Foundations of Modern Macroeconomics

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Chapter 16: Intergenerational Economics, I

Aims of this lecture

- Study "work-horse" model of modern macroeconomics which is based on overlapping generations. Motivation for doing this:
  - Ricardian equivalence may be inappropriate [the chain of bequests may not be fully operational]
  - Tractable way to introduce [and study consequences of] heterogeneous agents
  - Contains Ramsey model as a special case
- Show some applications of the Blanchard-Yaari model
  - fiscal policy [crowding out effects of public consumption]
  - debt neutrality revisited

- Extend the BY model in a number of directions:
  - endogenous labour supply [distorting aspects of taxation]
  - age-dependent productivity [mimic life-cycle; reintroduces possibility of dynamic inefficiency–“oversaving”]
  - more attractive model of the small open economy
- Punchlines

Yaari's lessons

- *Key questions studied*:
  - how does a household behave if it faces *lifetime uncertainty*?
  - what kind of institutions exist to insure oneself against risk of death?
- Up to now we have only studied models without lifetime uncertainty:
  - in the two-period consumption model the agent knows he/she will expire at the end of period 2 [certain death]
  - in the Ramsey model the agent has an infinite horizon [certain eternal life]

- A more realistic scenario:
  - agent has a finite life
  - date of death is uncertain [but demographic data exist]
- Model complications: if date of death is uncertain then ...

(A) ... the agent faces a stochastic decision problem. Hence, the *expected utility hypothesis* must be used

(B) ... the restriction on terminal assets becomes more complicated. If $A(T)$ is real assets at time T and T is the [stochastic] time of death, then the terminal condition is that $A(T) \geq 0$ with probability one.

- Yaari solves complication (A) as follows:
  - even though T is stochastic we have a good idea about the distribution of T in the population [ask the demographers]. The PDF of T is:

$$f(T) \geq 0, \ \forall T \geq 0, \quad \int_0^{\bar{T}} f(T) dT = 1 \qquad (*)$$

    - densities are non-negative
    - T is non-negative
    - $\bar{T}$ is the maximum lifetime
  - define lifetime utility as:

$$\Lambda(T) \equiv \int_0^T U[C(\tau)] e^{-\rho\tau} d\tau \qquad (\#)$$

– but since T is stochastic, an agent has the following objective function:

$$E\Lambda(T) \equiv \int_0^{\bar{T}} f(T)\Lambda\left[T\right]dT$$

– Using (*) and (#) we can derive a simple expression for expected lifetime utility:

$$\begin{aligned} E\Lambda(T) &\equiv \int_0^{\bar{T}} f(T)\Lambda\left[T\right]dT \\ &= \int_0^{\bar{T}} \left[\int_{\tau}^{\bar{T}} f(T)dT\right] U\left[C(\tau)\right]e^{-\rho\tau}d\tau \\ &= \int_0^{\bar{T}} \left[1-F(\tau)\right]U\left[C(\tau)\right]e^{-\rho\tau}d\tau \qquad (**) \end{aligned}$$

where $1-F(\tau)$ is the probability that the consumer will still be alive at time τ:

$$1 - F(\tau) \equiv \int_{\tau}^{\bar{T}} f(T)dT$$

- The key thing to note about (**) is that **lifetime uncertainty merely affects the rate at which felicity, $U[C(\tau)]$, is discounted!**

- Yaari solves complication (B) as follows:
  - first he derives the appropriate terminal condition on real assets in the presence of lifetime uncertainty [but in the absence of insurance opportunities]:

$$A(\bar{T}) = 0 \quad \text{(a)}$$

$$C(\tau) \leq W(\tau) \text{ whenever } A(\tau) = 0. \quad \text{(b)}$$

(a) assets must be zero if agent reaches maximum age

(b) if agent hits constraint in period τ then he/she must start saving [$\dot{A} > 0$] immediately to avoid defaulting

– second he shows that the consumption Euler equation is:

$$\frac{\dot{C}(\tau)}{C(\tau)} = \sigma\left[C(\tau)\right]\left[r(\tau) - \rho - \beta(\tau)\right]$$

where $\beta(\tau)$ is the instantaneous probability of death at time τ:

$$\beta(\tau) \equiv \frac{f(\tau)}{1 - F(\tau)}$$

NOTE: as we saw above, the lifetime uncertainty shows up as a heavier discounting of future felicity [one may not be around to enjoy felicity!]. This is Yaari's first insight.

– third, he argues that in reality all kind of insurance instruments exist. He introduces the so-called *actuarial note*

  * carries instantaneous yield $r^A(\tau)$
  * if you buy $1 of such notes: yield of $r^A(\tau)$ while you are alive; you lose the principal when you die; yield must be higher than yield on other instruments [$r^A > r$] ANNUITY
  * if you sell such a note: get $1 from life insurance company; pay premium of r^A while you are alive; debt is cancelled when you die; premium must compensate risk of the LIC [$r^A > r$] LIFE-INSURED LOAN

– under *actuarial fairness* the rate of return on the two types of instruments satisfy a no-arbitrage condition:

$$r^A(\tau) = r(\tau) + \beta(\tau) \quad (*)$$

The yield on actuarial notes equals the interest rate on traditional assets plus the instantaneous probability of death.

– fourth, Yaari shows that the household will always fully insure, i.e. will hold real wealth in the form of actuarial notes. This means that ...

∗ the budget identity is:

$$\dot{A}(\tau) = r^A(\tau)A(\tau) + W(\tau) - C(\tau)$$

∗ the terminal asset condition is trivially met [WHY?],

∗ and the consumption Euler equation is:

$$\frac{\dot{C}(\tau)}{C(\tau)} = \sigma\left[C(\tau)\right]\left[r^A(\tau) - \rho - \beta(\tau)\right] \quad (\#)$$

– fifth, combining (*) and (#) we derive **Yaari's second lesson**:

$$\frac{\dot{C}(\tau)}{C(\tau)} = \sigma\left[C(\tau)\right]\left[r(\tau) - \rho\right]$$

With fully insured [actuarially fair] lifetime uncertainty, the death rate drops out of the consumption Euler equation altogether!! [**NOTE**: the level of consumption is affected by the death rate]

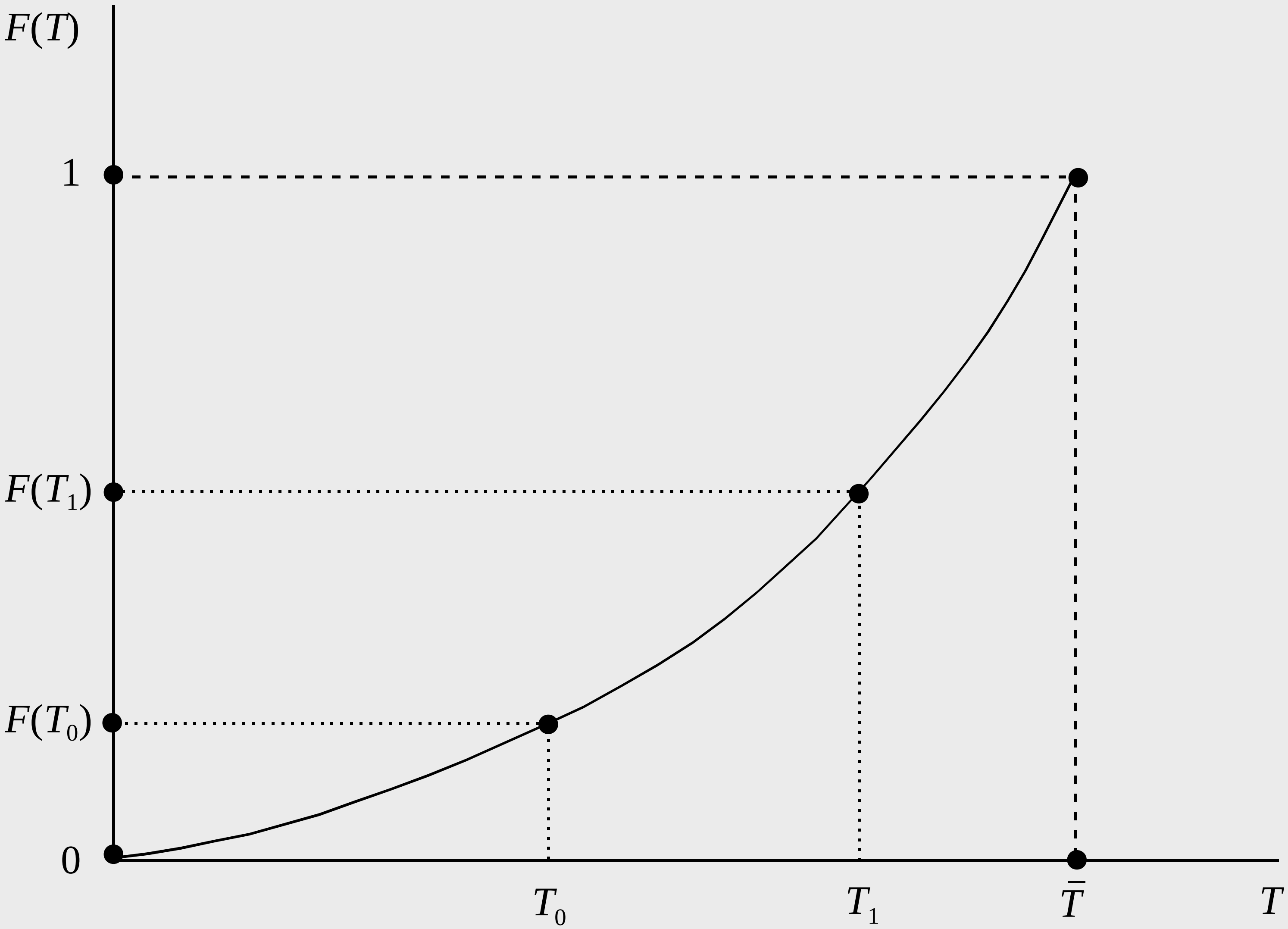

Figure F: Cumulative Probability of Death

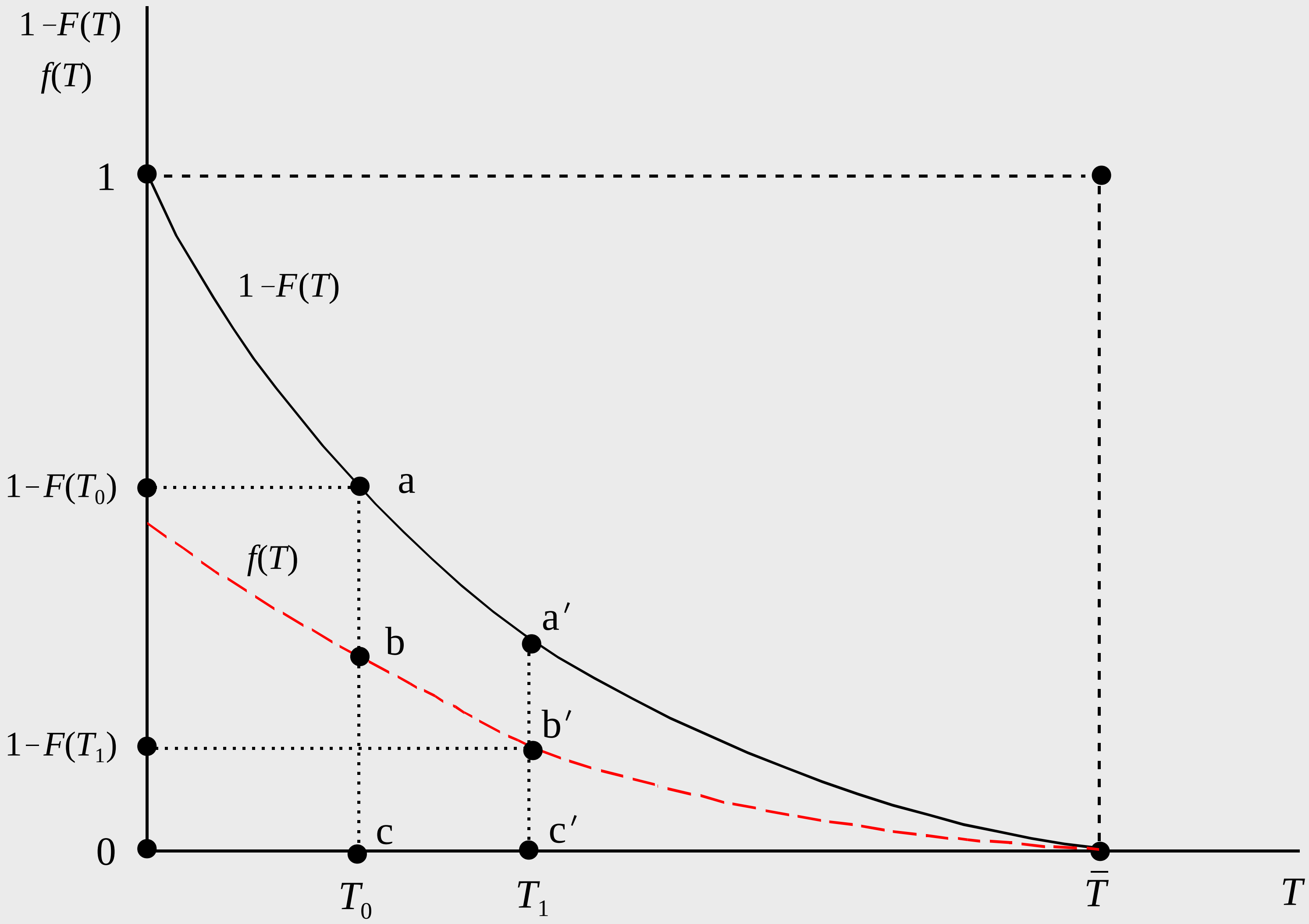

Figure G: Cumulative Probability of Survival and Hazard Rate

Digression on No-Arbitrage Equation (16.11)

- See **Figure K**
- Probability that agent is alive at time τ ($< \bar{T}$) is:

$$1 - F(\tau) = \int_{\tau}^{\bar{T}} f(T)\, dT$$

- For period $\tau + d\tau$ this probability is:

$$1 - F(\tau + d\tau) = \int_{\tau + d\tau}^{\bar{T}} f(T)\, dT$$

- The *conditional* probability that the agent is alive at time $\tau + d\tau$, given that he is alive at time τ is given by:

$$\Pr\left(\text{alive at time } \tau + d\tau \middle| \text{alive at time } \tau\right) \equiv \frac{1 - F(\tau + d\tau)}{1 - F(\tau)}$$

- In Figure K we have:
  - distance ab is equal to $[1-F(\tau+d\tau)]-[1-F(\tau)]$
  - distance bc is equal to $d\tau$
  - hence we can write:

$$
\begin{aligned}
\frac{d}{d\tau}[1-F(\tau)] &= -f(\tau) \approx \frac{[1-F(\tau+d\tau)]-[1-F(\tau)]}{d\tau} \quad \Leftrightarrow \\
1-F(\tau+d\tau) &= 1-F(\tau)-f(\tau)\,d\tau \qquad (1)
\end{aligned}
$$

- Equation (16.10) says:

$$
\begin{aligned}
\left[1+r^A(\tau)d\tau\right]\left(\frac{1-F(\tau+d\tau)}{1-F(\tau)}\right) &= 1+r(\tau)d\tau \\
r^A(\tau)\left(\frac{1-F(\tau+d\tau)}{1-F(\tau)}\right)d\tau &= \frac{[1-F(\tau)]-[1-F(\tau+d\tau)]}{1-F(\tau)}+r(\tau)d\tau
\end{aligned}
$$

- Using (1) to simplify this expression yields:

$$r^A(\tau)\left(\frac{1-F(\tau+d\tau)}{1-F(\tau)}\right)d\tau = \frac{f(\tau)}{1-F(\tau)}d\tau + r(\tau)d\tau \quad \Leftrightarrow$$

$$r^A(\tau)\left(\frac{1-F(\tau+d\tau)}{1-F(\tau)}\right) = \frac{f(\tau)}{1-F(\tau)} + r(\tau) \qquad (2)$$

- Letting $d\tau \longrightarrow 0$ we find:

$$r^A(\tau) = \frac{f(\tau)}{1-F(\tau)} + r(\tau)$$

Q.E.D.

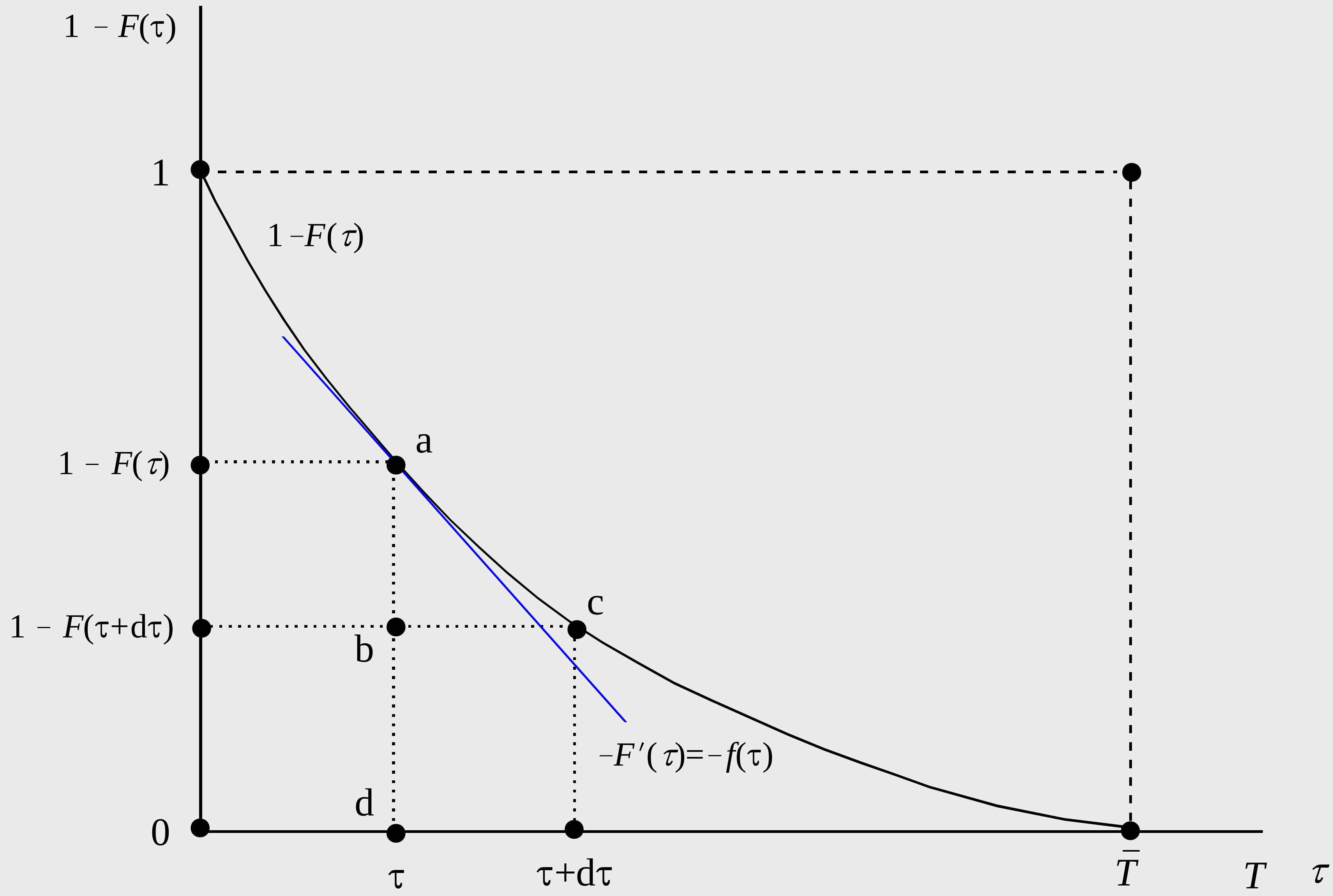

Figure K: Deriving No-Arbitrage Condition

Digression on Multiple Integral

- Clear treatment found in Chapter 9 of M.R. Spiegel (1974), *Advanced Calculus* (New York: McGraw-Hill)

- Provided the region of integration is of the right form we have:

$$\int_{x=a}^{b}\left[\int_{y=f_1(x)}^{f_2(x)} F(x,y)\,dy\right]dx = \int_{y=c}^{d}\left[\int_{x=g_1(y)}^{g_2(y)} F(x,y)\,dx\right]dy \tag{1}$$

- See **Figure J** for the type of region of integration we can handle.
  - line ACB is $y = f_1(x)$ and line ADB is $y = f_2(x)$
  - line CAD is $x = g_1(y)$ and line CBD is $x = g_2(y)$

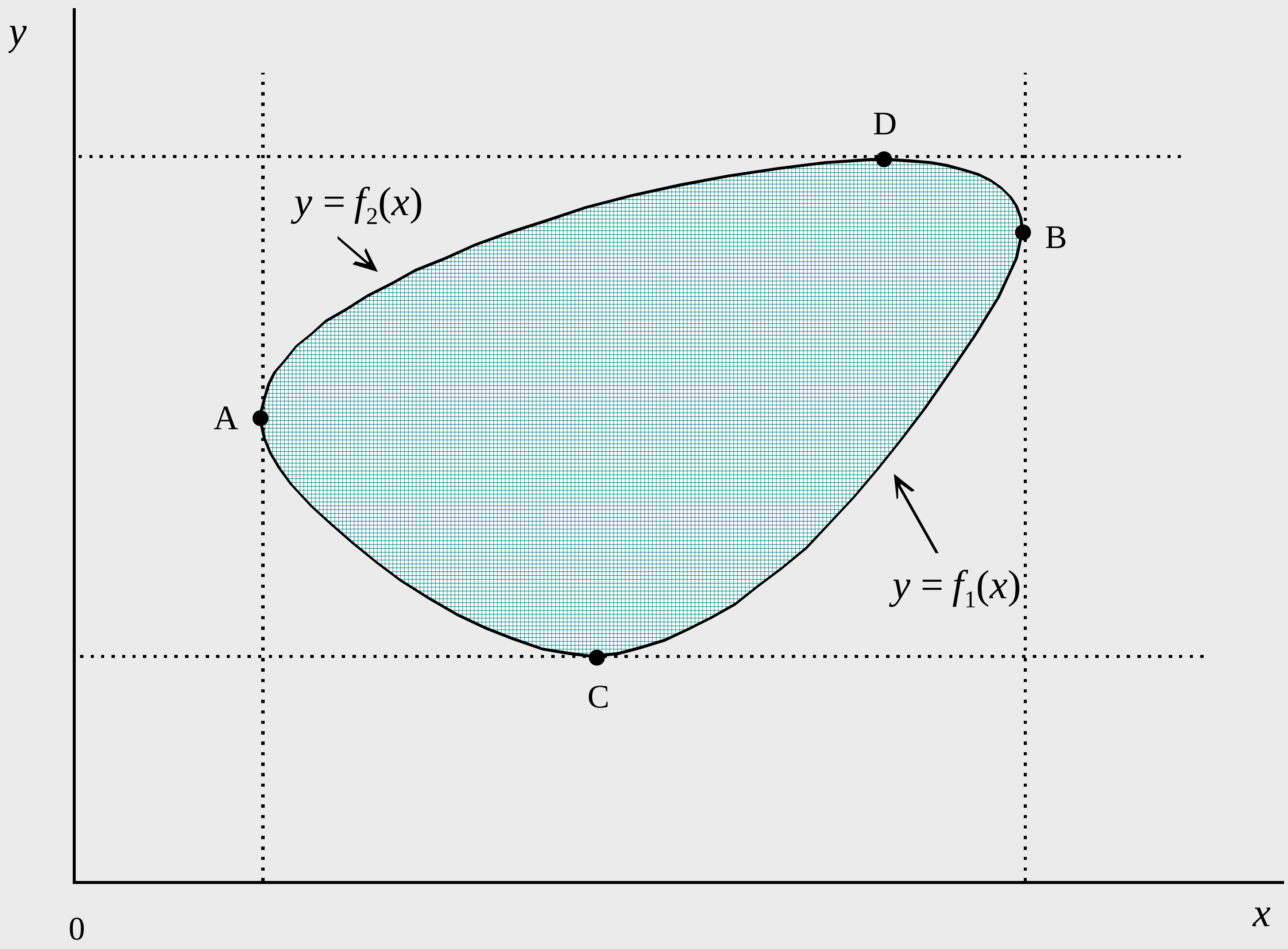

Figure J: Area of Integration in Multiple Integral

- In our case we have:

$$E\Lambda(T) = \int_0^{\bar{T}} f(T) \left[\int_0^T g(\tau)\, d\tau \right] dT$$
$$= \int_0^{\bar{T}} \left[\int_0^T f(T)\, g(\tau)\, d\tau \right] dT, \qquad (2)$$

where $g(\tau)$ is defined as:

$$g(\tau) \equiv U[C(\tau)]\, e^{-\rho\tau}$$

- The original region of integration is identified in **Figure H**:
  - inner integral in (2): take all values of τ between 0 and T (horizontal shading)
  - outer integral in (2): take all values of T between 0 and $\bar{T}$ (vertical shading)
  - hence: area of integration is the double shaded area

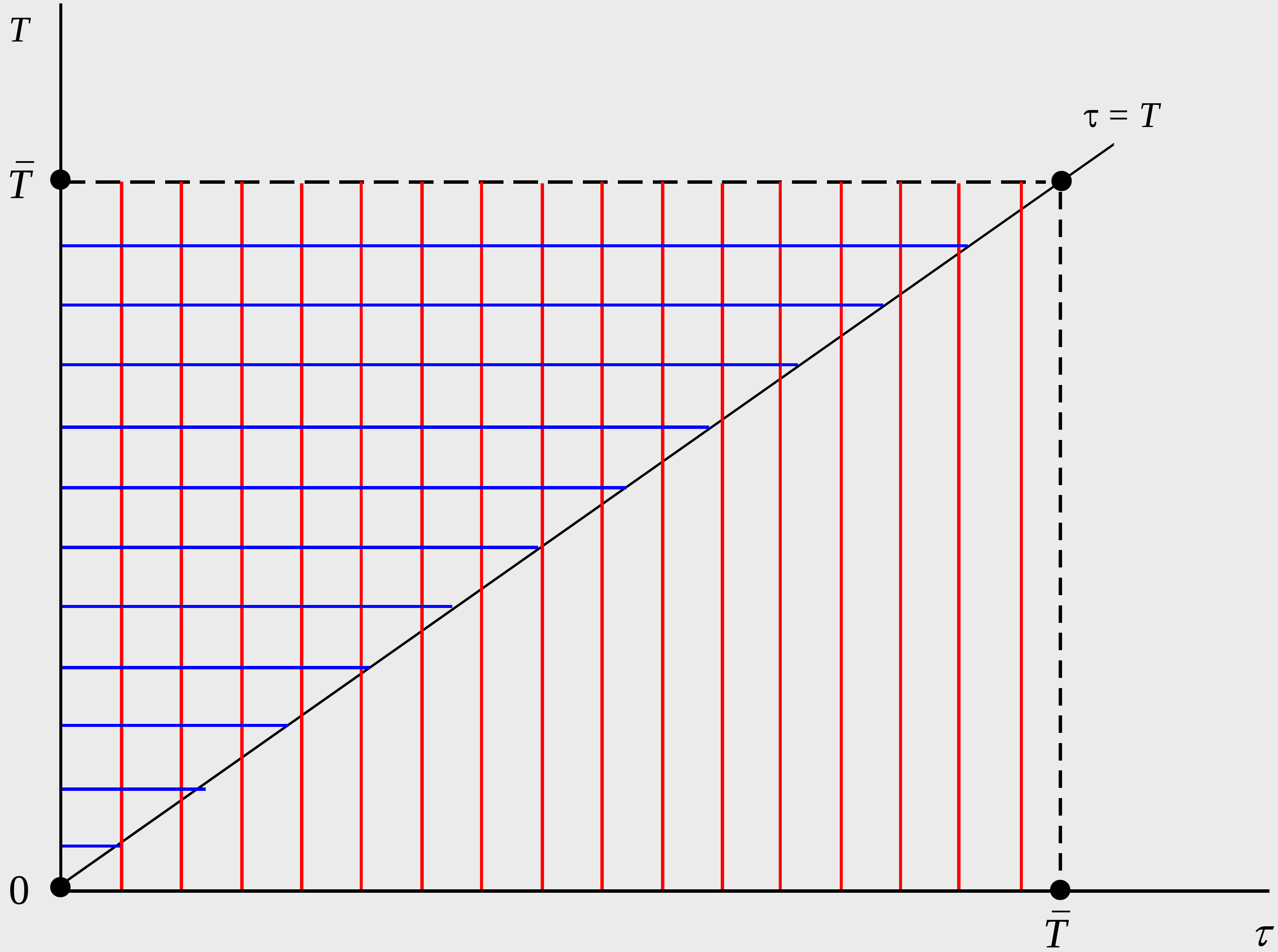

Figure H: Area of Integration

- Changing order of integration in (2) we get:

$$E\Lambda(T) = \int_{?}^{?} \left[\int_{?}^{?} f(T)\, dT \right] g(\tau)\, d\tau \tag{3}$$

  and we need to establish the region of integration.

- Equivalent region is obtained in **Figure I**:
  - inner integral in (3): take all values of T between τ and $\bar{T}$ (vertical shading)
  - outer integral in (3): take all values if τ between 0 and $\bar{T}$ (horizontal shading)
  - hence: area of integration is the double shaded area

- Using the results in (3) we get:

$$E\Lambda(T) = \int_{0}^{\bar{T}} \left[\int_{\tau}^{\bar{T}} f(T)\, dT \right] g(\tau)\, d\tau \tag{4}$$

  Q.E.D.

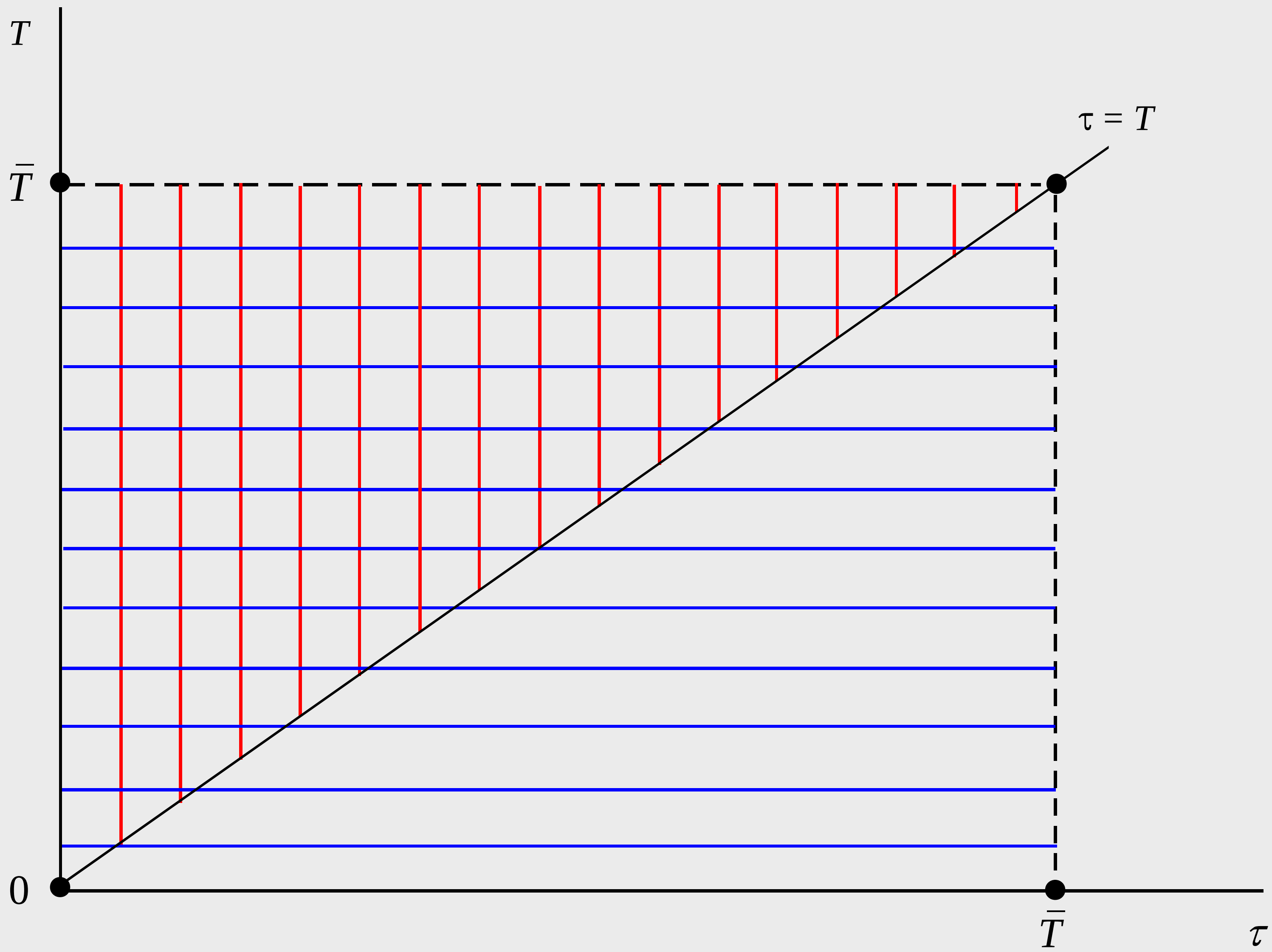

Figure I: Changing Order of Integration

Blanchard's model

- *Key idea*: Blanchard embedded Yaari's approach in a general equilibrium framework. He simplified the approach by assuming that the planning horizon is *age-independent* and is distributed *exponentially* ["perpetual youth" assumption]. Implications of this assumption
  - the death rate equals β [a constant],
  - the expected planning horizon equals $1/\beta$ in that case [**NOTE**: as $\beta = 0$ we have the Ramsey model again]
  - household decision rules linear in age parameter [see below]

- He furthermore assumed that at each instant a *large cohort* of agents in born [bare of any financial assets as they do not receive inheritance–unloved agents]. Implications:
  - denote the cohort born at time τ by $P(\tau, \tau) \equiv \beta P(\tau)$ [with $P(\tau)$ large]: the first index is the birth date and the second index is time
  - all agents face a probability of death of β so $\beta P(\tau)$ agents die at each instant [#births equals #deaths so population size is constant and $P(\tau)$ can be normalized to unity]
  - with large cohorts "probabilities and frequencies coincide" and given the first assumption we can trace the size of each cohort over time:

$$
\begin{aligned}
P(v, \tau) &= P(v, v) e^{\beta(v-\tau)} \\
&= \beta e^{\beta(v-\tau)}, \quad \tau \geq v
\end{aligned}
$$

- NOTES:
  - because we know cohort sizes we can aggregate all surviving households [nice for macro model]
  - eventually, as people die off the cohorts vanish.
  - We can now derive the implications for individual and aggregate household behaviour. Details are in the chapter. Sketch of the outcome here.

Individual household behaviour

- Expected lifetime utility of agent of cohort v in period t:

$$E\Lambda(v,t) \equiv \int_t^\infty [1-F(\tau-t)] \log C(v,\tau) e^{\rho(t-\tau)} d\tau$$
$$= \int_t^\infty \log C(v,\tau) e^{(\rho+\beta)(t-\tau)} d\tau$$

- Budget identity:

$$\dot{A}(v,\tau) = [r(\tau)+\beta] A(v,\tau) + W(\tau) - T(\tau) - C(v,\tau)$$

- No Ponzi Game (NPG) condition:

$$\lim_{\tau\to\infty} e^{-R^A(t,\tau)} A(v,\tau) = 0, \quad R^A(t,\tau) \equiv \int_t^\tau [r(s)+\beta]\, ds$$

- Decision rule for consumption:

$$C(v,t) = (\rho+\beta)\left[A(v,t)+H(t)\right] \qquad (*)$$
$$H(t) \equiv \int_t^{\infty} \left[W(\tau)-T(\tau)\right] e^{-R^A(t,\tau)} d\tau$$

- NOTES:
  - marginal propensity to consume out of total wealth is $\rho+\beta$ [does not feature an age index due to the perpetual youth assumption
  - human wealth discounted at the annuity rate of interest, $r(\tau)+\beta$.

Aggregate household behaviour

- We know that the size of cohort v at time t is $\beta e^{\beta(v-t)}$. This means that we can define aggregate variables by aggregating over all existing agents at time t. For example, aggregate consumption is:

$$C(t) \equiv \beta \int_{-\infty}^{t} e^{\beta(v-t)} C(v,t) dv$$

- In view of (*) *aggregate consumption* satisfies:

$$\begin{aligned} C(t) &\equiv \beta \int_{-\infty}^{t} e^{\beta(v-t)} (\rho + \beta) \left[A(v,t) + H(t) \right] dv \\ &= (\rho+\beta) \Big[\underbrace{\beta \int_{-\infty}^{t} e^{\beta(v-t)} A(v,t) dv}_{A(t)} + \underbrace{\beta \int_{-\infty}^{t} e^{\beta(v-t)} H(t) dv}_{H(t)} \Big] \\ &= (\rho+\beta) \left[A(t) + H(t) \right] \end{aligned}$$

- Similarly, the *aggregate budget identity* can be derived:

$$\dot{A}(t) = r(t)A(t) + W(t) - T(t) - C(t) \qquad (\#)$$

The market rate of interest (**not** the annuity rate) features in the aggregate budget identity: the term $\beta A(t)$ is a transfer–via the life insurance companies–from agents who die to agents who stay alive.

- The consumption Euler equation for individual agents is:

$$\frac{\dot{C}(v,t)}{C(v,t)} = r(t) - \rho$$

The "aggregate Euler equation" satisfies:

$$\begin{aligned}\frac{\dot{C}(t)}{C(t)} &= [r(t) - \rho] - \beta(\rho + \beta)\left(\frac{A(t)}{C(t)}\right) \\ &= \frac{\dot{C}(v,t)}{C(v,t)} - \beta\left(\frac{C(t) - C(t,t)}{C(t)}\right)\end{aligned}$$

- NOTE: aggregate consumption growth differs from individual consumption growth due to the turnover of generations. Newborns are poorer than the average household and therefore drag down aggregate consumption growth.

Key properties of the Blanchard-Yaari model

- We now have all the ingredients of the BY model [firm behaviour is standard; we allow for debt creation in the GBC]: see **Table 16.1**.
- In **Figure 16.1** we show the phase diagram for a special case of the BY model, under the assumption that there is no government at all [$T(t) = G(t) = B(t) = 0$]
- The $\dot{K} = 0$ line represents (C, K) combinations for which net investment is zero. It has the usual properties:
  - golden rule point at A_2
  - $\dot{K} > 0$ ($\dot{K} < 0$) for points below (above) the $\dot{K} = 0$ line [see horizontal arrows]

- The $\dot{C}=0$ line represents (C,K) combinations for which *aggregate* consumption is constant over time. It has some unusual properties:
  - it lies entirely to the left of the dashed line, representing the Keynes-Ramsey capital stock (for which $r^{KR}=\rho$). Why? Using the aggregate Euler equation for the BY model we get:

$$
\begin{aligned}
\frac{\dot{C}(t)}{C(t)} &= [r(t)-\rho]-\beta(\rho+\beta)\left(\frac{K(t)}{C(t)}\right)=0 \quad \Rightarrow \\
r^{BY}-\rho &= \beta(\rho+\beta)\left(\frac{K}{C}\right)^{BY} \quad \Rightarrow \\
r^{BY} &> \rho
\end{aligned}
$$

The interest rate strictly higher than ρ [due to generational turnover]. Hence, K^{BY} strictly smaller than K^{KR}

- the $\dot{C} = 0$ line is upward sloping. Can be understood by comparing points E_0, B, and C in **Figure 16.1**. In E_0 and B r is the same but K/C is higher in B. To restore $\dot{C} = 0$ we must have a move to point C, where K is lower than in B [r higher] and K/C is lower.
- for points above (below) the $\dot{C} = 0$ line, the generational turnover effect is too low (too strong), and aggregate consumption growth is positive (negative). See the vertical arrows in Figure 16.1.

- The BY model without a government features a unique equilibrium at E_0 which is saddle point stable.

Table 16.1. The Blanchard-Yaari model

$$\dot{C}(t) = [r(t) - \rho]\, C(t) - \beta(\rho + \beta)\, [K(t) + B(t)] \quad \text{(T1.1)}$$

$$\dot{K}(t) = F(K(t), L(t)) - C(t) - G(t) - \delta K(t) \quad \text{(T1.2)}$$

$$\dot{B}(t) = r(t)B(t) + G(t) - T(t) \quad \text{(T1.3)}$$

$$r(t) + \delta = F_K(K(t), L(t)) \quad \text{(T1.4)}$$

$$W(t) = F_L(K(t), L(t)) \quad \text{(T1.5)}$$

$$L(t) = 1 \quad \text{(T1.6)}$$

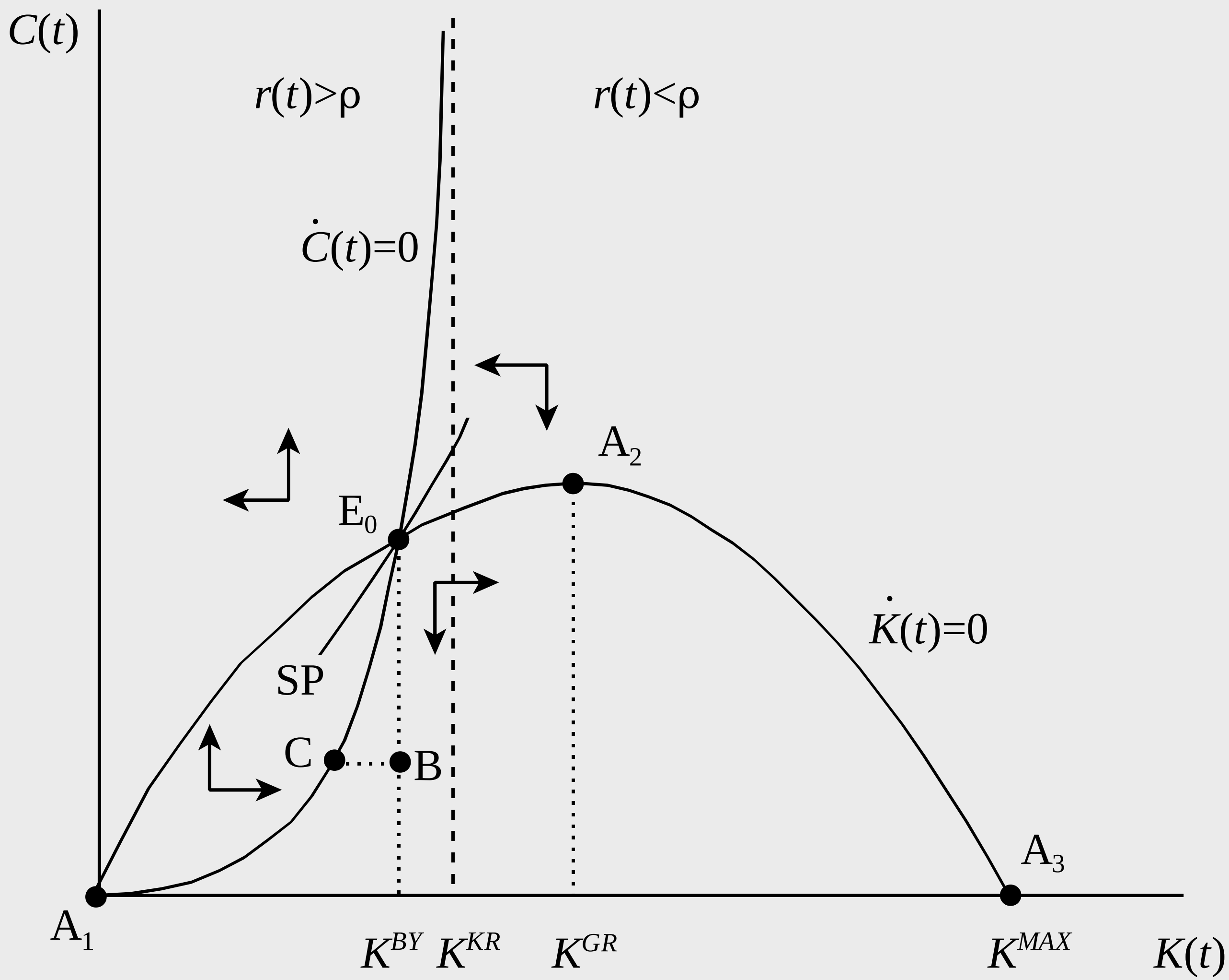

Figure 16.1: Phase Diagram Blanchard-Yaari Model

Applications of the BY model

(A) *Fiscal policy*: increase in government consumption financed by means of lump-sum taxes. Issues:

- crowding out
- intergenerational redistribution of resources

(B) *Non-neutrality of debt*

(A) Fiscal policy

- Unanticipated and permanent increase in G financed by increase in T [recall T is the same for all agents, regardless of their vintage]
- Abstract from government debt: $\dot{B} = B = 0$ and GBC is static, $G = T$
- The shock is analyzed in **Figure 16.2**.
  - The $\dot{K} = 0$ line shifts down by the amount of the shock
  - The $\dot{C} = 0$ line is unchanged [no supply effect of tax]
  - Steady state shifts from E_0 to E_1: $C(\infty) \downarrow$ and $K(\infty) \downarrow$ [the latter does not occur in Ramsey model]

– Transitional dynamics: jump from E_0 to A [at impact] followed by gradual move along saddle path from A to E_1 thereafter. [Recall: no t.d. in Ramsey model]

– Crowding out results:

$$
\begin{aligned}
-1 &< \frac{dC(0)}{dG} < 0 \\
\frac{dC(\infty)}{dG} &< -1
\end{aligned}
$$

Less than one-for-one at impact but more than one-for-one in the long run!

- Economic intuition: the $T \uparrow$ causes an intergenerational redistribution of resources away from future towards present generations.
  - at impact $C(v,0) \downarrow$ because $H(0) \downarrow$ [due to $T \uparrow$]
  - households discount net labour income stream, $W - T$, by annuity rate $r + \beta$ [higher than market interest rate, r]
  - hence, the drop in $C(v,0)$, $C(0)$, and $H(0)$ is not large enough, so that private investment in crowded out: $\dot{K}(0) \downarrow$
  - over time $K(t) \searrow$, so that $[W(t) - T] \searrow$, $r(t) \nearrow$, and $H(t) \searrow$
  - future newborns poorer than newborns in initial steady state [the former have less capital to work with]

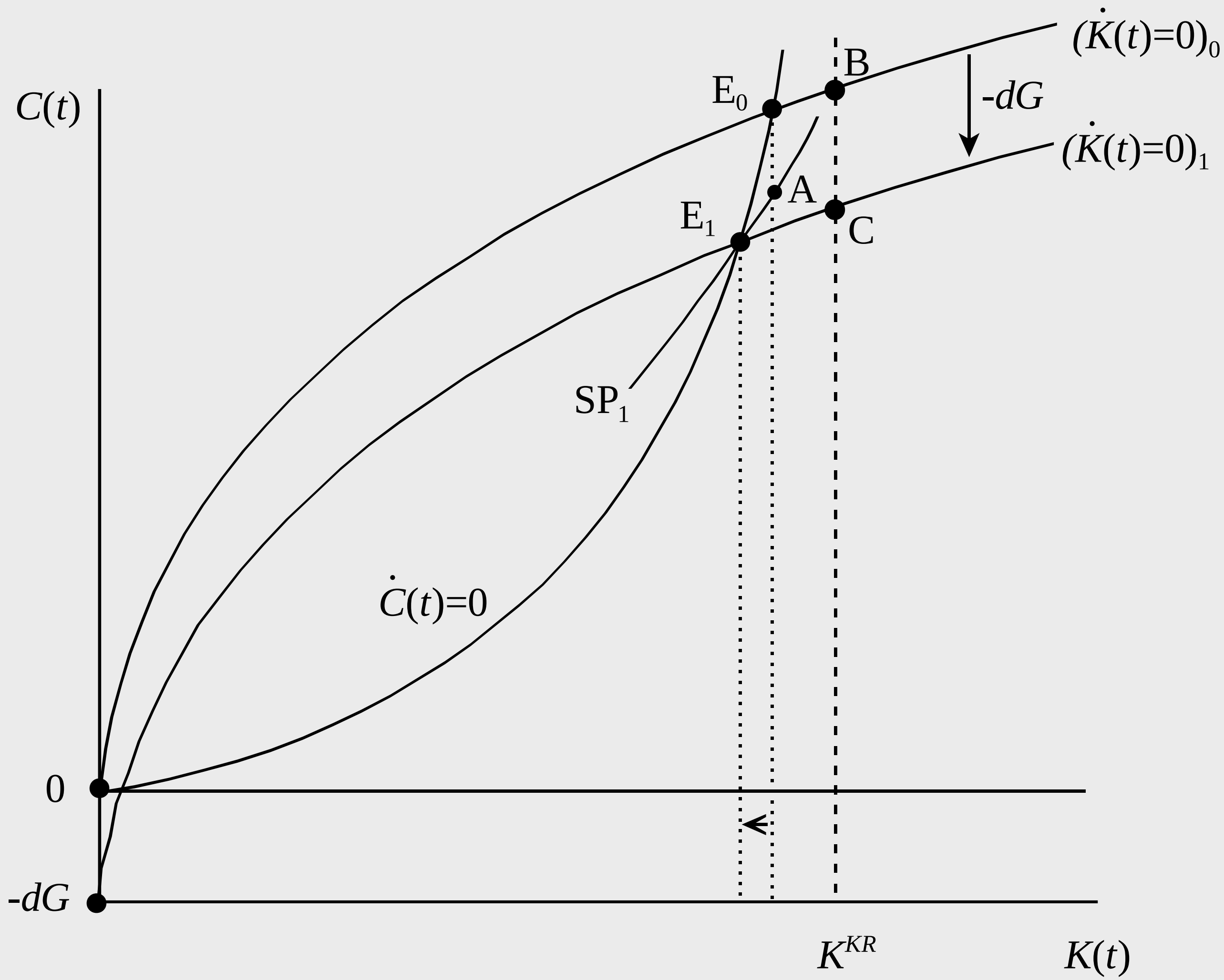

Figure 16.2: Fiscal Policy in the Blanchard-Yaari Model

(B) Non-neutrality of debt

- The fact that T causes intergenerational redistribution in the fiscal policy case hints at the non-neutrality of debt.

- Ricardian non-equivalence can be proven by looking at a simple accounting exercises: substitute the GBC into the HBC

- The aggregate wealth constraint facing household features the following definition for total wealth:

$$\begin{aligned} A(t)+H(t) &\equiv K(t)+B(t)+H(t) \\ &= K(t)+B(t)+\int_t^\infty [W(\tau)-T(\tau)]\, e^{-R^A(t,\tau)}d\tau \\ &= K(t)+\int_t^\infty [W(\tau)-G(\tau)]\, e^{-R^A(t,\tau)}d\tau+\Omega(t) \end{aligned}$$

- here $\Omega(t)$ is defined as:

$$\Omega(t) \equiv B(t) - \int_t^{\infty} [T(\tau) - G(\tau)] \underbrace{e^{-R^A(t,\tau)}}_{(a)} d\tau \qquad (\#)$$

NOTE: Ricardian equivalence holds iff $\Omega(t) \equiv 0$!

- Recall that the GBC can be written as:

$$0 = B(t) - \int_t^{\infty} [T(\tau) - G(\tau)] \underbrace{e^{-R(t,\tau)}}_{(b)} d\tau \qquad (\&)$$

- In (#) the stream of primary surpluses is discounted with the annuity rate [see (a)] whereas the market rate is used for discounting purposes in (&) [see (b)].
  - Hence, $\Omega(t)$ only vanishes iff the birth rate is zero, so that $R^A(t,\tau) = R(t,\tau)$, i.e. in the Ramsey model.
  - If $\beta > 0$ then $\Omega(t) \neq 0$ and Ricardian equivalence fails: the path of $T(\tau)$ and the initial debt level do not drop out of the aggregate HBC.

Extensions of the BY model

(A) *Endogenous labour supply*: introduce endogenous leisure choice into the household model. Issues:

- how do various taxes affect the labour supply decision?
- how do these taxes affect the aggregate economy and the intergenerational distribution of resources?

(B) *Age-dependent labour productivity*: can we mimic life-cycle phenomena in the perpetual youth model?

(C) *The open economy*: the effects of an oil shock on a small open economy facing perfect mobility of financial capital [self study].

(A) Endogenous labour supply

- *Motivation*: to make the model suitable for tax policy analysis it is important to have an endogenous labour supply decision.
- Individual households
  - lifetime utility is now:

$$E\Lambda(v,t) \equiv \int_t^{\infty} \log \left[C(v,\tau)^{\epsilon_C} \left[1 - L(v,\tau)\right]^{1-\epsilon_C} \right] e^{(\rho+\beta)(t-\tau)} d\tau$$

NOTE: unit intertemporal and intratemporal substitution elasticities just as in the RBC model.

– budget identity:

$$\begin{aligned}\dot{A}(v,\tau) &= [r(\tau)+\beta]\,A(v,\tau)+W(\tau)(1-t_L)L(v,\tau)+Z(\tau)\\ &\quad -(1+t_C)C(v,\tau)\\ &= [r(\tau)+\beta]\,A(v,\tau)+W(\tau)(1-t_L)+Z(\tau)\\ &\quad -X(v,\tau) \qquad \text{(*)}\end{aligned}$$

where $Z(\tau)$ is government transfers and $X(v,\tau)$ represents *full consumption*:

$$X(v,\tau) \equiv (1+t_C)C(v,\tau)+W(\tau)(1-t_L)\left[1-L(v,\tau)\right] \qquad \text{(\#)}$$

- problem can be solved by means of *two-stage budgeting*:

– Step (1) [static] for given level of full consumption choose consumption and leisure such that felicity is maximized. Optimality condition requires equality between the MRS between consumption and leisure and the after-tax wage rate:

$$\frac{(1-\epsilon_C)/\left[1-L(v,\tau)\right]}{\epsilon_C/C(v,\tau)} = W(\tau)\left(\frac{1-t_L}{1+t_C}\right) \quad \text{(a)}$$

NOTE: the tax on consumption (e.g. BTW) directly distorts the labour supply choice! Using (a) in (#) we get the "conditionally optimal" solutions:

$$\begin{aligned} (1+t_C)C(v,\tau) &= \epsilon_C X(v,\tau) && \text{(b)} \\ W(\tau)(1-t_L)\left[1-L(v,\tau)\right] &= (1-\epsilon_C)X(v,\tau) && \text{(c)} \end{aligned}$$

NOTE: due to CD assumption gross spending on consumption and leisure are constant proportion of full consumption.

– by substituting the "conditionally optimal" choices for consumption and leisure back into the felicity function we can rewrite lifetime utility in terms of full consumption and a true cost-of-living index:

$$E\Lambda(v,t) \equiv \int_t^{\infty} \left[\log X(v,\tau) - \log P_{\Omega}(\tau)\right] e^{(\rho+\beta)(t-\tau)} d\tau \qquad (\$)$$

$$P_{\Omega}(\tau) \equiv \left(\frac{1+t_C}{\epsilon_C}\right)^{\epsilon_C} \left(\frac{W(\tau)(1-t_L)}{1-\epsilon_C}\right)^{1-\epsilon_C}$$

– Step (2) [dynamic] the household now chooses the optimal time path for full consumption which maximizes ($) subject to the budget identity (*) [plus a NPG condition]. Outcome of this problem:

$$X(v,t) = (\rho+\beta)\left[A(v,t)+H(t)\right]$$

$$\frac{\dot{X}(v,\tau)}{X(v,\tau)} = r(\tau)-\rho, \quad \text{for } \tau \in [t,\infty)$$

$$H(t) \equiv \int_t^\infty \left[W(\tau)(1-t_L)+Z(\tau)\right] e^{-R^A(t,\tau)} d\tau$$

NOTE: full consumption proportional to total wealth, the Euler equation is now in terms of full consumption, and human wealth includes the government transfers.

- Aggregate households: aggregation just as before.
- Rest of the model unchanged. **Table 16.2** gives extended BY model.

Table 16.2. The extended Blanchard-Yaari model

$$\frac{\dot{C}(t)}{C(t)} = r(t) - \rho - \epsilon_C \beta(\rho + \beta) \left[\frac{K(t)}{(1 + t_C)C(t)} \right], \tag{T2.1}$$

$$\dot{K}(t) = Y(t) - C(t) - \delta K(t) \tag{T2.2}$$

$$Z(t) = t_L W(t) L(t) + t_C C(t) \tag{T2.3}$$

$$r(t) + \delta = (1 - \epsilon_L) \left(\frac{Y(t)}{K(t)} \right) \tag{T2.4}$$

$$W(t) = \epsilon_L \left(\frac{Y(t)}{L(t)} \right) \tag{T2.5}$$

$$W(t)\left[1 - L(t)\right] = \left(\frac{1 - \epsilon_C}{\epsilon_C} \right) \left(\frac{1 + t_C}{1 - t_L} \right) C(t), \quad 0 < \epsilon_C \leq 1. \tag{T2.6}$$

$$Y(t) = K(t)^{1-\epsilon_L} L(t)^{\epsilon_L}, \quad 0 < \epsilon_L < 1 \tag{T2.7}$$

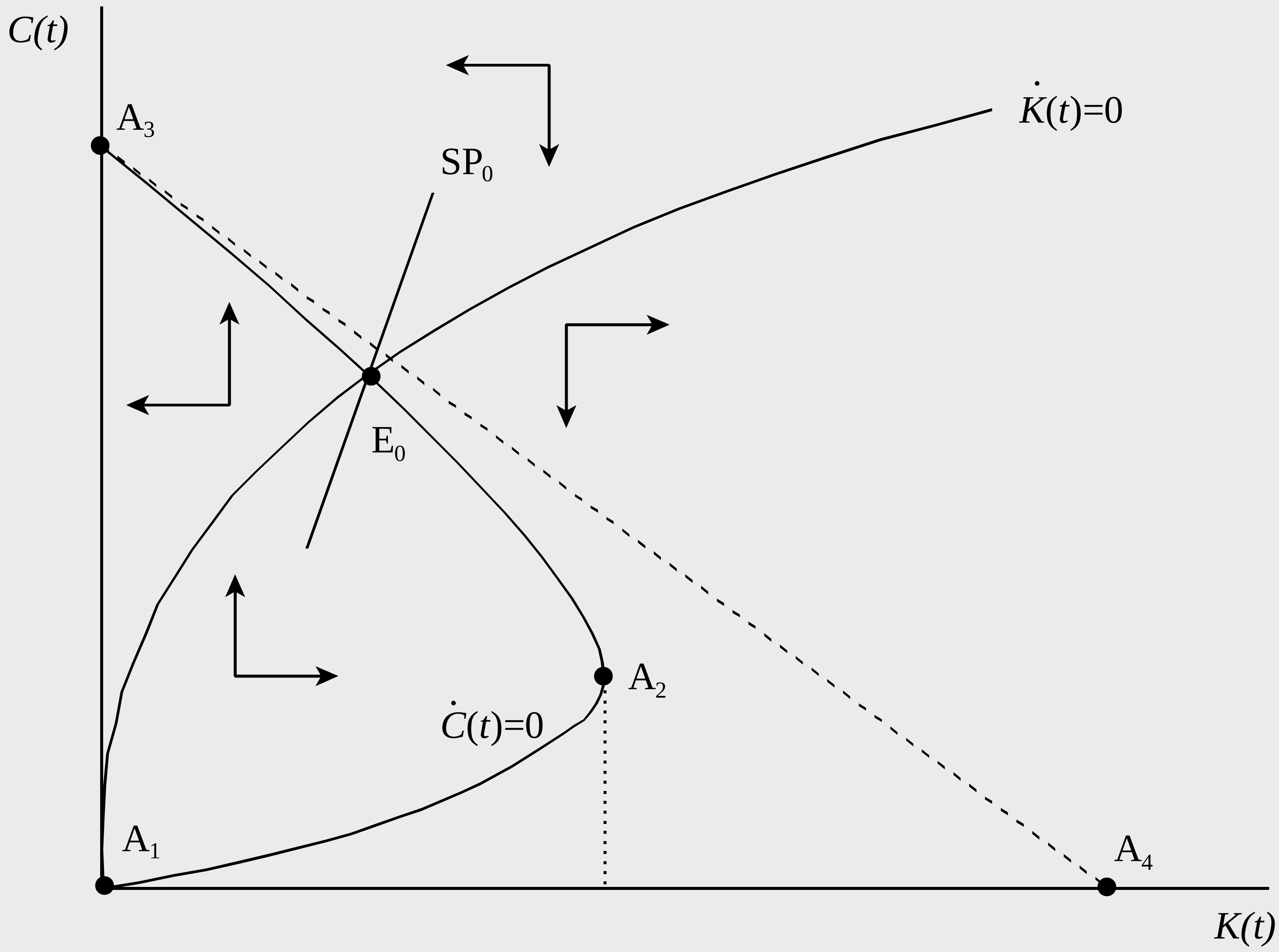

Figure 16.3: Phase Diagram Extended Blanchard-Yaari Model

- In the text we derive the phase diagram for the extended BY model. The resulting diagram is found in **Figure 16.3**. *Features*:

- the $\dot{K} = 0$ line is the same as in Chapter 15 [apart from the supply-side effects of the various tax rate: $\dot{K} > 0$ (< 0) for points below (above) the line–see the horizontal.

- the $\dot{C} = 0$ line combines two mechanisms:

  – **FS**: factor scarcity effect which explains the slope of the $\dot{C} = 0$ line for the representative agent model of Chapter 15.

  – **GT**: generational turnover effect which explains the slope of the $\dot{C} = 0$ line for the standard BY model with exogenous labour supply.

– in the lower branch of the $\dot{C}=0$ line, $L\approx 1$ and the GT effect dominates the FS effect. For points above the line $\dot{C}>0$:

$$\underbrace{\frac{\dot{C}}{C}}_{\uparrow} = \underbrace{r(C,K)}_{\downarrow} - \rho - \left(\frac{\beta\epsilon_C(\rho+\beta)}{1+t_C}\right)\underbrace{\left(\frac{K}{C}\right)}_{\downarrow\downarrow}$$

– in the upper branch of the $\dot{C}=0$ line, $L\approx 0$ and the FS effect dominates the GT effect. For points above the line $\dot{C}<0$

$$\underbrace{\frac{\dot{C}}{C}}_{\downarrow} = \underbrace{r(C,K)}_{\downarrow\downarrow} - \rho - \left(\frac{\beta\epsilon_C(\rho+\beta)}{1+t_C}\right)\underbrace{\left(\frac{K}{C}\right)}_{\downarrow}$$

- In the text we show how the model can be used to analyze the macroeconomic effects of a change in the consumption tax. *Key points*:
  - impact, transitional, and long-run effects can be obtained by using the log-linearized version of the model [small tax changes]
  - the steady state effect on the capital stock depends on the relative strength of the FS and GT effects. *Intuition*:
    - ∗ dominant GT effect: $t_C \uparrow$ causes $K(\infty) \uparrow$. Redistribution from old to young. [$C(v,0)$ high the older one is: old pay more on consumption tax]. r virtually unchanged [weak FS effect]. $C(0) \downarrow\downarrow$ and $Y(0) \downarrow$ so $\dot{K}(0) \uparrow$. Capital accumulation takes place.
    - ∗ dominant FS effect: $t_C \uparrow$ causes $K(\infty) \downarrow$. Great ratios: (K/L) virtually unchanged in long run. As $L(\infty) \downarrow$ so must $K(\infty)$.
  - simple as the extended BY model is, similar results are obtained in much more detailed *computable general equilibrium* models [e.g. Auerbach & Kotlikoff (1989)]

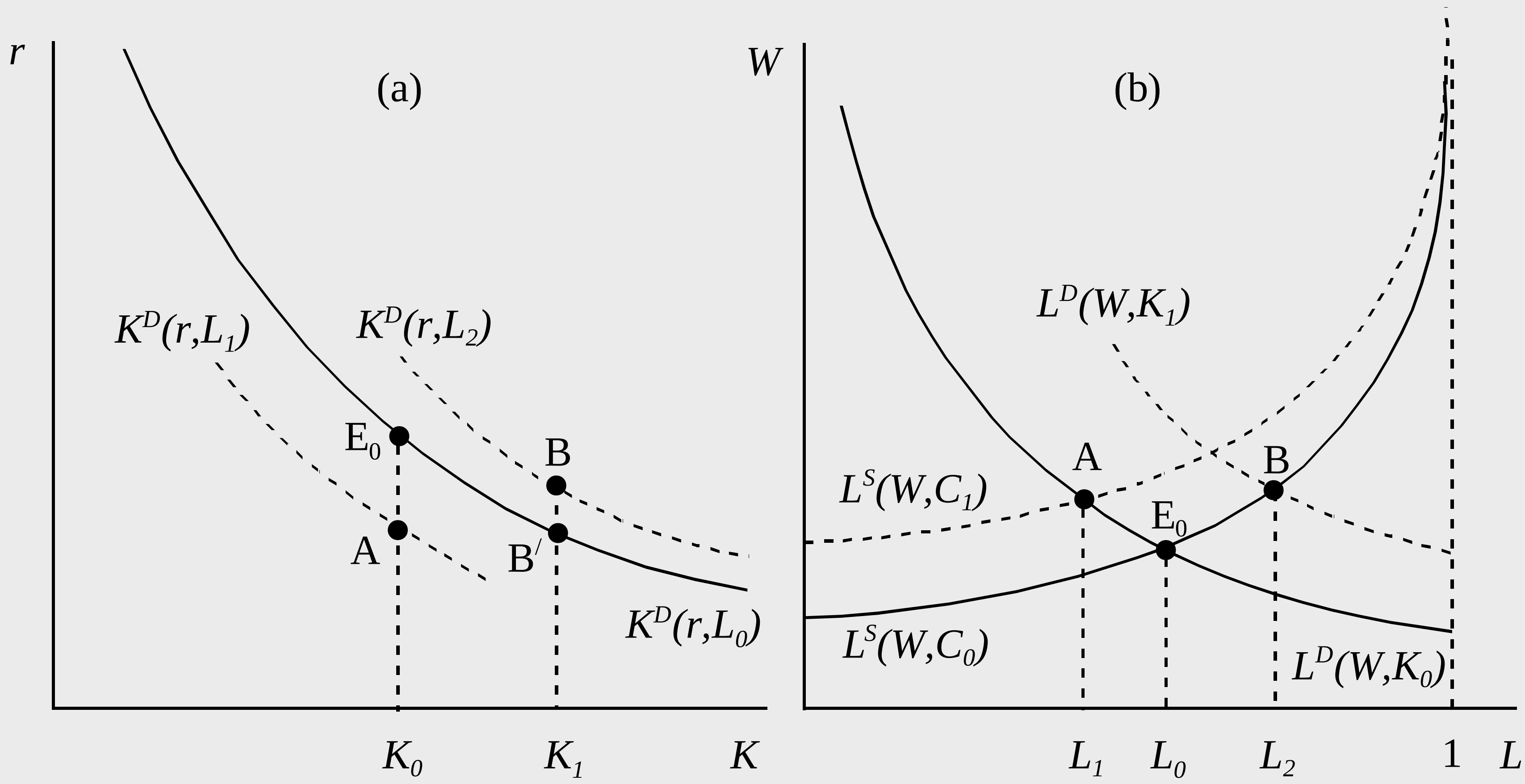

Figure 16.4: Factor Markets

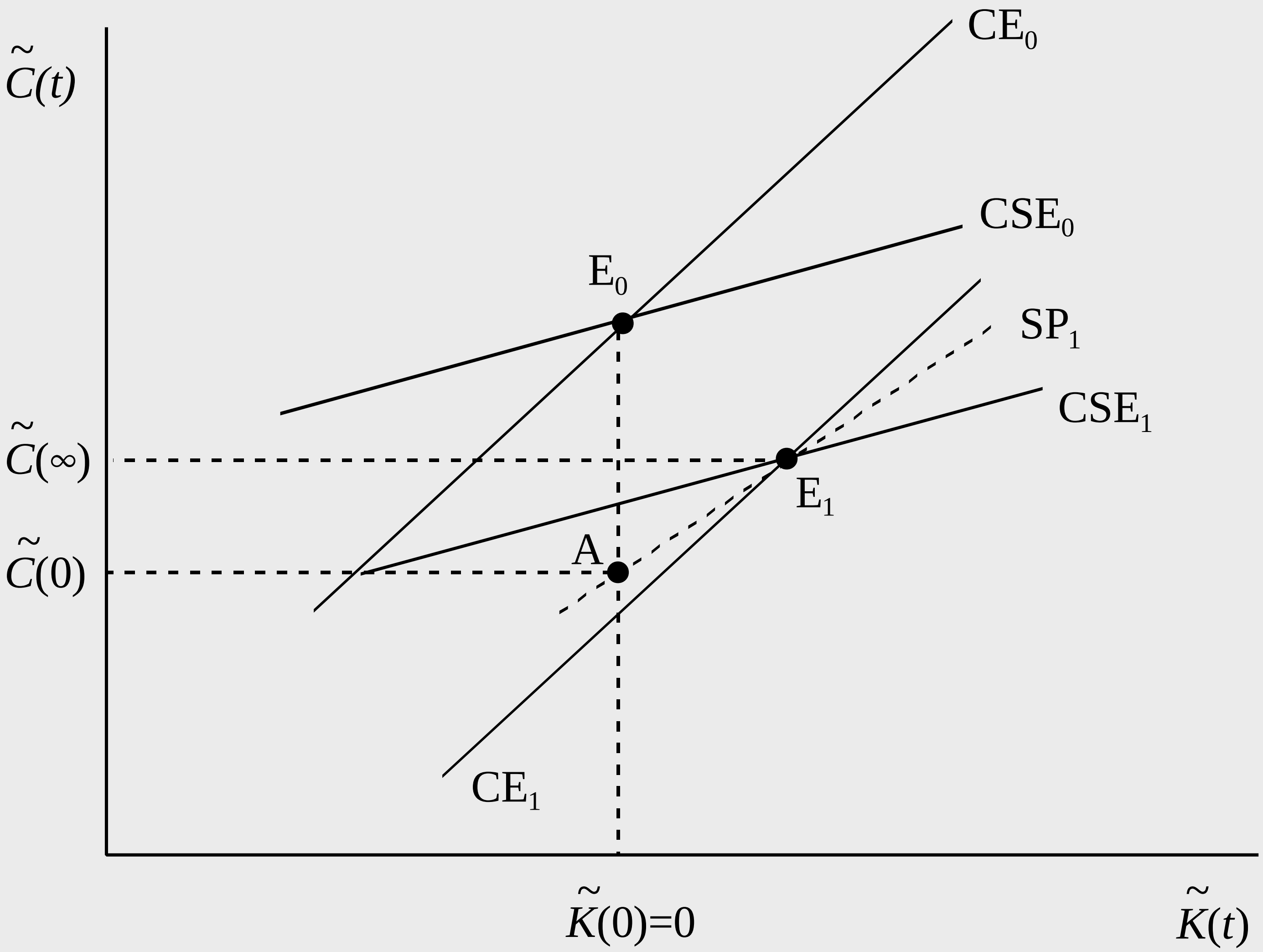

Figure 16.5: Consumption Tax [Dominant Generational-Turnover Effect]

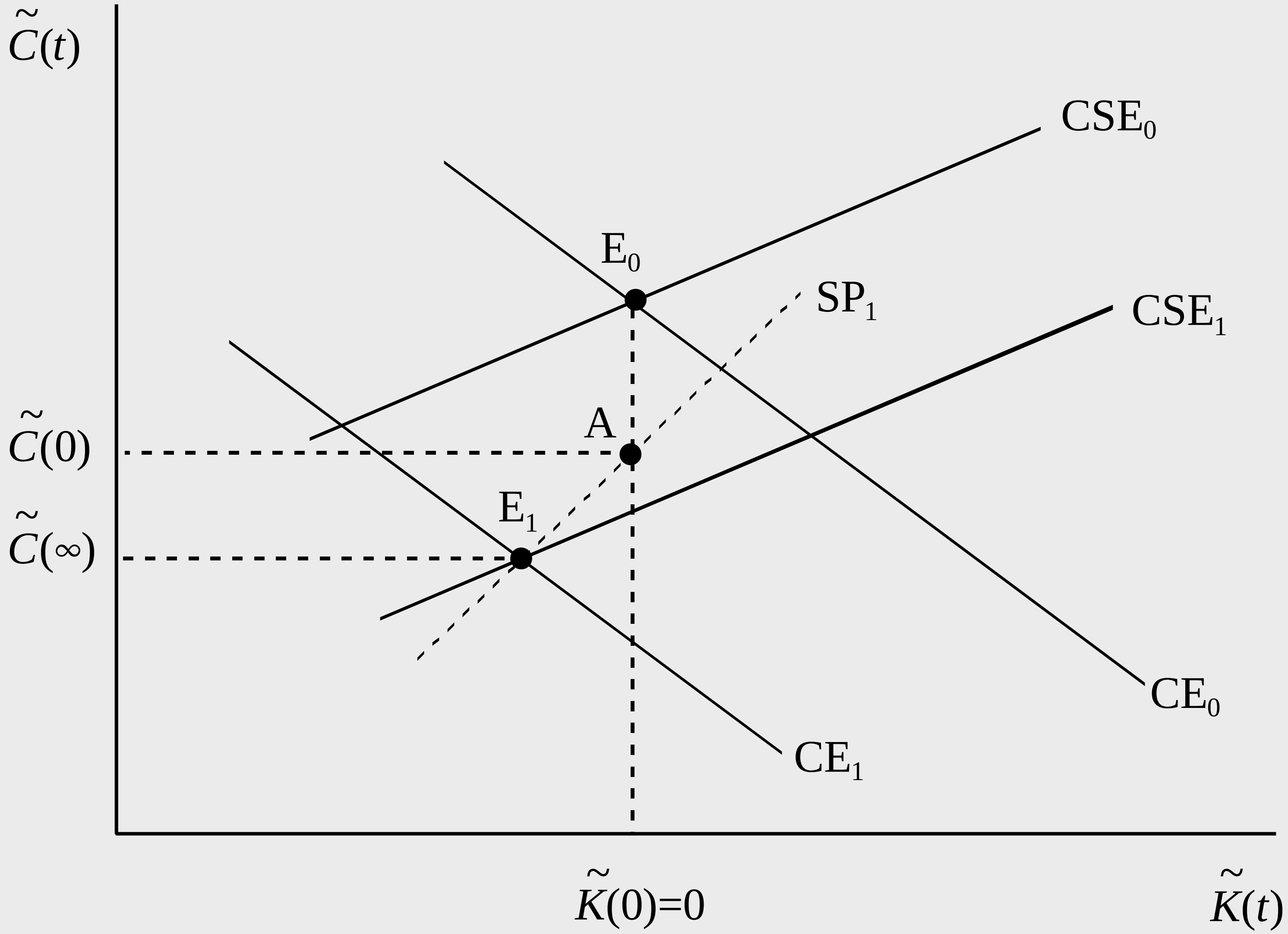

Figure 16.6: Consumption Tax [Dominant Factor-Scarcity Effect]

(B) Age-dependent labour productivity

- *Key idea*: one of the unattractive aspects of the standard BY model is the fact that all agents, regardless of their age, have the same expected remaining lifetime. [Agents enjoy a "perpetual youth"]. In reality households do age and plan to retire from the labour force. There is a life cycle in the pattern of income and one of the motives for saving is to provide for old age [*life-cycle saving*]

- One way to mimic the effects of the life-cycle saving motive is to assume that the household's productivity in the labour market depends on its age. Typically the productivity pattern is *hump shaped*, low early on and during old age and high in the middle.

- In the text we show the consequences of a simpler productivity pattern, one where skills are high early on but decline exponentially as the agent gets older. We embed this productivity profile in the standard BY model [with exogenous labour supply]. The worker's efficiency pattern is:

$$E(\tau - v) \equiv \left(\frac{\alpha + \beta}{\beta} \right) e^{-\alpha(\tau - v)}$$

where α thus measures the rate at which labour productivity declines as one gets older [so far we used $\alpha = 0$]

- The main results [intuitively]
  - Old worker less productive. Firms pay them lower wages. Labour supply exogenous so wage income declines during worker's life.
  - Motive to "save for a rainy day" [worker does not retire but will eventually work for close to nothing]
  - Human wealth is now age dependent [higher the younger one is]
  - Aggregate human wealth discounted more heavily because of the declining wage as one gets older:

$$H(t) \equiv \int_t^{\infty} W(\tau) \exp\left\{-\int_t^{\tau} [r(s) + \alpha + \beta] \, ds\right\} d\tau$$

  - The dynamic system characterizing the aggregate economy is also affected by the productivity-decline parameter:

$$\frac{\dot{C}(t)}{C(t)} = \underbrace{[r(t)-\rho]}_{(a)} + \underbrace{\left[\alpha-(\alpha+\beta)(\rho+\beta)\left(\frac{K(t)}{C(t)}\right)\right]}_{(b)}$$

$$\dot{K}(t) = F(K(t),1) - C(t) - \delta K(t),$$

$$r(t) \equiv F_K(K(t),1) - \delta$$

– NOTE: the aggregate consumption Euler equation is much more complex:

  * Item (a): individual consumption growth [Euler eqn for individual households]
  * Item (b): correction term due to generational turnover depends on interplay between two mechanisms. On the one hand newborns have higher human wealth than older agents and consume more on that account [$\dot{C}/C \uparrow$]. On the other hand, older households have positive real wealth [$\dot{C}/C \downarrow$].

- There is nothing to rule out a macroeconomic equilibrium which is dynamically inefficient, as in **Figure 16.7**. If productivity declines rapidly as one ages than young agents save ferociously to provide for old-age consumption. As a result the aggregate capital stock may become too large from a social welfare point of view.

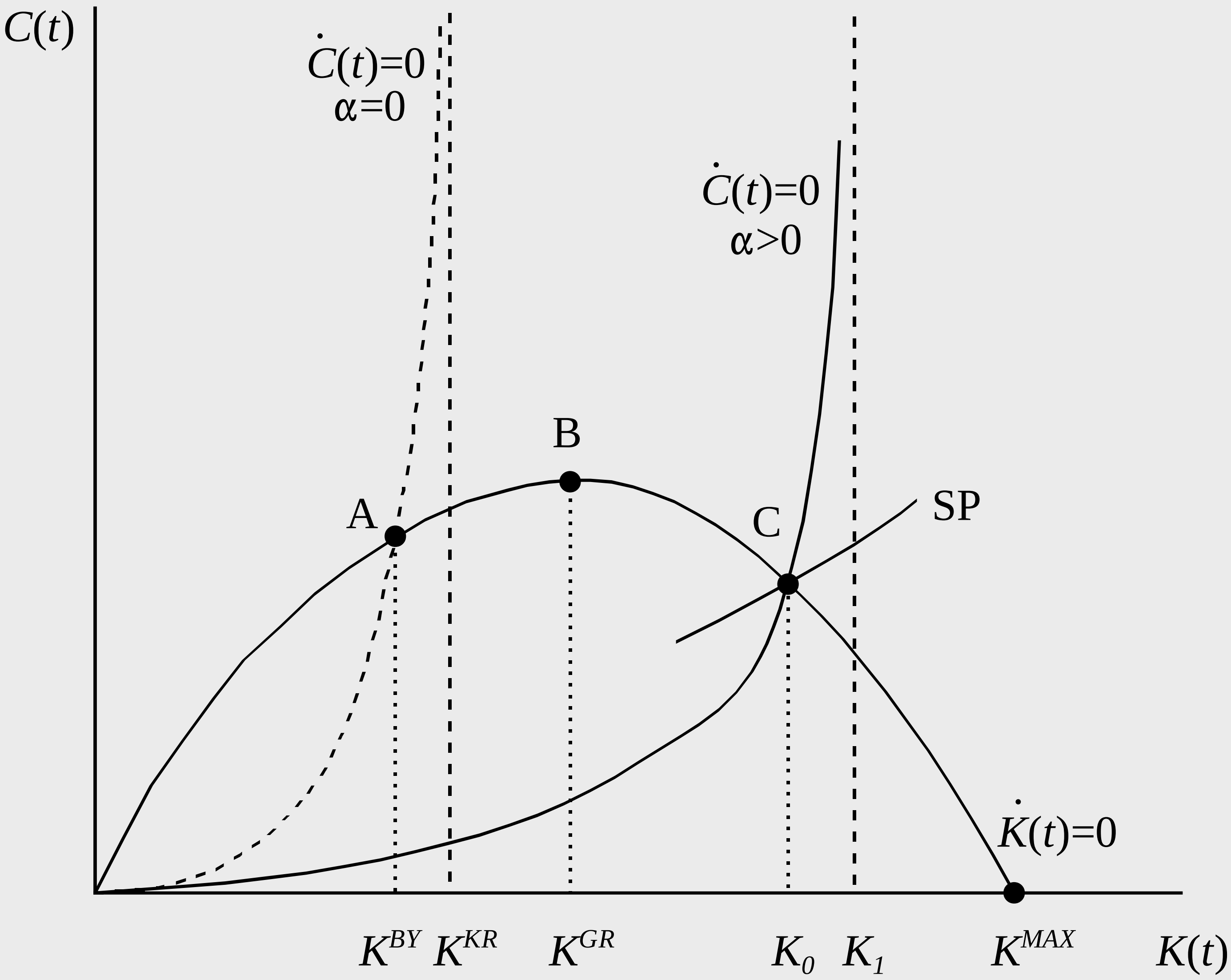

Figure 16.7: Dynamic Inefficiency [Declining Productivity]

(C) The open economy

- Key points of the open-economy BY model:
  - the generational turnover mechanism ensures that the model is well defined even if the world rate of interest is unequal to the [exogenous] pure rate of time preference. Hence, we can distinguish creditor and debtor nations. [In open economy Ramsey model only the knife-edge case yields a well-defined equilibrium, though one exhibiting hysteresis–see Chapter 14]
  - just as in the open-economy Ramsey model, adjustment costs on investment are needed to limit the international mobility of physical capital–see Chapter 14.
  - in the SOE the dynamics of (q, K) and (A, H) [or (C, A)] decouple. Model can be solved recursively. See text for a worked example involving an oil price shock.

- Here we discuss a simplified model of the open economy due to Blanchard (1985). It abstracts from physical capital altogether. The model is:

$$\begin{aligned} \dot{C}(t) &= (r-\rho)C(t) - \beta(\rho+\beta)A_F(t) \\ \dot{A}_F(t) &= rA_F(t) + W(t) - C(t) \end{aligned}$$

  where A_F is net foreign assets

- Technology is given by $Y(t) = Z(t)L(t)$ so that profit maximization leads to $W(t) = Z(t)$ (full employment also obtains and $L(t) = 1$)

- The system of differential equations is:

$$\begin{bmatrix} \dot{C}(t) \\ \dot{A}_F(t) \end{bmatrix} = \begin{bmatrix} r-\rho & -\beta(\rho+\beta) \\ -1 & r \end{bmatrix} \begin{bmatrix} C(t) \\ A_F(t) \end{bmatrix} + \begin{bmatrix} 0 \\ Z(t) \end{bmatrix}$$

- The determinant of the Jacobian is:

$$|\Delta| = r(r-\rho) - \beta(\rho+\beta) \qquad \text{(a)}$$

  - $\dot{C} = 0$ implies: $C = \beta(\rho+\beta)A_F/(r-\rho)$
  - $\dot{A}_F = 0$ implies: $C - rA_F = W$
  - using both results in (a) we find:

$$|\Delta| = -\beta(\rho+\beta)\,[W/C] < 0$$

  - provided the steady state exists, it is saddle-point stable!

- We can now look at several special cases of the model:
  - creditor nation ($r > \rho$) versus debtor nation ($r < \rho$)
  - non-saving nation ($r = \rho$)
  - representative-agent knife-edge case ($r = \rho$ *and* $\beta = 0$)

- In each case we study the effects of (permanent or temporary) productivity shocks

Creditor nation

- The phase diagram is illustrated in **Figure A** (not in book)
- The $\dot{C} = 0$ line is upward sloping:

$$C = \left(\frac{\beta(\rho + \beta)}{r - \rho} \right) A_F$$

- Consumption dynamics:

$$\frac{\partial \dot{C}}{\partial C} = r - \rho > 0$$

  See vertical arrows.

- The $\dot{A}_F = 0$ line is upward sloping (but flatter than $\dot{C} = 0$):

$$C = rA_F + Z$$

- Current account dynamics:

$$\frac{\partial \dot{A}_F}{\partial A_F} = r > 0$$

  See horizontal arrows.

- Model is saddle-point stable and features upward sloping saddle path.

- The effect of an unanticipated and permanent productivity shock are shown in **Figure B** (not in book)
  - impact effect: C jumps (human capital effect)
  - transitional dynamics: gradual increase in C and A_F
  - long-run effect: both C and A_F increase.

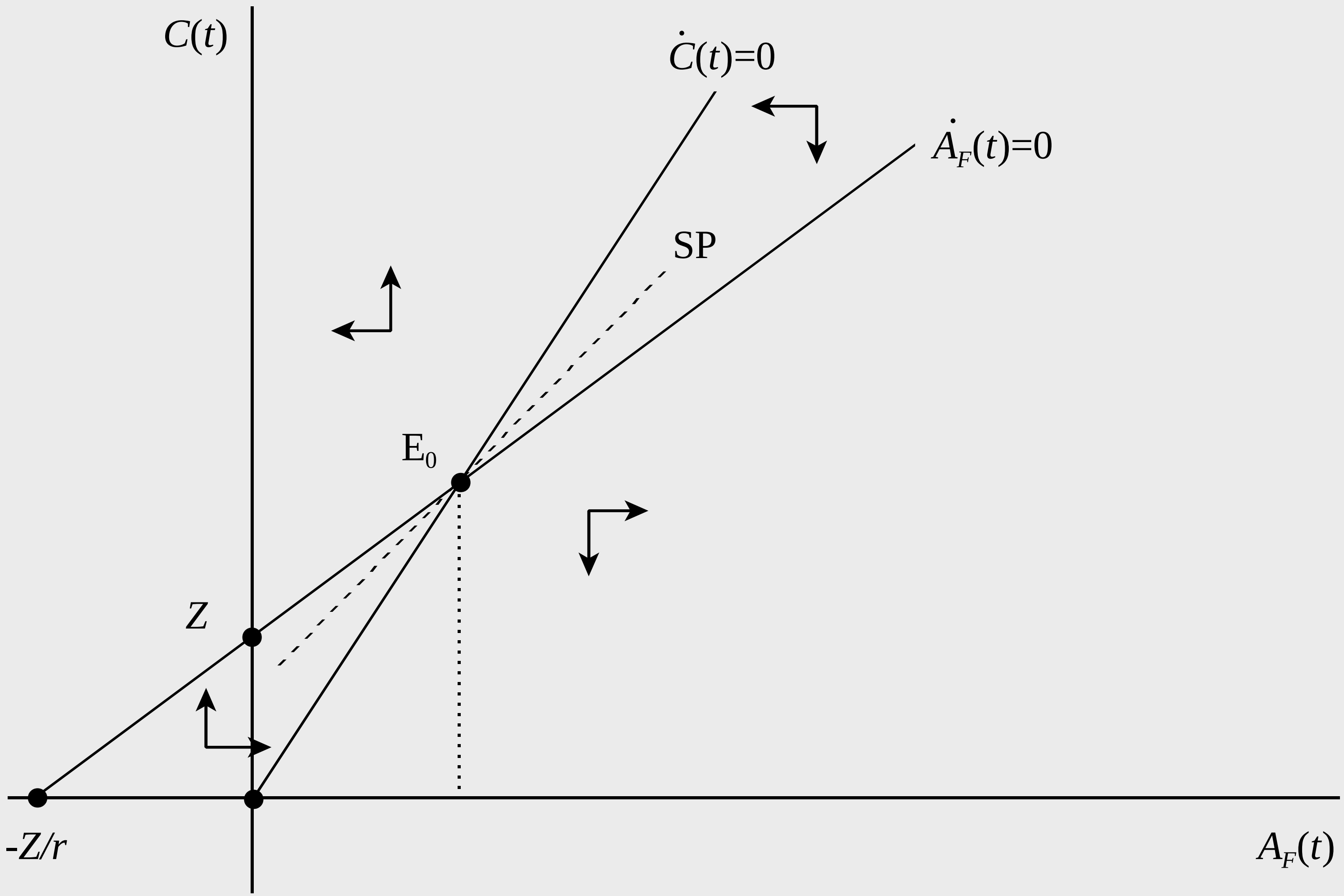

Figure A: A Patient Small Open Economy

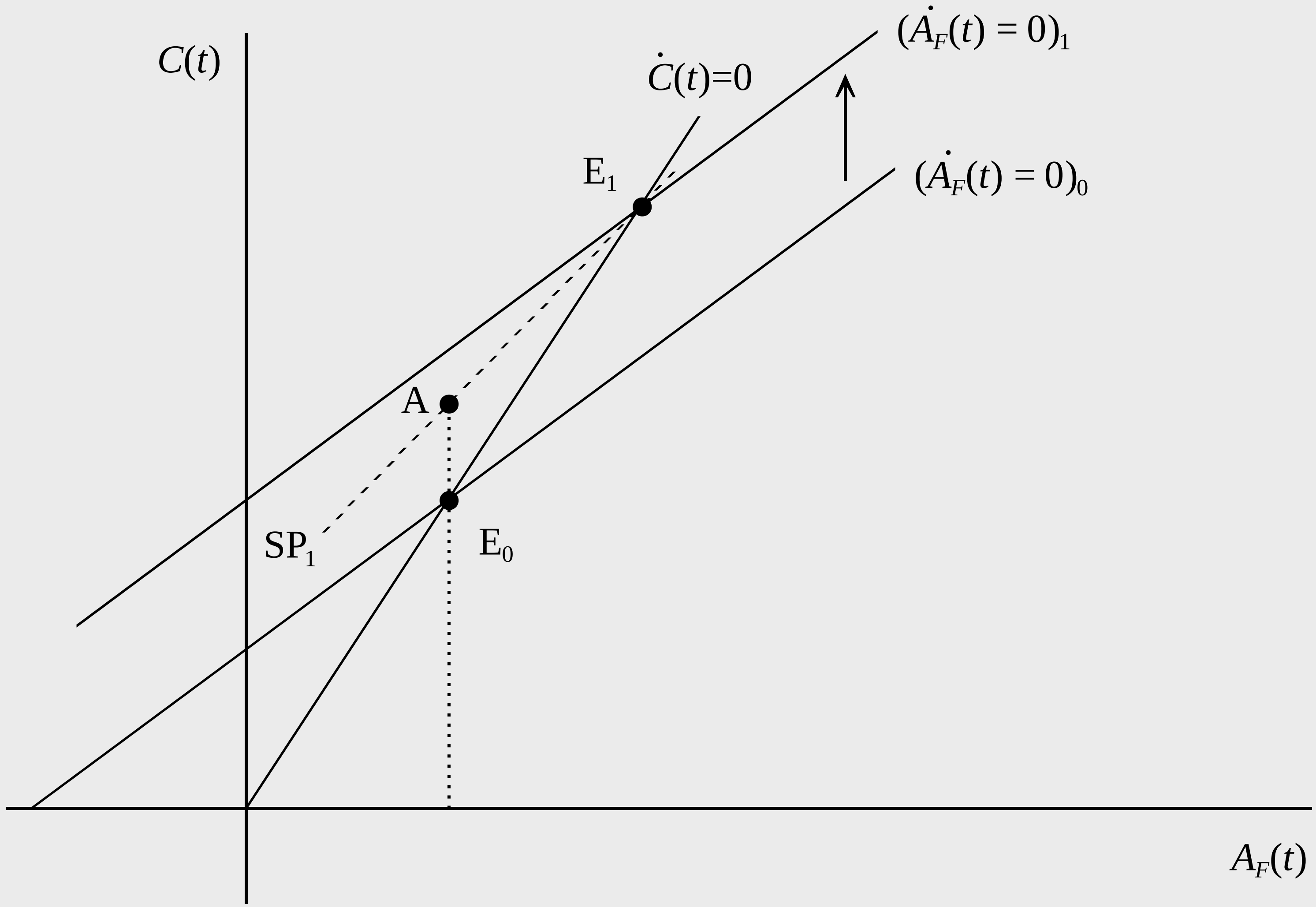

Figure B: A Productivity Shock

Debtor nation

- The phase diagram is illustrated in **Figure C** (not in book)
- The $\dot{C} = 0$ line is downward sloping:

$$C = \left(\frac{\beta(\rho + \beta)}{r - \rho} \right) A_F$$

- Consumption dynamics:

$$\frac{\partial \dot{C}}{\partial C} = r - \rho < 0$$

  See vertical arrows.

- The $\dot{A}_F = 0$ line is upward sloping:

$$C = rA_F + Z$$

- Current account dynamics:

$$\frac{\partial \dot{A}_F}{\partial A_F} = r > 0$$

See horizontal arrows.

- Model is saddle-point stable and features upward sloping saddle path.

- Effect of productivity shock left as an exercise.

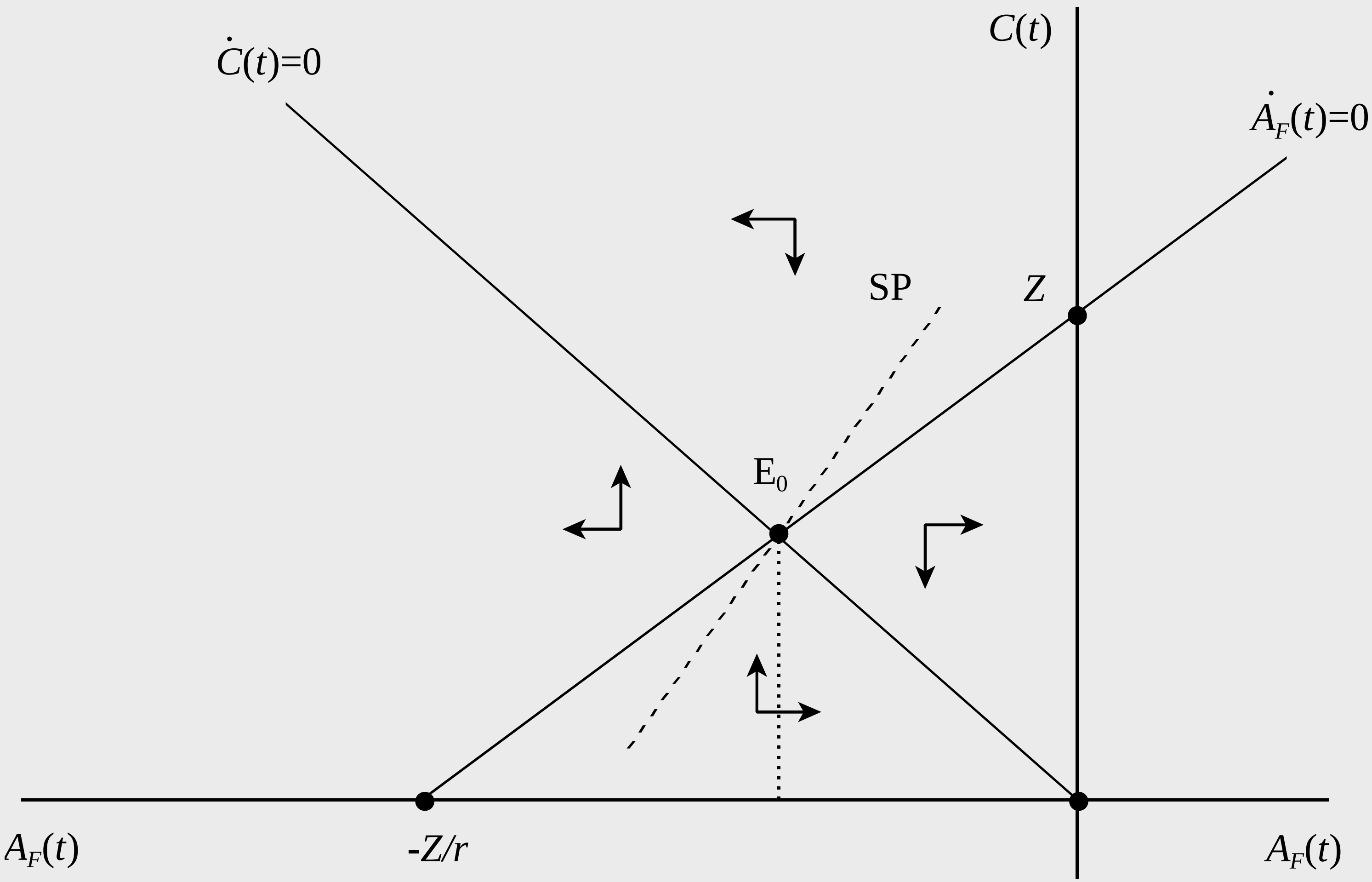

Figure C: An Impatient Small Open Economy

Non-saving nation

- Special case with $r = \rho$: aggregate Euler equation is:

$$\dot{C}(t) = -\beta(\rho + \beta)A_F(t)$$

- $\dot{C} = 0$ line coincides with the vertical axis in **Figure D** (not in book)

- Model is still saddle-point stable (as $|\Delta| = -\beta(\rho + \beta) < 0$) and the $\dot{C} = 0$ line is the saddle path.

- Effects of temporary productivity shock:
  - impact: upward jump in consumption (human wealth effect)
  - transition during high productivity: gradual decline in C and increase in A_F (saving to smooth consumption). Human wealth of newborns declines during transition.
  - transition after high productivity: gradual decline in both C and A_F
  - long run: no effect on C and A_F
- Temporary shock only has temporary effects (no hysteresis).

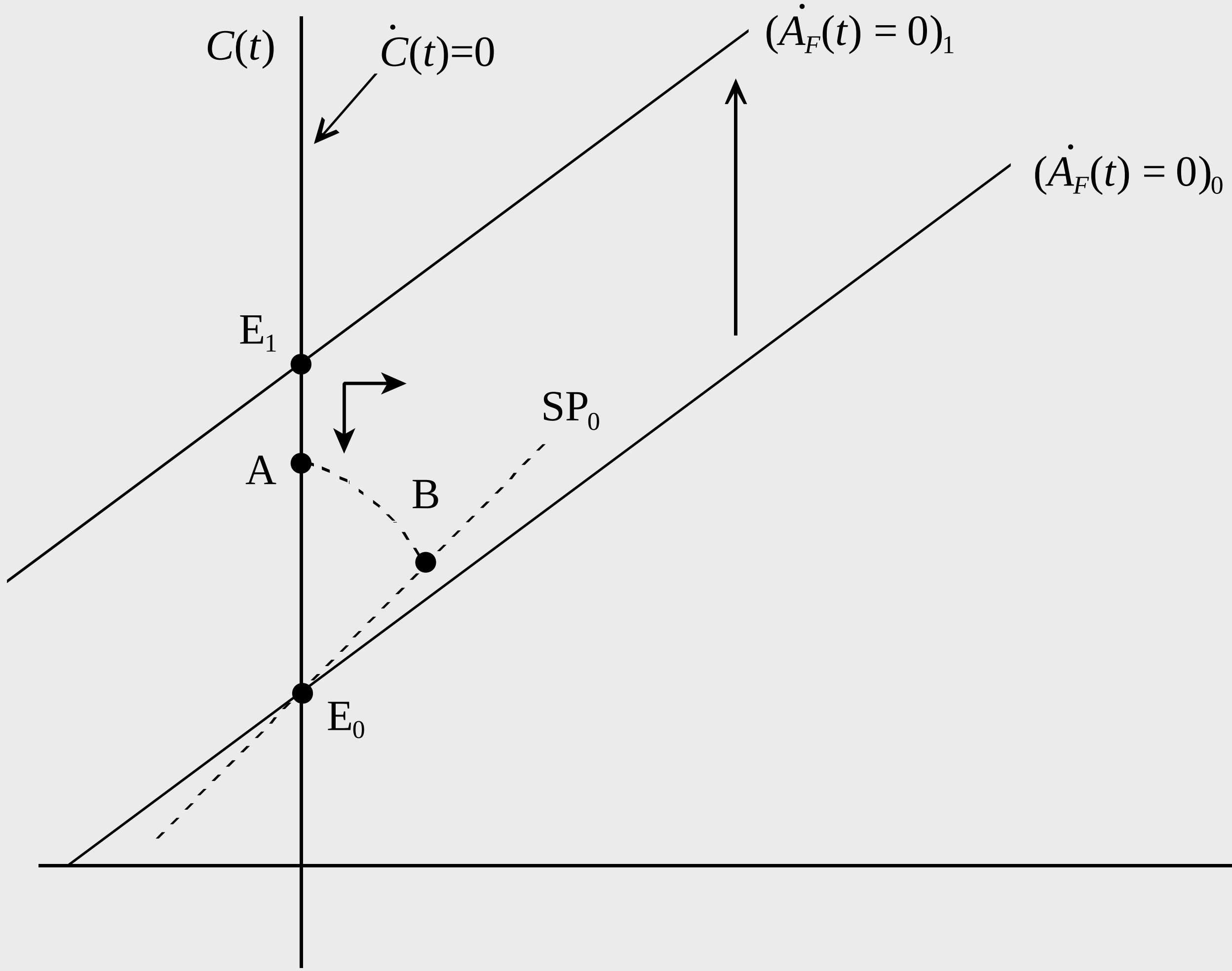

Figure D: A Temporary Productivity Shock in a Non-Saving Nation

Representative-agent model

- Special case with $r = \rho$ and $\beta = 0$: aggregate (and individual) Euler equation is:

$$\dot{C}(t) = 0$$

- Model features one unstable root ($\lambda_2 = r = \rho > 0$) and one zero root ($\lambda_1 = 0$): hysteresis.

- The consumption level is fully determined by the requirement of national solvency:

$$A_F(0) = \int_t^{\infty} [C(t) - Z(t)]\, e^{-\rho t} dt$$

- But $\dot{C}(t) = 0$ implies that (a) simplifies to:

$$C(0) = \rho \left[A_F(0) + \int_0^{\infty} Z(t) e^{-\rho t} dt \right]$$

- In **Figure E** (not in book) we show the effects of a temporary productivity shock under the assumption that the country holds no foreign assets initially ($A_F(0) = 0$).
  - impact: upward jump in C (human wealth effect)
  - during transition, dynamics of E$_1$ dictates adjustment: net saving takes place (from A to B)
  - long-run effect: consumption and net foreign assets permanently higher
- Temporary shock has permanent effects (hysteresis).

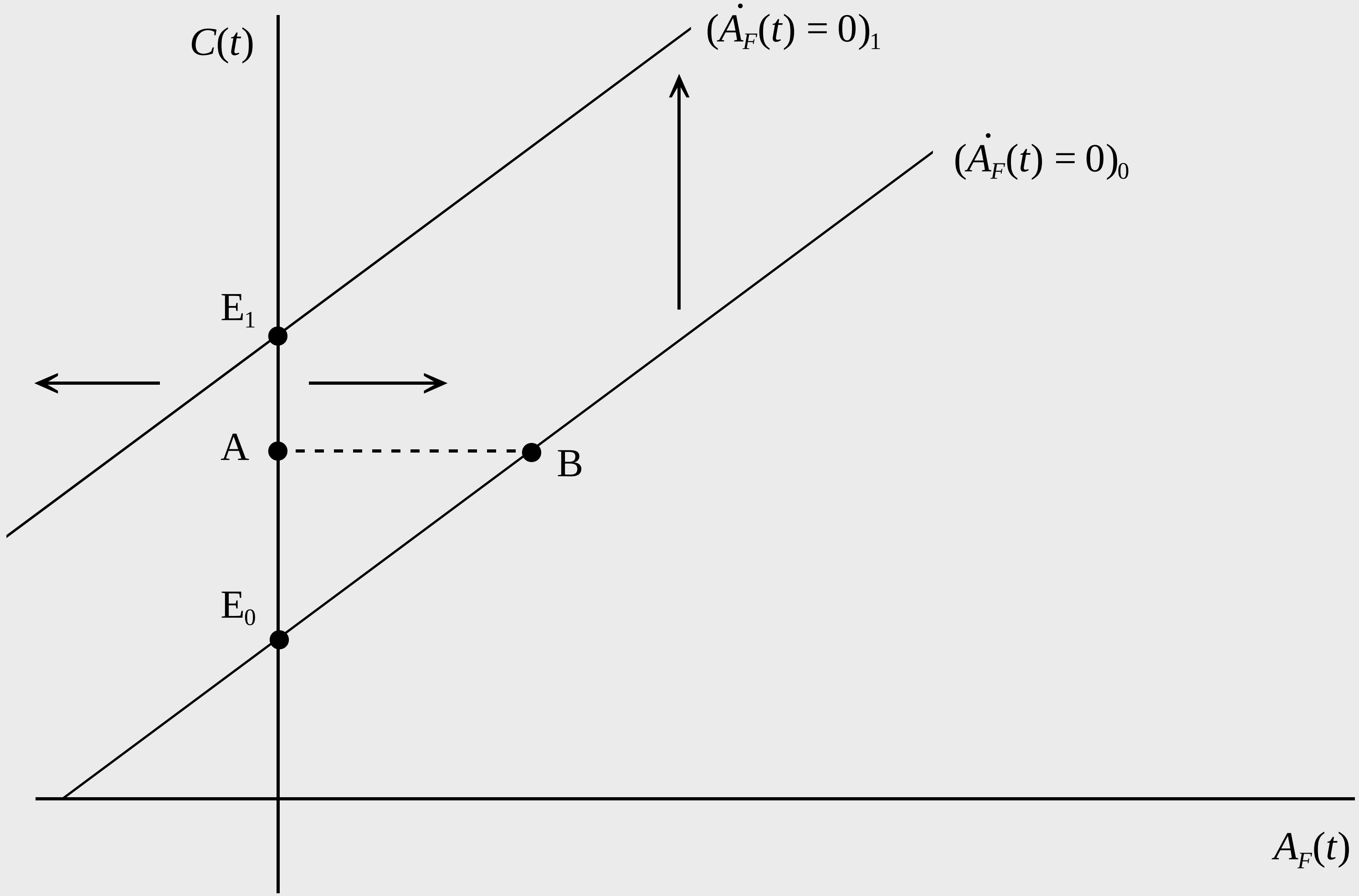

Figure E: A Temporary Productivity Shock in the Representative-Agent Model

Punchlines

- Insights of Yaari:
  - positive death rate leads to more severe discounting of future felicity
  - with actuarially fair life insurance the household can fully insure against the unpleasant aspects associated with the loss of life
- Blanchard shows that Yaari's consumption model can be embedded in general equilibrium framework.
  - fully tractable GE model with heterogeneous agents
  - both efficiency and intergenerational distribution matter
  - Ricardian equivalence not valid

- BY model can be easily extended
  - endogenous labour supply [tax distortions]
  - saving for rainy day [dynamic inefficiency]
  - open economy [non-degenerate dynamics]
- Approach fully deserves its current “work horse” status

Table 16.3. The loglinearized extended model

$$\dot{\tilde{C}}(t) = r\tilde{r}(t) + (r-\rho)\left[\tilde{C}(t) + \tilde{t}_C - \tilde{K}(t)\right] \tag{T3.1}$$

$$\dot{\tilde{K}}(t) = (\delta/\omega_I)\left[\tilde{Y}(t) - \omega_C\tilde{C}(t) - \omega_I\tilde{K}(t)\right] \tag{T3.2}$$

$$\tilde{Z}(t) = (1+t_C)\omega_C\left[\tilde{t}_C + \left(\frac{t_C}{1+t_C}\right)\tilde{C}(t)\right] + \epsilon_L t_L \tilde{Y}(t) \tag{T3.3}$$

$$r\tilde{r}(t) = (r+\delta)\left[\tilde{Y}(t) - \tilde{K}(t)\right] \tag{T3.4}$$

$$\tilde{W}(t) = \tilde{Y}(t) - \tilde{L}(t) \tag{T3.5}$$

$$\tilde{L}(t) = \omega_{LL}\left[\tilde{W}(t) - \tilde{t}_C - \tilde{C}(t)\right] \tag{T3.6}$$

$$\tilde{Y}(t) = \epsilon_L\tilde{L}(t) + (1-\epsilon_L)\tilde{K}(t) \tag{T3.7}$$

Table 16.4. The birth rate and the GT effect

β	$1/\beta$	ρ	*GT effect*	*FS effect*
0.005	200	0.0156	0.000312	0.0457
0.01	100	0.0151	0.000762	0.0457
0.02	50	0.0138	0.002054	0.0457
0.04	25	0.0098	0.006051	0.0457
0.07229	13.83	0	0.015868	0.0457

Table 16.5. The small open economy model

$$\dot{K}(t) = \left[\phi\left(\frac{I(t)}{K(t)}\right) - \delta\right] K(t) \tag{T5.1}$$

$$\dot{q}(t) = \left[r + \delta - \phi\left(\frac{I(t)}{K(t)}\right)\right] q(t) + \frac{I(t)}{K(t)} - F_K(L(t), K(t), O(t)) \tag{T5.2}$$

$$W(t) = F_L(L(t), K(t), O(t)) \tag{T5.3}$$

$$P_O(t) = F_O(L(t), K(t), O(t)) \tag{T5.4}$$

$$L(t) = 1 \tag{T5.5}$$

$$1 = q(t)\phi'\left(\frac{I(t)}{K(t)}\right) \tag{T5.6}$$

$$Y(t) = F(L(t), K(t), O(t)) = L(t)^{\epsilon_L} K(t)^{\epsilon_K} O(t)^{\epsilon_O} \tag{T5.7}$$

$$\dot{H}(t) = (r + \beta)H(t) - W(t) \tag{T5.8}$$

$$\dot{A}(t) = (r - \rho - \beta)A(t) - (\rho + \beta)H(t) + W(t) \tag{T5.9}$$

$$A_F(t) = A(t) - q(t)K(t) \tag{T5.10}$$

Table 16.6. The loglinearized small open economy model

$$\dot{\tilde{K}}(t) = \left(\frac{r\omega_I}{\omega_V}\right)\left[\tilde{I}(t) - \tilde{K}(t)\right] \quad \text{(T6.1)}$$

$$\dot{\tilde{q}}(t) = r\tilde{q}(t) - \left(\frac{r\epsilon_K}{\omega_V}\right)\left[\tilde{Y}(t) - \tilde{K}(t)\right] \quad \text{(T6.2)}$$

$$\tilde{W}(t) = \tilde{Y}(t) \quad \text{(T6.3)}$$

$$\tilde{P}_O(t) = \tilde{Y}(t) - \tilde{O}(t) \quad \text{(T6.4)}$$

$$\tilde{q}(t) = \sigma_A\left[\tilde{I}(t) - \tilde{K}(t)\right] \quad \text{(T6.5)}$$

$$\tilde{Y}(t) = \epsilon_K\tilde{K}(t) + \epsilon_O\tilde{O}(t) \quad \text{(T6.6)}$$

$$\dot{\tilde{H}}(t) = (r+\beta)\tilde{H}(t) - r\epsilon_L\tilde{W}(t) \quad \text{(T6.7)}$$

$$\dot{\tilde{A}}(t) = (r-\rho-\beta)\tilde{A}(t) - (\rho+\beta)\tilde{H}(t) + r\epsilon_L\tilde{W}(t) \quad \text{(T6.8)}$$

$$\tilde{A}_F(t) = \tilde{A}(t) - \omega_V\left[\tilde{K}(t) + \tilde{q}(t)\right] \quad \text{(T6.9)}$$

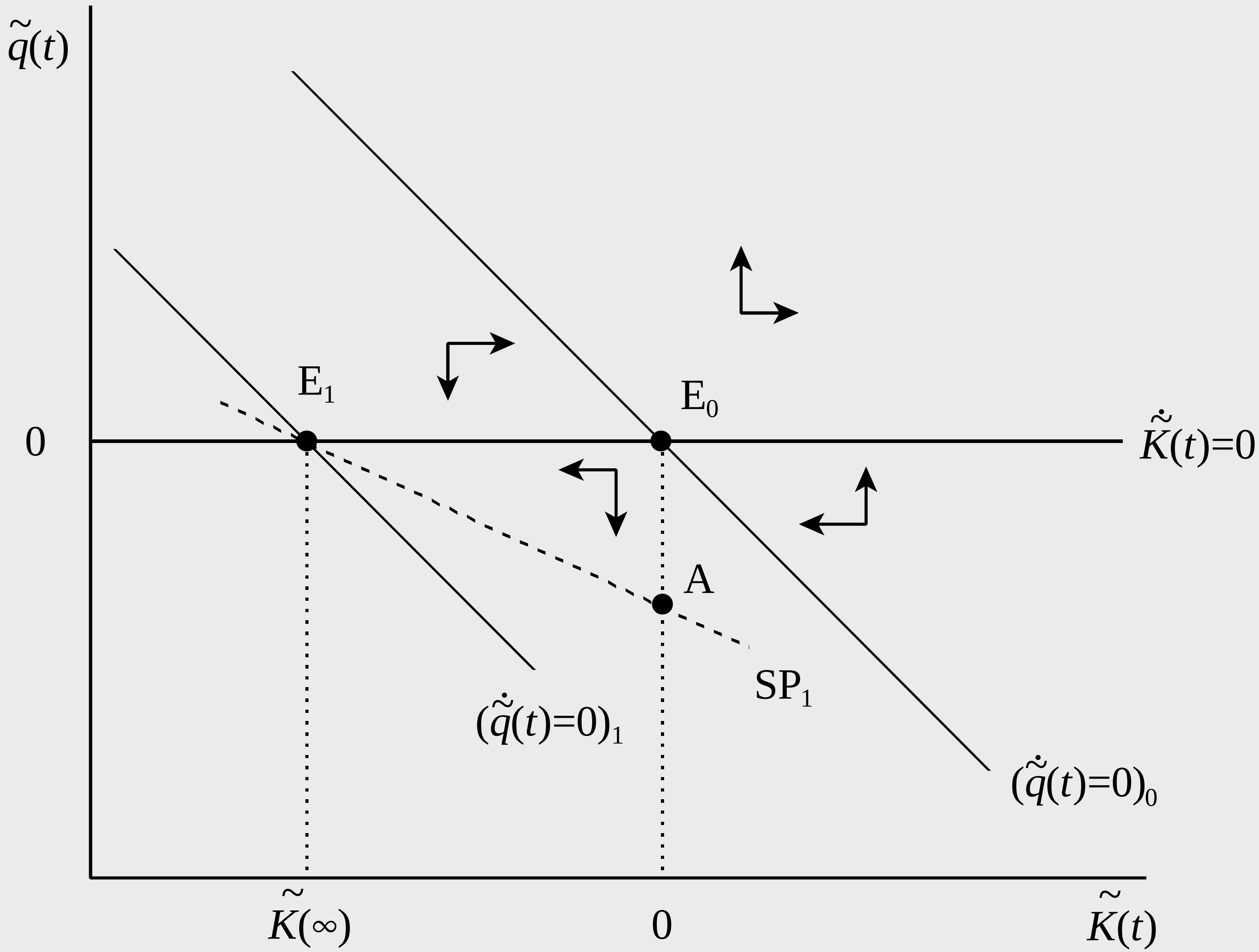

Figure 16.8: An Oil Shock in the Small Open Economy