i = G(E_t^i) H_t^i$
  - *level* of training explains *growth rate* in human capital
  - knowledge/technical skills are disembodied (live on after agent has died)
  - endogenous growth vanishes if knowledge/technical skills are embodied

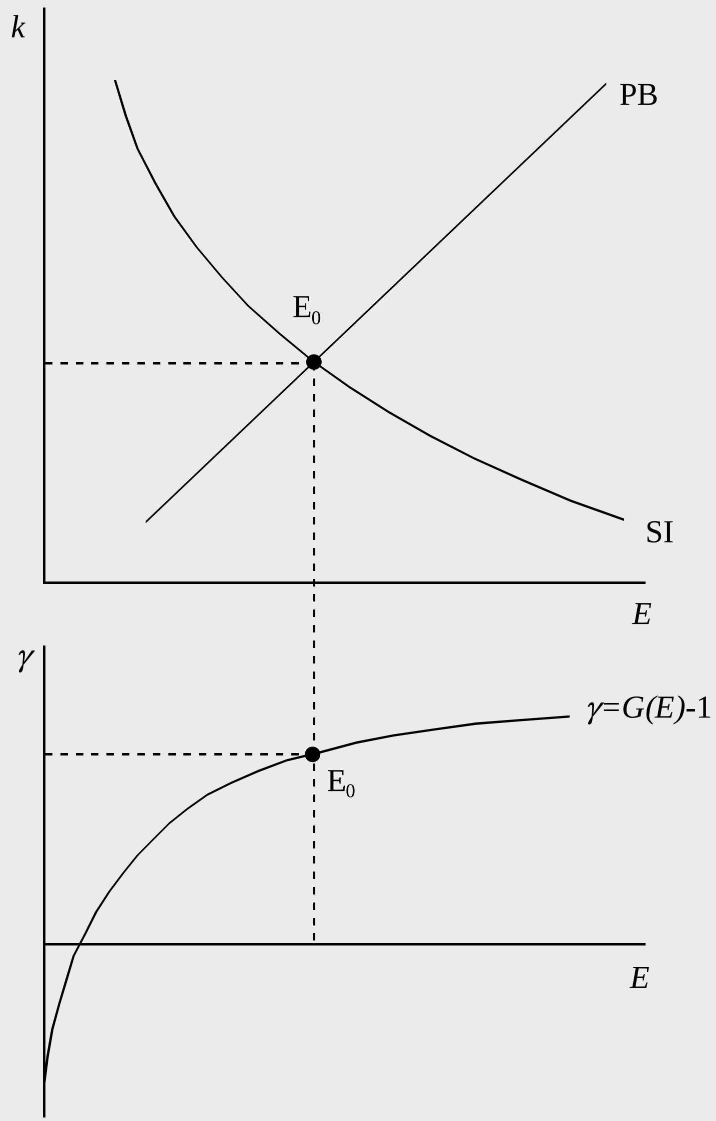


Figure 17.5: Endogenous Growth due to Human Capital Formation

# Application of the A-D model

- Eckstein-Zilcha: why do we have compulsory education systems?
- *Key idea*: parents may under-invest in the human capital of their children (*intra*-family external effect)
- We discuss a simple version of the E-Z model to demonstrate underinvestment result
- Utility function:

$$\Lambda_t^Y \equiv \Lambda^Y(C_t^Y, C_{t+1}^O, Z_t, O_{t+1})$$

  - $C_t^Y$ is consumption when young
  - $C_{t+1}^O$ is consumption when old
  - $Z_t$ is leisure during youth
  - $O_{t+1} \equiv (1+n)H_{t+1}$ is total human capital of the agent's offspring ($H_{t+1}$ is human capital per child, $1+n$ is the number of children)

- In-house training technology (no schools)

$$H_{t+1} = G(E_t) H_t^{\beta}$$

  - $E_t$ is educational effort per child
  - $G(\cdot)$ is the training curve (satisfying $0 < G(0) \leq 1$, $G(1) > 1$, $G' > 0 \geq G''$)
  - positive but diminishing marginal product of human capital: $0 < \beta \leq 1$.
  - NOTE difference with A-D model: now parent chooses human capital of children (costs and benefits accrue to different agents)

- Time endowment: households has two units of time available:
  - one unit is supplied inelastically to the labour market
  - one is divided over leisure and training: $Z_t + (1+n)E_t = 1$
- Household's lifetime budget constraint:

$$C_t^Y + \frac{C_{t+1}^O}{1+r_{t+1}} = W_t H_t$$

  - $W_t = F_N\left(K_t, N_t\right)$ where $N_t \equiv L_t H_t$ (efficiency units of labour)
  - $r_t + \delta = F_K\left(K_t, N_t\right)$

- Household chooses $C_t^Y$, $C_{t+1}^O$, $Z_t$, $E_t$, and $H_{t+1}$ to maximize lifetime utility subject to (a) the training technology, (b) the time constraint, and (c) the consolidated budget constraint. Key first-order conditions:

$$\begin{aligned} \frac{\partial \Lambda^Y / \partial C_t^Y}{\partial \Lambda^Y / \partial C_{t+1}^O} &= 1 + r_{t+1} \\ \frac{\partial \Lambda^Y}{\partial O_t} G'(E_t) H_t^\beta - \frac{\partial \Lambda^Y}{\partial Z_t} &< 0 \quad \Longrightarrow \quad E_t = 0 \\ \frac{\partial \Lambda^Y}{\partial O_t} G'(E_t) H_t^\beta - \frac{\partial \Lambda^Y}{\partial Z_t} &= 0 \quad \Longleftarrow \quad E_t > 0 \qquad \text{(A)} \end{aligned}$$

  - corner solution if the net marginal benefit of training is negative
  - for interior solution training provided until net marginal benefit of training is zero (all gains exhausted)

- Assume that interior solution (A) obtains. Note that (A) only contains costs and benefits of the parent! First hint at underinvestment problem. Not all benefits are taken into account.
- Formal analysis of underinvestment problem: Social Welfare Function approach
- The Social Welfare Function is:

$$SW_0 \equiv \sum_{t=0}^{\infty} \lambda_t \Lambda_t^Y = \sum_{t=0}^{\infty} \lambda_t \Lambda^Y (C_t^Y, C_{t+1}^O, Z_t, O_{t+1})$$

  - $SW_0$ is social welfare in the planning period ($t = 0$)
  - $\{\lambda_t\}_{t=0}^{\infty}$ is a positive monotonically decreasing sequence of weights attached to the different generations (which satisfies $\sum_{t=0}^{\infty} \lambda_t < \infty$)

- resource constraint:

$$C_t^Y + \frac{C_t^O}{1+n} + (1+n)k_{t+1} = F(k_t, H_t) + (1-\delta)k_t$$

where $k_t \equiv K_t/L_t$

- social planner chooses sequences for consumption ($\{C_t^Y\}_{t=0}^{\infty}$ and $\{C_{t+1}^O\}_{t=0}^{\infty}$), the stocks of human and physical capital ($\{K_{t+1}\}_{t=0}^{\infty}$ and $\{H_{t+1}\}_{t=0}^{\infty}$), and the educational effort ($\{E_t\}_{t=0}^{\infty}$) in order to maximize $SW_0$ subject to (a) the training technology, (b) the time constraint, and (c) the resource constraint.

- Most interesting (for our purposes) first-order condition:

$$
\begin{aligned}
\frac{\partial \Lambda^Y(\hat{x}_t)}{\partial Z_t} \quad = \quad & G'(\hat{E}_t)\hat{H}_t^{\beta}\left[\left(\frac{\partial \Lambda^Y(\hat{x}_t)}{\partial O_t}\right)\right. \\
& + \left(\frac{\partial \Lambda^Y(\hat{x}_t)}{\partial C_{t+1}^O}\right) F_N(\hat{k}_{t+1}, \hat{H}_{t+1}) \\
& \left. + \frac{\beta(1+n)\hat{H}_{t+2}}{G'(\hat{E}_{t+1})\hat{H}_{t+1}^{1+\beta}} \left(\frac{\partial \Lambda^Y(\hat{x}_t)}{\partial C_{t+1}^O}\right) \left(\frac{\partial \Lambda^Y(\hat{x}_{t+1})/\partial Z_{t+1}}{\partial \Lambda^Y(\hat{x}_{t+1})/\partial C_{t+1}^Y}\right)\right]
\end{aligned}
$$

  – marginal social costs (LHS) must be equated to marginal social benefits (RHS)

  – marginal social benefits consist of three terms:

    ∗ "own" term, affecting decision maker directly (line 1)

    ∗ "induced" term, affecting earning power of children (line 2)

    ∗ "induced" term, affecting incentive of children to educate *their* children provide (line 3)

- Second and third effects are ignored by parents which leads to under-investment in human capital. Policy options:
  - complex set of incentives (taxes/subsidies) to correct the parent's behaviour
  - compulsory education

# Public investment

- Empirical work by Aschauer prompts a number of questions:
  - What are the macroeconomic effects of public investment?
  - How much public capital should a country possess?
- Study these questions with a modified D-S model
  - exogenous labour supply / lump-sum taxes
  - public capital is a stock variable
  - public capital affects factor productivity (e.g. bridges, roads, airports, etc.)

- Accumulation identity:

$$G_{t+1} - G_t = I_t^G - \delta_G G_t$$

  - $G_{t+1}$ is public capital stock
  - $I_t^G$ is public investment
  - $\delta_G$ is depreciation rate on public capital

- Technology:

$$Y_t = F(K_t, L_t, g_t)$$

  where $g_t \equiv G_t / L_t$

  - CRTS in $(K_t, L_t)$
  - positive but diminishing marginal product of public capital, $F_g > 0$, $F_{gg} < 0$

- Competitive production yields rental expressions:

$$\begin{aligned} r_t &= r(k_t, g_t) \equiv f_k(k_t, g_t) - \delta, \\ W_t &= W(k_t, g_t) \equiv f(k_t, g_t) - k_t f_k(k_t, g_t), \end{aligned}$$

  where $f(k_t, g_t) \equiv F(K_t/L_t, 1, g_t)$ is the intensive-form production function. By assumption, $g_t$ affects $r_t$ and $W_t$ positively (Example: $Y_t = K_t^{1-\epsilon_L} L_t^{\epsilon_L} g_t^{\eta}$ with $0 < \eta < \epsilon_L$)

- Household utility:

$$\Lambda_t^Y = \log C_t^Y + \left(\frac{1}{1+\rho}\right) \log C_{t+1}^O$$

- Household lifetime budget constraint:

$$\hat{W}_t \equiv W_t - T_t^Y - \frac{T_{t+1}^O}{1+r_{t+1}} = C_t^Y + \frac{C_{t+1}^O}{1+r_{t+1}}$$

- Savings function:

$$S_t = (1 - c)\left(W_t - T_t^Y\right) + c\left(\frac{T_{t+1}^O}{1 + r_{t+1}}\right)$$

where $c \equiv \frac{1+\rho}{2+\rho}$

# Model summary

- Model is:

$$(1+n)g_{t+1} = i_t^G + (1-\delta_G)g_t \qquad \text{(a)}$$

$$i_t^G = T_t^Y + \frac{T_t^O}{1+n} \qquad \text{(b)}$$

$$(1+n)k_{t+1} = (1-c)\left[W(k_t, g_t) - T_t^Y\right] + \frac{cT_{t+1}^O}{1+r(k_{t+1}, g_{t+1})} \qquad \text{(c)}$$

  - (a): accumulation identity for public capital per worker ($i_t^G \equiv I_t^G / L_t$)
  - (b): government budget constraint
  - (c): link between savings and private capital formation

- Immediately obvious that financing method critically affects the model: who pays for $i_t^G$ affects (c) and thus the private capital stock and the rest of the economy

- In **Figure 17.6** we consider the case in which the old are untaxed ($T_t^O = 0$ for all $t$)
  - GE line: $(g, k)$ combinations for which $g_{t+1} = g_t$.
    - horizontal
    - $g_t$ rises (falls) for points above (below) the GE line–see the vertical arrows.
  - KE line: $(g, k)$ combinations for which $k_{t+1} = k_t$.
    - slope is negative (positive) for low (high) private capital stock [Intuition: slope determined by $1 + n - (1 - c) W_k$; $W_k$ high (low) if $k$ is low (high)]
    - $k_t$ increases (decreases) for points above (below) the KE line–see the horizontal arrows.
  - two steady-state equilibria:
    - $E_0$: stable node (stable monotonic or cyclical adjustment)
    - A: saddle point (unstable because both $g_t$ and $k_t$ are predetermined variables)

- Focus on equilibrium E$_0$: what happens if $i_t^G$ is increased?
  - both GE and KE shift up
  - effect on $g$ unambiguously positive
  - effect on $k$ depends on relative scarcity of public capital ($W \uparrow$ but $T_t^Y \uparrow$ so net effect ambiguous): $k$ rises (falls) if $i^G/y < \eta\epsilon_L$ ($> \eta\epsilon_L$), i.e. if public capital is initially relatively scarce (abundant).

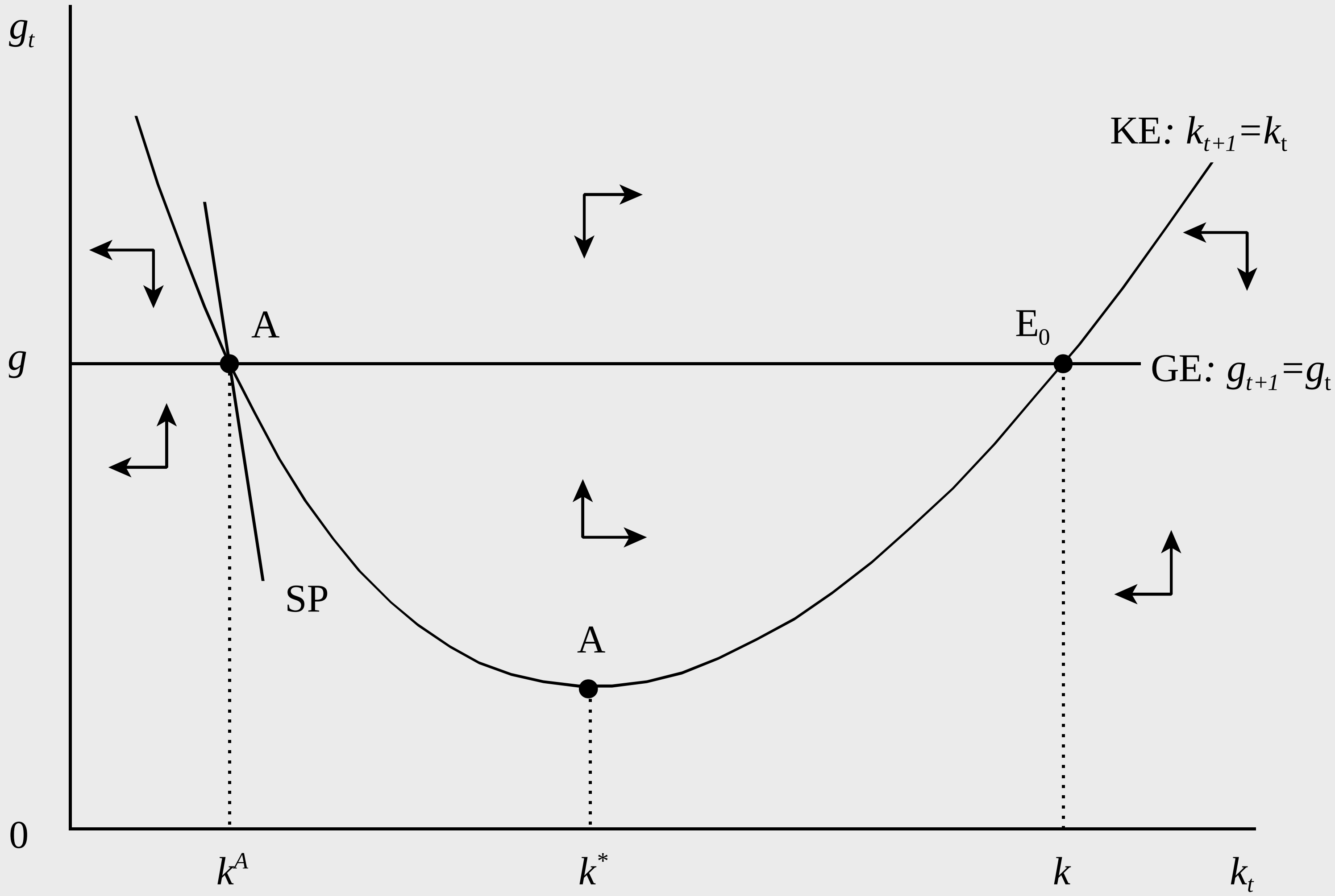


Figure 17.6: Public and Private Capital

# How much public capital should we have?

- SWF approach in an OLG setting
- Social Welfare Function:

$$SW_0 \equiv \left(\frac{1+n}{1+\rho_G}\right)^{-1} \Lambda^Y(C_{-1}^Y, C_0^O) + \sum_{t=0}^{\infty} \left(\frac{1+n}{1+\rho_G}\right)^t \Lambda^Y(C_t^Y, C_{t+1}^O)$$

  - Benthamite format ("...greatest happiness of the greatest number...")
  - $\rho_G$ is the planner's discount rate ($\rho_G > n$). May or may not equal $\rho$.
  - special treatment of current old generation to avoid dynamic inconsistency of the social optimum (see Intermezzo)

- Resource constraint:

$$C_t^Y + \frac{C_t^O}{1+n} + (1+n)\left[k_{t+1} + g_{t+1}\right] = f(k_t, g_t) + (1-\delta)k_t + (1-\delta_G)g_t$$

- Social planner chooses $\{C_t^Y\}_{t=0}^{\infty}$, $\{C_t^O\}_{t=0}^{\infty}$), $\{g_{t+1}\}_{t=0}^{\infty}$, and $\{k_{t+1}\}_{t=0}^{\infty}$), in order to maximize $SW_0$ subject to the resource constraint, taking $k_0$ and $g_0$ as given.

- Key first-order conditions for the social optimum:

$$\frac{\partial \Lambda^Y(\hat{x}_t)/\partial C_t^Y}{\partial \Lambda^Y(\hat{x}_t)/\partial C_{t+1}^O} = 1 + \hat{r}_{t+1} \quad \text{(a)}$$

$$\hat{r}_{t+1} = f_k(\hat{k}_{t+1}, \hat{g}_{t+1}) - \delta = f_g(\hat{k}_{t+1}, \hat{g}_{t+1}) - \delta_G \quad \text{(b)}$$

$$\frac{\partial \Lambda^Y(\hat{x}_t)/\partial C_t^Y}{\partial \Lambda^Y(\hat{x}_{t-1})/\partial C_t^O} = 1 + \rho_G \quad \text{(c)}$$

  - (a): socially optimal Euler equation; MRS between present and future consumption equated to gross interest factor
  - (b): yields on private and public capital should be equalized (efficient investment)

– (c): $\rho_G$ determines optimal *intra*temporal division of consumption. With additively separable preferences we get:

$$\frac{U'(\hat{C}_t^Y)}{U'(\hat{C}_t^O)} = \frac{1+\rho_G}{1+\rho}$$

* if $\rho_G > \rho$ then planner ensures that $U'(\hat{C}_t^Y) > U'(\hat{C}_t^O)$ i.e. that $\hat{C}_t^Y < \hat{C}_t^O$ (favour the old).
* if $\rho_G = \rho$ then planner ensures that $U'(\hat{C}_t^Y) = U'(\hat{C}_t^O)$ i.e. that $\hat{C}_t^Y = \hat{C}_t^O$ (egalitarian solution)
* if $\rho_G < \rho$ then planner ensures that $U'(\hat{C}_t^Y) < U'(\hat{C}_t^O)$ i.e. that $\hat{C}_t^Y > \hat{C}_t^O$ (favour the young).

# Some final remarks:

- In the steady state, $\hat{r}_t = \rho_G$ so (b) simplifies to:

$$[\hat{r} \equiv] \quad f_k(k,g) - \delta = \rho_G = f_g(k,g) - \delta_G$$

  Hence, modified golden rules for private and public capital accumulation feature the social planner's discount rate

- First-best social optimum can be decentralized if and only if the right policy instruments are available:
  - $i^G$ (and thus $g$) is set correctly
  - *age-specific* lump-sum taxes are available (even stronger requirement than in the representative-agent model)

- If one or more of the policy variables is not available, the problem becomes a second-best optimization problem (Ramsey taxation, modified Samuelson rule)

# Intergenerational accounting

- How should we measure the government's fiscal stance?
- Government budget deficit is fundamentally ambiguous
  - how do we treat public investment
  - how to treat future commitments (medicare, social security, etc.)
  - how to treat government assets
- Auerbach, Gokhale, and Kotlikoff:
  - "...every dollar the government takes in or pays out is labeled in a manner that is economically arbitrary"
  - propose method of generational accounting
  - concepts of "deficit" and "intergenerational distribution" are twin sisters

- Stated advantages of method:
  - invariant to accounting labels
  - bring out zero-sum aspect of GBC (some generation has to pay)
  - allows evaluation of effects of different policies
- We use a simplified D-S model to illustrate (and criticize) the method
  - constant population ($L_{t-1} = L_t = 1$)
  - public goods for young and old (e.g. swimming pools and old-age homes)

# Government

- Flow budget identity

$$B_{t+1} = (1 + r_t)B_t + G_t^O + G_t^Y - T_t^O - T_t^Y$$

  - $B_t$ is government debt
  - $G_t^O$ is pure public good provided to old (free of charge)
  - $G_t^Y$ is pure public good provided to young (free of charge)
  - $T_t^O$ lump-sum tax levied on old
  - $T_t^Y$ lump-sum tax levied on young

- The government budget restriction (implied by the solvency condition) is:

$$B_t = \sum_{\tau=0}^{\infty} R_{t-1,\tau} \left[ T_{t+\tau}^{O} + T_{t+\tau}^{Y} - \left( G_{t+\tau}^{O} + G_{t+\tau}^{Y} \right) \right]$$

where $R_{t-1,\tau}$ is a discount factor:

$$R_{t-1,\tau} = \prod_{s=0}^{\tau} \left( \frac{1}{1 + r_{t+s}} \right)$$

  – pre-existing debt must eventually be covered in present-value terms by future primary surpluses

  – solvency requirement: debt growth dominated by discounting effect

# Households

- Standard assumptions
- Key expressions:

$$\begin{aligned} C_t^Y + S_t &= W_t - T_t^Y \\ C_{t+1}^Y &= (1 + r_{t+1}) S_t - T_{t+1}^O \\ S_t &= B_{t+1} + K_{t+1} \end{aligned}$$

- Lifetime budget constraint:

$$C_t^Y + \frac{C_{t+1}^O}{1+r_{t+1}} = W_t - T_{t,t}$$

  – $T_{t,t}$ is the present value of (lump-sum) taxes that a generation born in period $t$ (second subscript) must pay over the course of its life seen from the perspective of period $t$ (first subscript):

$$T_{t,t} \equiv T_t^Y + \frac{T_{t+1}^O}{1+r_{t+1}}.$$

# Generational accounts

- The GBC can be rewritten in terms of tax bills paid by present and future generations (see book for details):

$$
\begin{aligned}
B_t \quad = \quad & \underbrace{\left(\frac{1}{1+r_t}\right) T_t^O}_{(a)} + \underbrace{\left(\frac{1}{1+r_t}\right)\left[T_t^Y + \frac{T_{t+1}^O}{1+r_{t+1}}\right]}_{(b)} \\
& + \underbrace{\left(\frac{1}{1+r_t}\right)\left(\frac{1}{1+r_{t+1}}\right)\left[T_{t+1}^Y + \frac{T_{t+2}^O}{1+r_{t+2}}\right]}_{(c)} \\
& + \left(\frac{1}{1+r_t}\right)\left(\frac{1}{1+r_{t+1}}\right)\left(\frac{1}{1+r_{t+2}}\right)\left[T_{t+2}^Y + \frac{T_{t+3}^O}{1+r_{t+3}}\right] + \ldots. \\
& - \sum_{\tau=0}^{\infty} R_{t-1,\tau}\left[G_{t+\tau}^O + G_{t+\tau}^Y\right]
\end{aligned}
$$

- (a): remaining tax to be paid by old at time $t$
- (b): lifetime tax bill of young at time $t$
- (c): lifetime tax bill of young at time $t+1$
- NOTE: all expressed in present value terms (discounted back to the end of period $t-1$)

• A more compact statement of the generational accounts is:

$$B_t + \sum_{\tau=0}^{\infty} R_{t-1,\tau}\left[G^O_{t+\tau} + G^Y_{t+\tau}\right] = \sum_{k=t-1}^{\infty} T_{t-1,k},$$

where the $T_{t-1,k}$ terms are defined as follows:

$$T_{t-1,t-1} \equiv \left(\frac{1}{1+r_t}\right) T_t^O \qquad \text{(existing old)}$$

$$T_{t-1,t} \equiv \left(\frac{1}{1+r_t}\right) T_{t,t} \qquad \text{(existing young)}$$

$$= \left(\frac{1}{1+r_t}\right) \left[T_t^Y + \frac{T_{t+1}^O}{1+r_{t+1}}\right]$$

$$T_{t-1,k} \equiv R_{t-1,k-1} T_{k,k} \qquad \text{(future generations)}$$

$$= R_{t-1,k-t} \left[T_k^Y + \frac{T_{k+1}^O}{1+r_{k+1}}\right]$$

- Auerbach, Kotlikoff and various co-workers have constructed empirical implementations of the generational accounting methods. In **Table 17.2** we show the male generational accounts (for the US, in 1991). Key features:
  - Row "0": newborns in 1991: net payment to the government of almost US $79,000 in (PV terms)
  - Row "40": 40-year old males in 1991: net payment is US $180,000!
  - Row "70": 70-year old males in 1991: net *transfer* of US $81,000!
  - Row "Future": the typical future newborn male must pay US $166,500 (compared to US $79,000 for a newborn male in 1991)
  - A-K: there is a striking generational imbalance in US fiscal policy.

## Table 17.2. Male generational accounts

| *Generation's age in 1991* | *Net payments* $\times \$1000$ | *Tax payments* $\times \$1000$ | *Transfer receipts* $\times \$1000$ |
|---|---|---|---|
| 0 | 78.9 | 99.3 | 20.4 |
| 10 | 125.0 | 155.3 | 30.3 |
| 20 | 187.1 | 229.6 | 42.5 |
| 30 | 205.5 | 258.5 | 53.0 |
| 40 | 180.1 | 250.0 | 69.9 |
| 50 | 97.2 | 193.8 | 96.6 |
| 60 | -23.0 | 112.1 | 135.1 |
| 70 | -80.7 | 56.3 | 137.0 |
| 80 | -61.1 | 30.2 | 91.3 |
| 90 | -3.5 | 8.8 | 12.3 |
| Future | 166.5 | | |

# Discussion

- Restate some of the key points suggested by Buiter
- True, deficit is meaningless indicator
- Method of Generational Accounting also has its own problems:
  - lives or dies with validity of the life-cycle model (Ricardian equivalence invalid)
  - method says nothing about effects of public goods
  - method ignores general equilibrium effects
- In *principle* one could build a CGE model to take all these effects into account. Practice, however, will prove recalcitrant.

# Punchlines

- Studied workhorse model of macroeconomics and public finance
  - life-cycle saving
  - dynamic inefficiency quite possible
  - wide set of applications
- Pensions
  - fully funded: neutral (saving by the government)
  - PAYG: not neutral (welfare and crowding-out effects)
  - transitional problems
  - population ageing may lead to extra saving under PAYG system

- Human capital
  - osmotic transfer and growth
  - mandatory education
- Public capital
  - macroeconomic effects
  - some more golden rules
- The pros and cons of generational accounting