

Foundations of Modern Macroeconomics

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Chapter 14: Theories of Economic Growth

Aims of this lecture

- Stylized facts of economic growth
- How well does the Solow-Swan model explain these stylized facts?
- Adding human capital to the Solow-Swan model.
- Growth models based on dynamically optimizing consumers.
- Fiscal policy and Ricardian equivalence in various traditional growth models?
- Endogenous growth models
 - capital fundamentalism
 - human capital formation
 - endogenous technology

Kaldor's stylized facts of economic growth

- (1) (*) Output per worker shows continuing growth “with no tendency for a falling rate of growth of productivity.”
 - (2) Capital per worker shows continuing growth.
 - (3) The rate of return on capital is steady.
 - (4) (*) The capital-output ratio is steady.
 - (5) (*) Labour and capital receive constant shares of total income.
 - (6) (*) There are wide differences in the rate of productivity growth across countries.
- **Note:** not all these stylized facts are independent:
 - (SF1) and (SF4) imply (SF 2)
 - (SF4) and (SF5) imply (SF3).
 - Hence, the starred facts are fundamental.

Romer's additional stylized facts of economic growth

- (7) In cross section, the mean growth rate shows no variation with the level of per capita income.
- (8) The rate of growth of factor inputs is not large enough to explain the rate of growth of output; that is, growth accounting always finds a residual.
- (9) Growth in the volume of trade is positively correlated with growth in output.
- (10) Population growth rates are negatively correlated with the level of income.
- (11) Both skilled and unskilled workers tend to migrate toward high-income countries.

The neoclassical growth model: Solow-Swan

- *Key notion*: capital and labour are substitutable
- Technology [neoclassical part of the model]:

$$Y(t) = F [K(t), L(t), t]$$

- t is time-dependent shift in technology
- CRTS:

$$F [\lambda K(t), \lambda L(t), t] = \lambda F [K(t), L(t), t], \quad \text{for } \lambda > 0 \quad (\text{P1})$$

- Saving [“Keynesian” part of the model]:

$$S(t) = Y(t) - C(t) = sY(t), \quad 0 < s < 1$$

where s is the constant propensity to save (exogenous)

- Goods market (closed economy, no government consumption):

$$Y(t) = C(t) + I(t)$$

- Gross investment:

$$I(t) = \delta K(t) + \dot{K}(t)$$

where $\delta K(t)$ is replacement investment (δ is the constant depreciation rate), and $\dot{K}(t)$ is net addition to the capital stock

- Labour supply is exogenous but the population grows as a whole at a constant exponential rate n_L :

$$\frac{\dot{L}(t)}{L(t)} = n_L \quad \Leftrightarrow \quad L(t) = L(0)e^{n_L t}$$

where we can normalize $L(0) = 1$.

Case 1: No technical progress

- Time drops out of technology:

$$Y(t) = F [K(t), L(t)]$$

- positive and diminishing marginal products:

$$F_K, F_L > 0, \quad F_{KK}, F_{LL} < 0, \quad F_{KL} > 0 \quad (\text{P2})$$

- Inada conditions:

$$\lim_{K \rightarrow 0} F_K = \lim_{L \rightarrow 0} F_L = +\infty, \quad \lim_{K \rightarrow \infty} F_K = \lim_{L \rightarrow \infty} F_L = 0 \quad (\text{P3})$$

- Solve model by writing in per capita form, i.e. $y(t) \equiv Y(t)/L(t)$,
 $k(t) \equiv K(t)/L(t)$, etcetera. Here are some steps:

– step 1: savings equal investment:

$$\begin{aligned}
 S(t) &= I(t) \\
 sY(t) &= \delta K(t) + \dot{K}(t) \\
 sF [K(t), L(t)] &= \delta K(t) + \dot{K}(t) \quad \Rightarrow \\
 s \left(\frac{F [K(t), L(t)]}{L(t)} \right) &= \delta \left(\frac{K(t)}{L(t)} \right) + \frac{\dot{K}(t)}{L(t)} \quad (a)
 \end{aligned}$$

– step 2: since $k(t) \equiv K(t)/L(t)$ it follows that:

$$\begin{aligned}
 \dot{k}(t) &\equiv \frac{\dot{K}(t)}{L(t)} - \frac{K(t)}{L(t)} \frac{\dot{L}(t)}{L(t)} \quad \Rightarrow \\
 \frac{\dot{K}(t)}{L(t)} &= \dot{k}(t) + k(t)n_L \quad (b)
 \end{aligned}$$

– *step 3*: since $F [K(t), L(t)]$ features CRTS we have:

$$\begin{aligned}
 Y(t) &= F [K(t), L(t)] = L(t) F \left[\frac{K(t)}{L(t)}, 1 \right] && \Rightarrow \\
 y(t) &= f(k(t)) && \text{(c)}
 \end{aligned}$$

[e.g. Cobb-Douglas $Y = K^\alpha L^{1-\alpha}$ implies $y = k^\alpha$]

- By substituting (b) and (c) into (a) we obtain the *fundamental differential equation* (FDE) for $k(t)$:

$$\dot{k}(t) = \underbrace{sf(k(t))}_{(a)} - \underbrace{(\delta + n_L)k(t)}_{(b)}$$

(a) per capita saving

(b) to maintain constant $k(t)$ one must replace and expand the *level* of the capital stock

- In **Figure 14.1** we illustrate the FDE. The Inada conditions imply:

- $f(k(t))$ vertical for $k(t) \rightarrow 0$
- $f(k(t))$ horizontal for $k(t) \rightarrow \infty$
- unique steady state at E_0
- stable equilibrium

- In the balanced growth path (BGP):

$$\frac{\dot{Y}(t)}{Y(t)} = \frac{\dot{K}(t)}{K(t)} = \frac{\dot{I}(t)}{I(t)} = \frac{\dot{S}(t)}{S(t)} = \frac{\dot{L}(t)}{L(t)} = n_L$$

Hence the name *exogenous growth*.

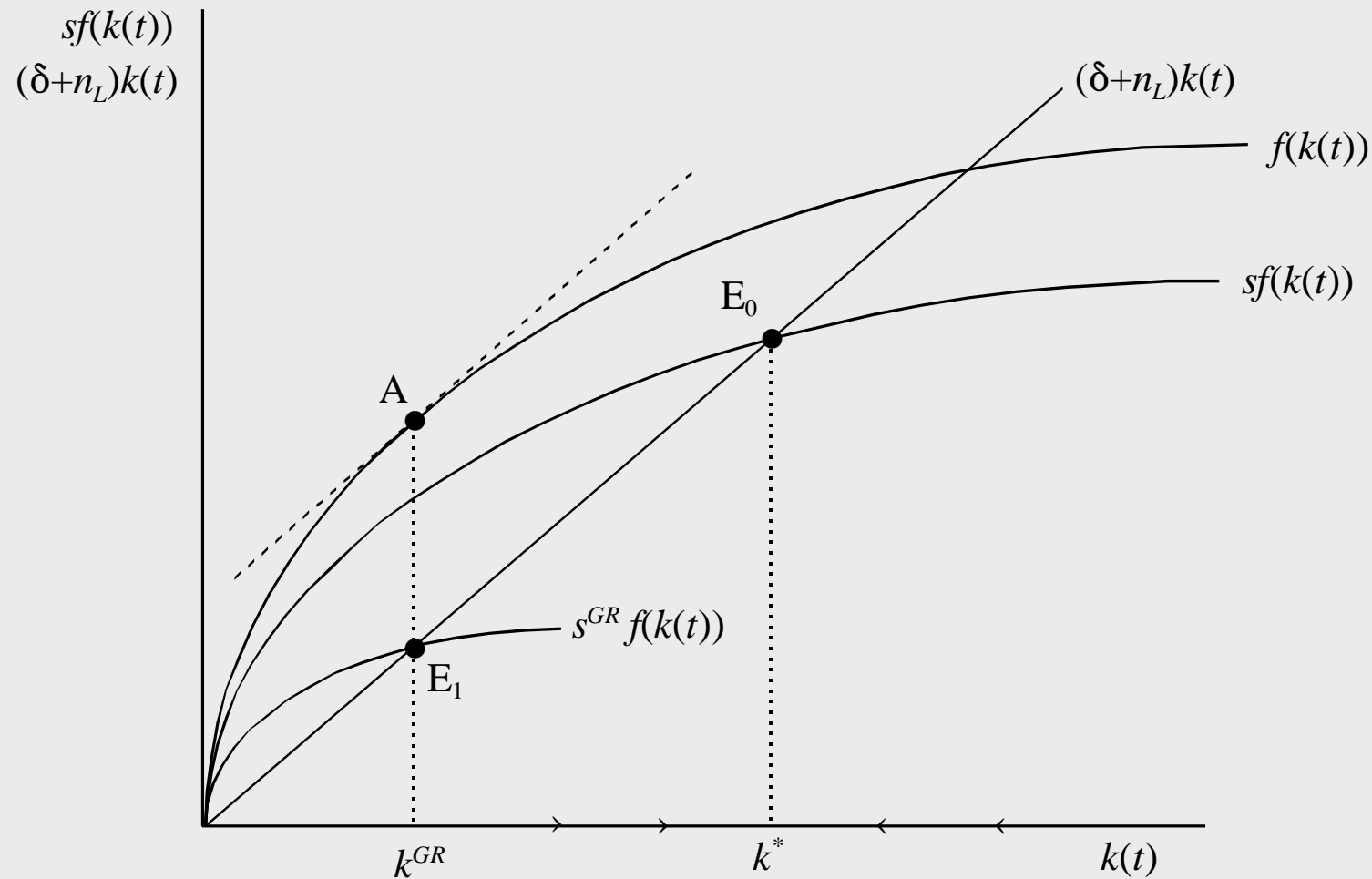


Figure 14.1: The Solow-Swan Model

Case 2: With technological progress

- Focus on *disembodied* technological progress

$$Y(t) = F \left[\underbrace{A_K(t)K(t)}_{(a)}, \underbrace{A_L(t)L(t)}_{(b)} \right]$$

(a) “effective” capital input

(b) “effective” labour input

- Three types of progress:
 - Harrod neutral: $A_K(t) \equiv 1$
 - Hicks neutral: $A_K(t) \equiv A_L(t)$
 - Solow neutral: $A_L(t) \equiv 1$
- Cases are indistinguishable for Cobb-Douglas

- For non-CD case progress must be Harrod neutral to have a steady state with constant growth rate [otherwise one of the shares goes to zero, contra (SF5)]
- Define $N(t) \equiv A(t)L(t)$ and assume that technical progress occurs at a constant exponential rate:

$$\frac{\dot{A}(t)}{A(t)} = n_A, \quad A(t) = A(0)e^{n_A t},$$

so that the *effective* labour force grows at a constant exponential rate $n_L + n_A$.

- Measuring output and capital per unit of effective labour, i.e. $y(t) \equiv Y(t)/N(t)$ and $k(t) \equiv K(t)/N(t)$, the FDE for $k(t)$ is obtained:

$$\dot{k}(t) = sf(k(t)) - (\delta + n_L + n_A)k(t)$$

- In the BGP we have:

$$\frac{\dot{Y}(t)}{Y(t)} = \frac{\dot{K}(t)}{K(t)} = \frac{\dot{I}(t)}{I(t)} = \frac{\dot{S}(t)}{S(t)} = \frac{\dot{N}(t)}{N(t)} = \frac{\dot{L}(t)}{L(t)} + \frac{\dot{A}(t)}{A(t)} = n_L + n_A$$

- Exogenous growth rate now equals $n_L + n_A$.

Further properties of the Solow-Swan model

Overview:

- (A) The golden rule of capital accumulation: dynamic inefficiency possible.
- (B) Transitional dynamics: conditional growth convergence seems to hold.
- (C) Speed of adjustment: too fast. Model can be rescued.
- (D) Rescuing the Solow-Swan model

(A) The golden rule

- Golden rule: maximum steady-state consumption per capita
- For each savings rate there is a unique *steady-state* capital-labour ratio [assume $n_A = 0$ for simplicity]:

$$k^* = k^*(s),$$

with $dk^*/ds = y^*/[\delta + n - sf'(k^*)] > 0$. The higher is s , the larger is k^* .

- Since $C(t) \equiv (1 - s)Y(t)$ we have for per capita consumption:

$$\begin{aligned} c(s) &= (1 - s)f[k^*(s)] \\ &= f[k^*(s)] - (\delta + n_L)k^*(s) \end{aligned}$$

- The golden-rule savings rate is such that $c(s)$ is maximized:

$$\frac{dc(s)}{ds} = \left[f' [k^*(s)] - (\delta + n_L) \right] \left(\frac{dk^*(s)}{ds} \right) = 0$$

- Since $\frac{dk^*(s)}{ds} > 0$ we get that:

$$f' [k^*(s^{GR})] = \delta + n_L \tag{*}$$

One interpretation: the produced asset (the physical capital stock) yields an own-rate of return equal to $f' - \delta$, whereas the non-produced primary good (labour) can be interpreted as yielding an own-rate of return n_L . Intuitively, the efficient outcome occurs if the rate of return on the two assets are equalized

- Recall that in the steady state::

$$s^{GR} f [k^*(s^{GR})] = (\delta + n_L)k^*(s^{GR}) \tag{\#}$$

- By using (*) we can rewrite (#) in terms of a national income share:

$$s^{GR} = \frac{(\delta + n_L)k^*(s^{GR})}{f[k^*(s^{GR})]} = \frac{k^*(s^{GR})f'[k^*(s^{GR})]}{f[k^*(s^{GR})]}$$

[e.g. for Cobb-Douglas $f(.) = k(t)^\alpha$, α represents the capital income share so that the golden rule savings rate equals $s^{GR} = \alpha$.]

- In **Figures 14.2-14.3** we illustrate the possibility of *dynamic inefficiency* [oversaving]:
If $s_0 > s^{GR}$ then a Pareto-improving transition from E_0 to E_1 is possible.

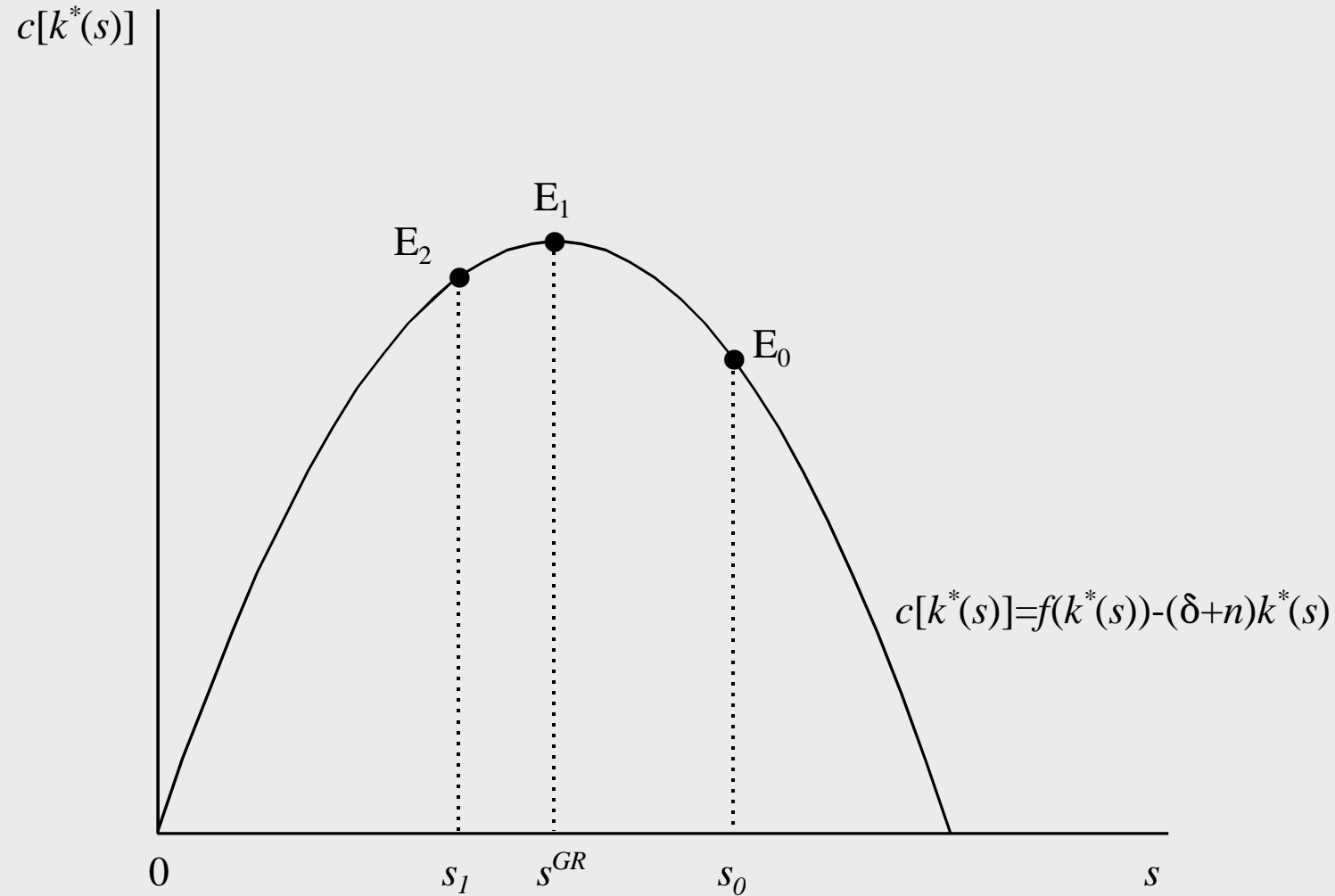


Figure 14.2: Per Capita Consumption and the Savings Rate

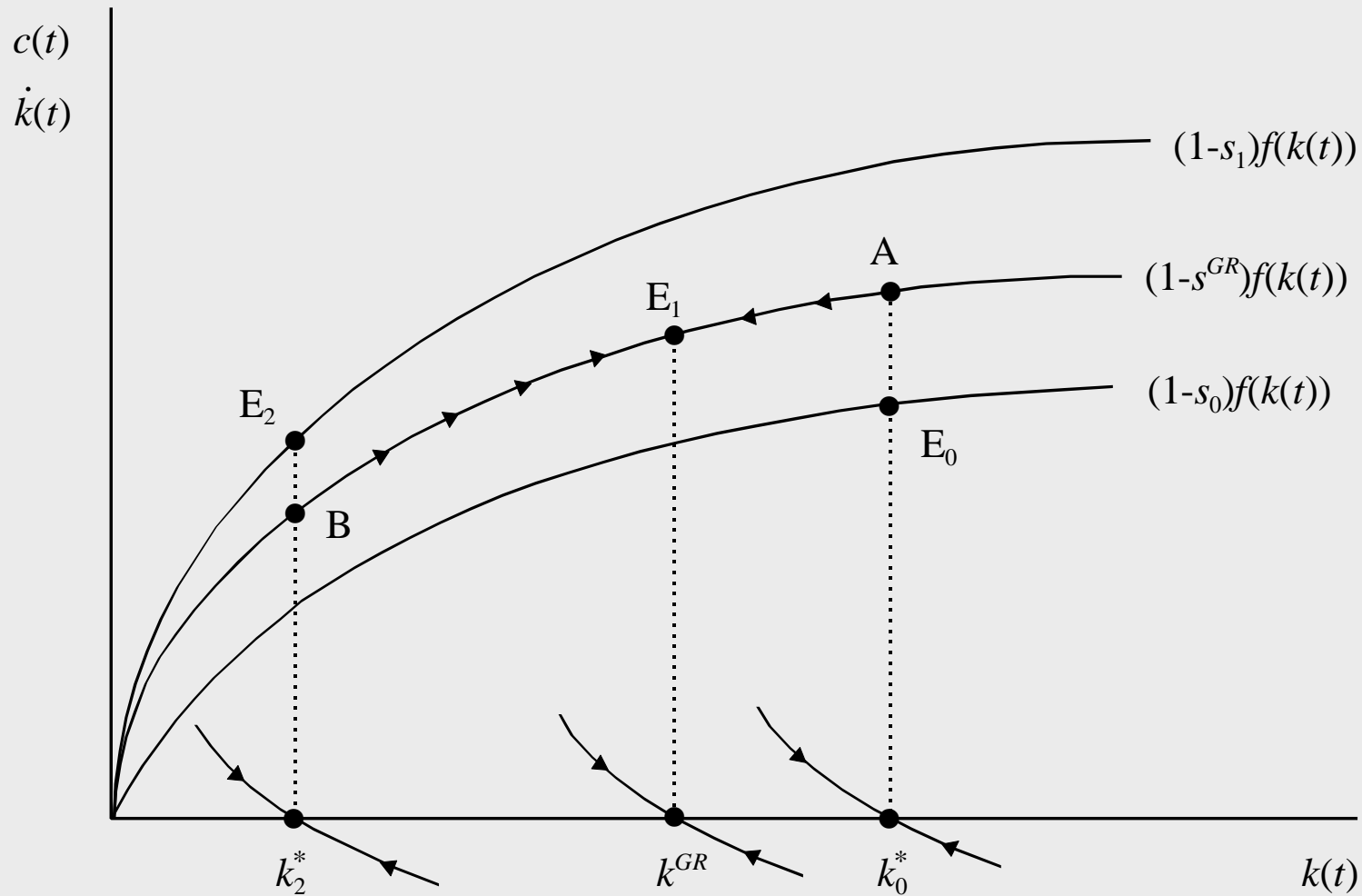


Figure 14.3: Per Capita Consumption During Transition to Golden Rule

(B) Transitional dynamics towards the steady state

- Defining the growth rate of $k(t)$ as $\gamma_k(t) \equiv \dot{k}(t)/k(t)$, we derive from the FDE:

$$\gamma_k(t) \equiv s \left(\frac{f(k(t))}{k(t)} \right) - (\delta + n),$$

where $n \equiv n_L + n_A$.

- In **Figure 14.4** this growth rate is represented by the vertical difference between the two lines. [The Inada conditions ensure that $\lim_{k \rightarrow 0} s f(k)/k = \infty$ and $\lim_{k \rightarrow \infty} s f(k)/k = 0$.]
- Countries with little capital (in efficiency units) grow faster than countries with a lot of capital. In other words, poor and rich countries should converge! [link between $\gamma_k(t)$ and $\gamma_y(t)$ is easily established, especially for the CD case]

- This suggests that there is a simple empirical test of the Solow-Swan model which is based on the convergence property of output in a cross section of many different countries.
 - *Absolute convergence hypothesis* (ACH) then suggests that poor countries should grow faster than rich countries. Barro and Sala-i-Martin (1995, p. 27) show the results of regressing of $\gamma_y(t)$ on $\log y(t)$ for a sample of 118 countries. The results are dismal: instead of finding a negative effect as predicted by the ACH, they find a slight positive effect. Absolute convergence does not seem to hold and (Romer's) stylized fact (SF7) is verified by the data.
 - More refined test: *Conditional convergence hypothesis* (CCH) according to which *similar* countries should converge. In **Figure 14.5** we show case where poor country is closer to its steady state than the rich country is to its own steady state. Hence, rich country grows at a faster rate. According to Barro and Sala-i-Martin the data confirm the CCH

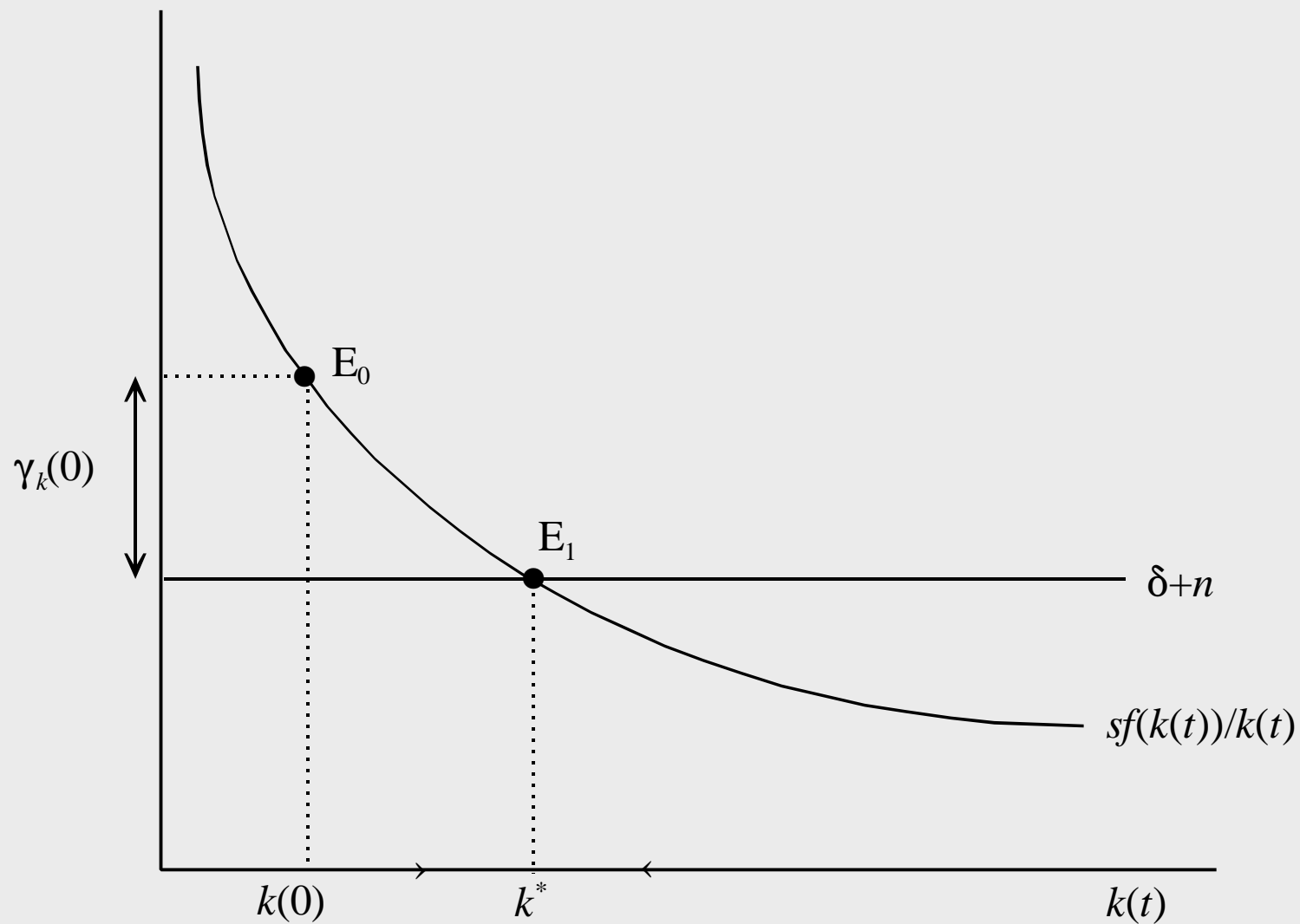


Figure 14.4: Growth Convergence

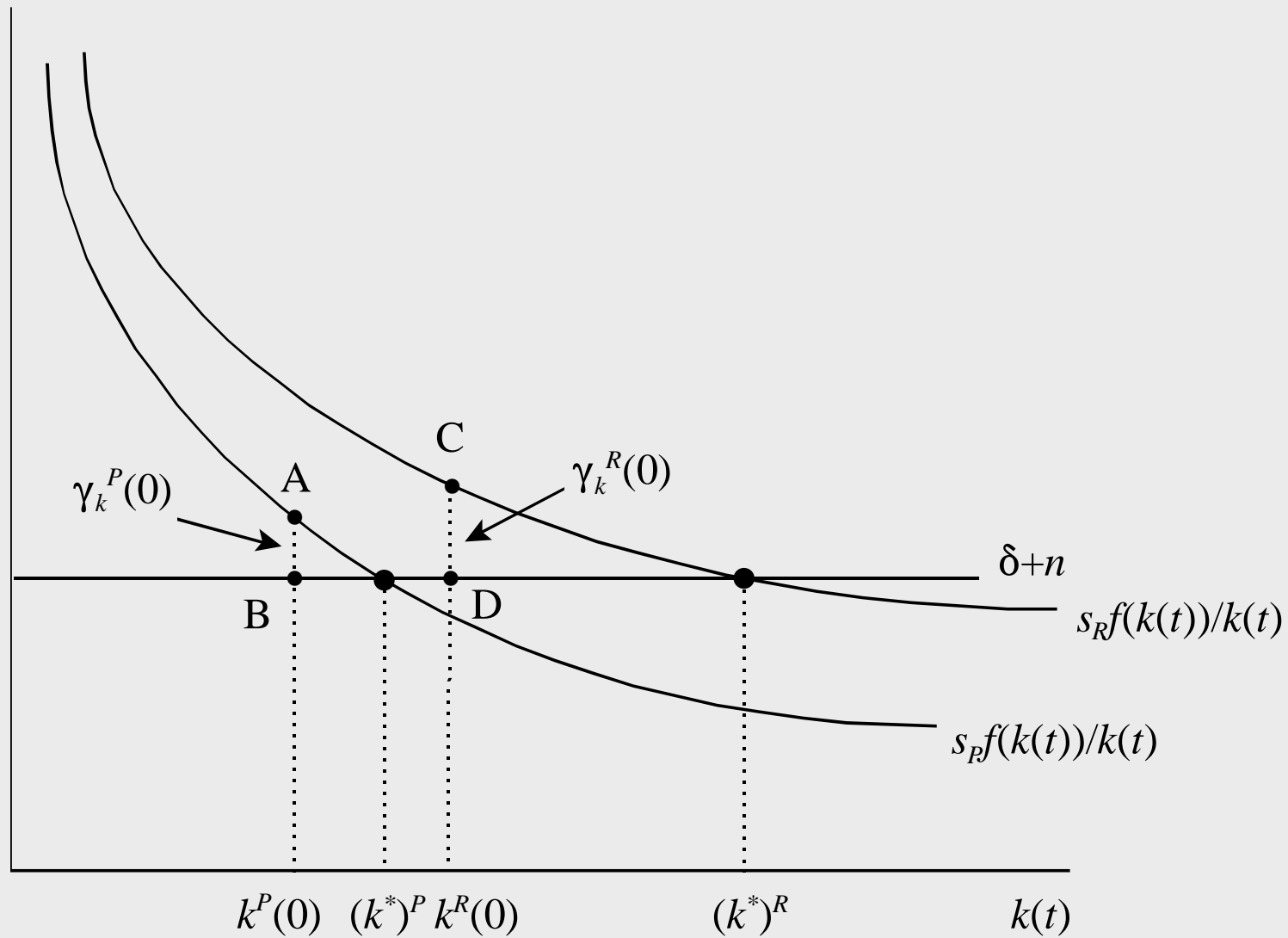


Figure 14.5: Conditional Growth Convergence

(C) Speed of adjustment

- How *fast* is the convergence in a Solow-Swan economy?
- Focus on the Cobb-Douglas case for which $f(.) = k(t)^\alpha$, and the FDE is:

$$\dot{k}(t) = sk(t)^\alpha - (\delta + n)k(t) \quad (*)$$

- An exact solution to the FDE can be obtained by using a transformation of variables, i.e. by rewriting (*) in terms of the capital-output ratio,

$$x(t) \equiv k(t)/y(t) = k(t)^{1-\alpha}:$$

$$\dot{x}(t) = (1 - \alpha) [s - (\delta + n)x(t)] \quad (**)$$

- The solution to (**) is obtained by standard methods:

$$x(t) = x(\infty) + [x(0) - x(\infty)] e^{-\beta t}$$

- where $x(\infty) \equiv s/(\delta + n)$ is the steady-state capital-output ratio to which the economy converges in the long run
- $\beta \equiv (1 - \alpha)(\delta + n)$ measures the speed of convergence. The interpretation of β is as follows: $\zeta \times 100\%$ of the divergence between $x(t)$ and $x(\infty)$ is eliminated after a time interval of t_ζ :

$$t_\zeta \equiv -(1/\beta) \log(1 - \zeta).$$

Hence, the half-life of the divergence ($\zeta = \frac{1}{2}$) equals

$$t_{1/2} = \log 2/\beta = 0.693/\beta.$$

- Back-of-the-envelope computations: $n_L = 0.01$ (per annum), $n_A = 0.02$, $\delta = 0.05$, and $\alpha = 1/3$ yield the value of $\beta = 0.0533$ (5.33 percent per annum) and an estimated half-life of $t_{1/2} = 13$ years. Fast transition.

- Estimate is way too high to accord with empirical evidence: actual β is in the range of 2 percent per annum (instead of 5.33 percent).
- Problem with the Solow-Swan model. Solutions:
 - assume high capital share (for $\alpha = \frac{3}{4}$ we get $\beta = 0.02$)!
 - assume a broad measure of capital to include human as well as physical capital [Mankiw, Romer, and Weil (1992)].

(D) Rescuing the Solow-Swan model

- *Key idea*: add human capital to the model
- Technology:

$$Y(t) = K(t)^{\alpha_K} H(t)^{\alpha_H} [A(t)L(t)]^{1-\alpha_K-\alpha_H}, \quad 0 < \alpha_K + \alpha_H < 1$$

where $H(t)$ is the stock of human capital and α_K and α_H are the efficiency parameters of the two types of capital ($0 < \alpha_K, \alpha_H < 1$).

- In close accordance with the Solow-Swan model, productivity and population growth are both exponential ($\dot{A}(t)/A(t) = n_A$ and $\dot{L}(t)/L(t) = n_L$)

- The accumulation equations for the two types of capital can be written in effective labour units as:

$$\begin{aligned}\dot{k}(t) &= s_K y(t) - (\delta + n)k(t), \\ \dot{h}(t) &= s_H y(t) - (\delta + n)h(t),\end{aligned}$$

where $h(t) \equiv H(t)/[A(t)L(t)]$, $n \equiv n_A + n_L$, and s_K and s_H represent the propensities to accumulate physical and human capital, respectively.

- Since there are decreasing returns to the two types of capital in combination ($\alpha_K + \alpha_H < 1$) the model possesses a steady state for which $\dot{k}(t) = \dot{h}(t) = 0$, $k(t) = k^*$, and $h(t) = h^*$:

$$k^* = \left(\frac{s_K^{1-\alpha_H} s_H^{\alpha_H}}{\delta + n} \right)^{1/(1-\alpha_K-\alpha_H)}$$

$$h^* = \left(\frac{s_K^{\alpha_K} s_H^{1-\alpha_K}}{\delta + n} \right)^{1/(1-\alpha_K-\alpha_H)}$$

- By substituting k^* and h^* into the (logarithm of the) production function we obtain an estimable expression for per capita output along the balanced growth path:

$$\begin{aligned} \log [Y(t)/L(t)] = & \log A(0) + n_A t - \left(\frac{\alpha_K + \alpha_H}{1 - \alpha_K - \alpha_H} \right) \log(n + \delta) \\ & + \left(\frac{\alpha_K}{1 - \alpha_K - \alpha_H} \right) \log s_K + \left(\frac{\alpha_H}{1 - \alpha_K - \alpha_H} \right) \log s_H \end{aligned}$$

- Mankiw et al. (1992, p. 417) suggest approximate guesses for $\alpha_K = \frac{1}{3}$ and α_H between $\frac{1}{3}$ and $\frac{4}{9}$.
- the extended Solow-Swan model is much better equipped to explain large cross-country income differences for relatively small differences between savings rates (s_K and s_H) and population growth rates (n). [multiplier factor is $\frac{1}{1-\alpha_K-\alpha_H}$ instead of $\frac{1}{1-\alpha_K}$.]
- the inclusion of a human capital variable works pretty well empirically; the estimated coefficient for α_H is highly significant and lies between 0.28 and 0.37
- The convergence property of the augmented Solow-Swan model is also much better. The convergence speed is now defined as $\beta \equiv (1 - \alpha_K - \alpha_H)(n + \delta)$ which can be made in accordance with the observed empirical estimate of $\hat{\beta} = 0.02$ without too much trouble.
- Hence, by this very simple and intuitively plausible adjustment the Solow-Swan model can be salvaged from the dustbin of history. The speed of convergence it implies can be made to fit the real world.

Macroeconomic applications of the Solow-Swan model

Overview:

- (A) Fiscal policy: long-run crowding out of private by public consumption.
- (B) Ricardian non-equivalence: government debt is not neutral.

(A) Fiscal policy in the Solow-Swan model

- The government consumes $G(t)$ units of output so that aggregate demand in the goods market is:

$$Y(t) = C(t) + I(t) + G(t)$$

- Aggregate saving is proportional to after-tax income:

$$S(t) = s [Y(t) - T(t)]$$

where $T(t)$ is the lump-sum tax

- Since $S(t) \equiv Y(t) - C(t) - T(t)$ any primary government deficit must be compensated for by an excess of private saving over investment, i.e.

$$G(t) - T(t) = S(t) - I(t)$$

- The government budget identity is given by:

$$\dot{B}(t) = r(t)B(t) + G(t) - T(t)$$

where $B(t)$ is government debt and $r(t)$ is the real interest rate

- Under the competitive conditions the interest rate equals the net marginal productivity of capital [see also below]:

$$r(t) = f'(k(t)) - \delta$$

- By writing all variables in terms of effective labour units, the model can be condensed to the following two equations:

$$\begin{aligned} \dot{k}(t) &= f(k(t)) - (\delta + n)k(t) - c(t) - g(t) \\ &= sf(k(t)) - (\delta + n)k(t) + (1 - s)\tau(t) - g(t), \end{aligned} \tag{a}$$

$$\dot{b}(t) = \left[f'(k(t)) - \delta - n \right] b(t) + g(t) - \tau(t), \tag{b}$$

where $\tau(t) \equiv T(t)/N(t)$, $g(t) \equiv G(t)/N(t)$, and $b(t) \equiv B(t)/N(t)$.

- Under pure tax financing we have $\dot{b}(t) = b(t) = 0$ so that government budget identity (b) reduces to $g(t) \equiv \tau(t)$. The FDE becomes:

$$\dot{k}(t) = sf(k(t)) - (\delta + n)k(t) - sg(t)$$

- In **Figure 14.6** we show stability and illustrate the effects of an increase in government consumption.
- Qualitative effects:

$$\frac{dy(\infty)}{dg} = \frac{f' dk(\infty)}{dg} = \frac{sf'}{sf' - (\delta + n)} < 0,$$

$$\frac{dc(\infty)}{dg} = (1 - s) \left[1 - \frac{dy(\infty)}{dg} \right] = \frac{(1 - s)(\delta + n)}{sf' - (\delta + n)} < 0.$$

Capital and consumption are both crowded out in the long run! Model is Classical in the long run [despite its Keynesian consumption function]

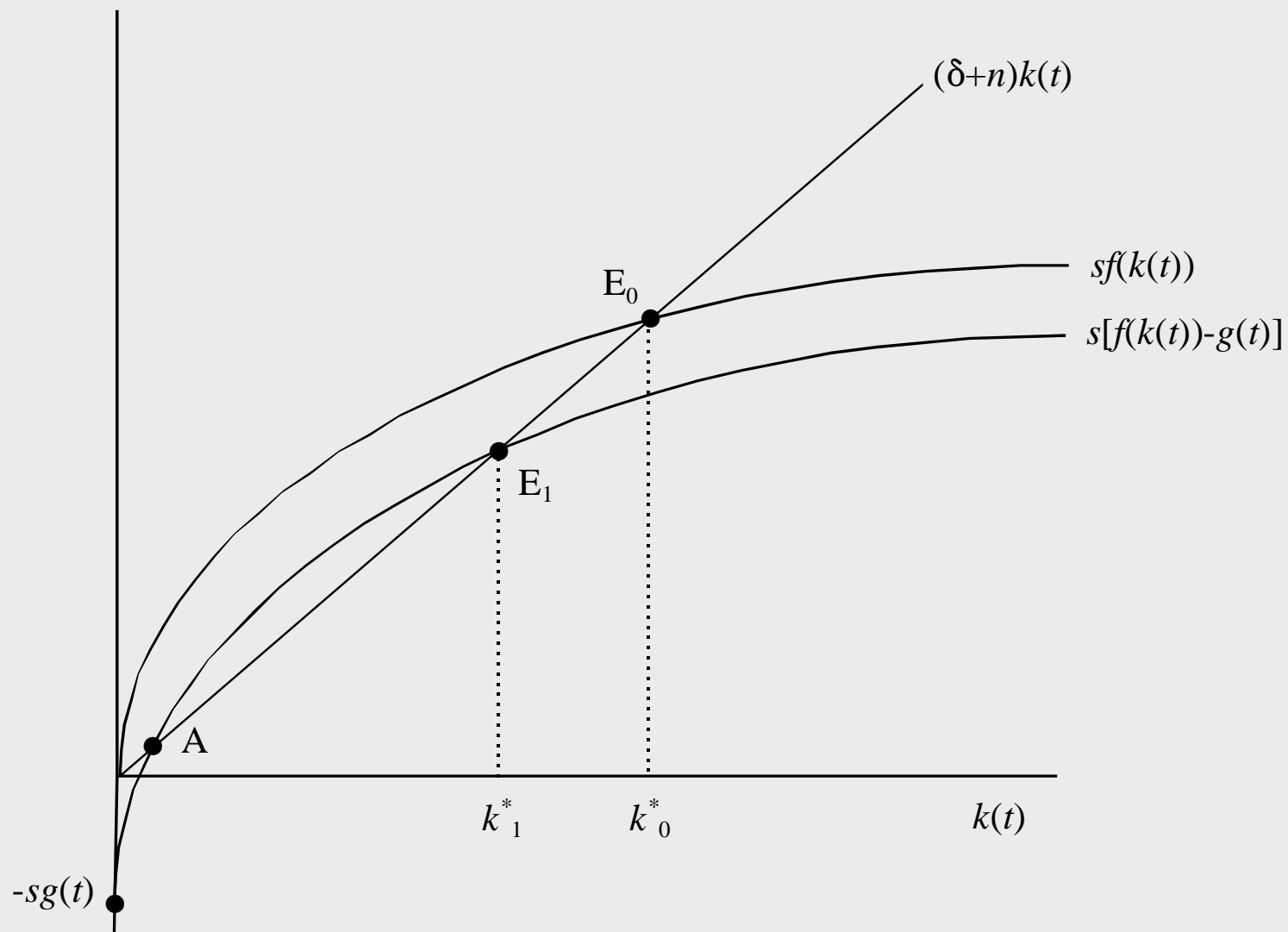


Figure 14.6: Fiscal Policy in the Solow-Swan Model

(B) Ricardian non-equivalence in the Solow-Swan model

- If the economy is dynamically efficient we have:

$$r(t) \equiv f'(k(t)) - \delta > n$$

- This means that debt process in (b) is inherently unstable [explosive]; an economically uninteresting phenomenon
- *Buiter rule* ensures that debt is stabilized:

$$\tau(t) = \tau_0 + \xi b(t), \quad \xi > r - n$$

- The system of fundamental differential equations becomes:

$$\dot{b}(t) = \left[f'(k(t)) - \delta - n - \xi \right] b(t) + g(t) - \tau_0$$

$$\dot{k}(t) = sf(k(t)) - (\delta + n)k(t) + (1 - s) [\tau_0 + \xi b(t)] - g(t)$$

- The model can be analyzed graphically with the aid of **Figure 14.7**.
- The $\dot{k} = 0$ line:
 - upward sloping in (b, k) space
 - points above (below) the line are associated with positive (negative) net investment, i.e. $\dot{k} > 0$ (< 0).
- The $\dot{b} = 0$ line:
 - downward sloping in (b, k) space
 - for points above (below) the $\dot{b} = 0$ line there is a government surplus (deficit) so that debt falls (rises).
- Equilibrium at E_0 is inherently stable.

- *Ricardian experiment*: postponement of taxation.
 - In the model this amounts to a reduction in τ_0 . This creates a primary deficit at impact ($g(t) > \tau_0$) so that government debt starts to rise.
 - In terms of **Figure 14.7**, both the $\dot{k} = 0$ line and the $\dot{b} = 0$ line shift up, the former by more than the latter.
 - In the long run, government debt, the capital stock, and output (all measured in efficiency units of labour) rise as a result of the tax cut.

$$\frac{dy(\infty)}{d\tau_0} = \frac{f' dk(\infty)}{d\tau_0} = - \frac{(1-s)(r-n)f'}{|\Delta|} < 0,$$

$$\frac{db(\infty)}{d\tau_0} = \frac{sf' - (\delta + n) + (1-s)bf''}{|\Delta|} < 0,$$

- Ricardian equivalence does not hold in the Solow-Swan model. A temporary tax cut boosts consumption, depresses investment and thus has real effects.

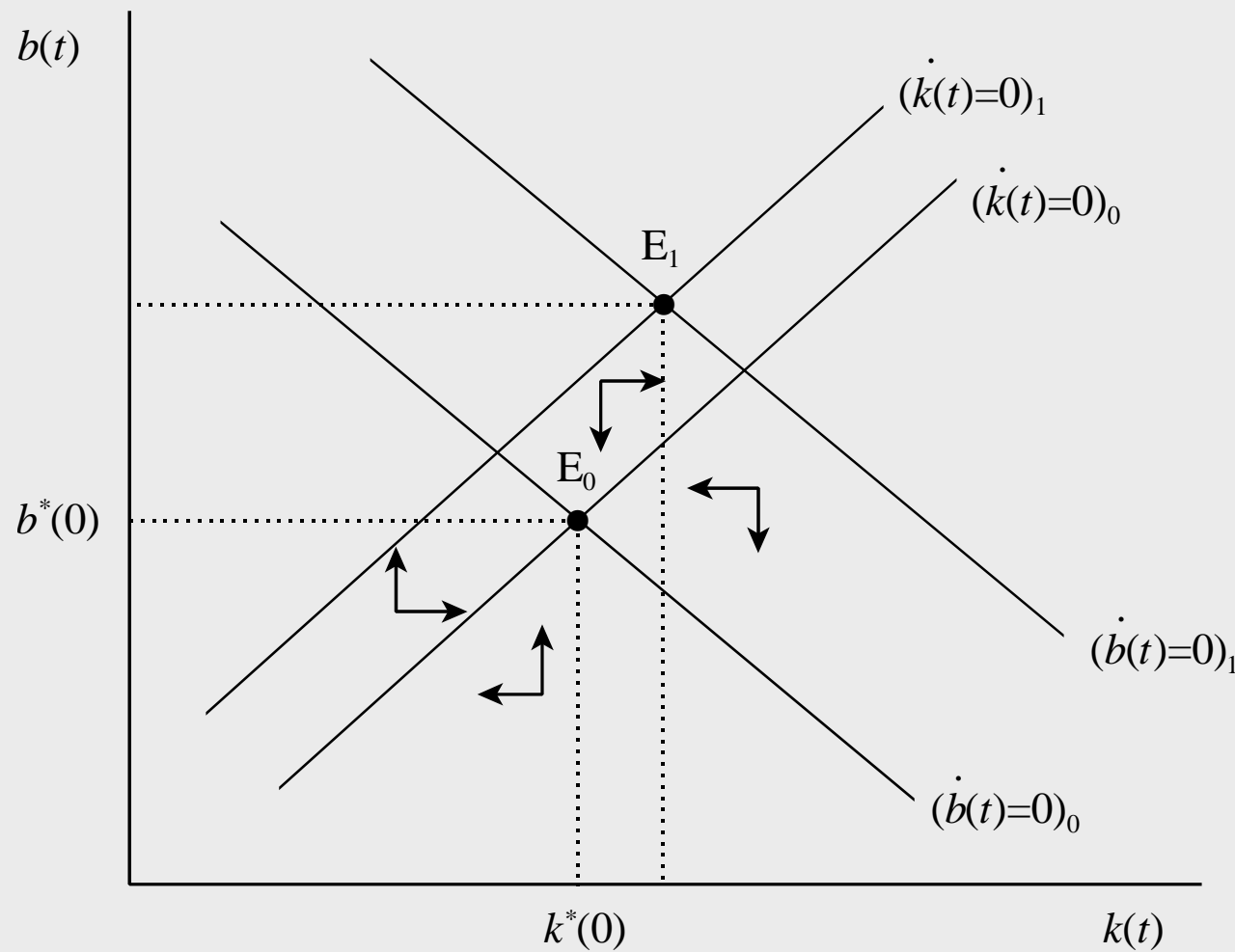


Figure 14.7: Ricardian Non-Equivalence in the Solow-Swan Model

The Ramsey model

- *Key idea*: replace ad hoc savings [consumption] function by forward-looking theory based on utility maximization
- Names: Frank Ramsey, Tjalling Koopmans, David Cass

Overview:

- (A) Representative consumer
- (B) Representative firm
- (C) Model properties
 - Phase diagram
 - Efficiency

(D) Growth properties

- Exogenous growth
- Transitional dynamics

(E) Fiscal policy

(F) Ricardian equivalence

(G) Conclusions

(A) Representative consumer

- Infinitely lived
- Identical
- Perfect foresight
- “Felicity function” [instantaneous utility]:

$$U(c(t))$$

with properties:

- positive but diminishing marginal felicity of consumption:

$$U'(c(t)) > 0$$

$$U''(c(t)) < 0$$

- “Inada-style” curvature conditions:

$$\lim_{c(t) \rightarrow 0} U'[c(t)] = +\infty$$

$$\lim_{c(t) \rightarrow \infty} U'[c(t)] = 0$$

[Draw on board]

- Labour supply is exogenous and grows exponentially:

$$\frac{\dot{L}(t)}{L(t)} = n_L$$

[May also be interpreted as dynastic family which grows exogenously]

- Utility functional is discounted integral of present and future felicity:

$$\Lambda(0) \equiv \int_0^{\infty} U[c(t)]e^{-\rho t} dt, \quad \rho > 0$$

where $t = 0$ is the planning period (“today”) and ρ is the rate of pure time preference. (“Millian” welfare function—utility of representative family member)

- The budget identity is:

$$\underbrace{C(t) + \dot{A}(t)}_{(a)} \equiv \underbrace{r(t)A(t) + W(t)L(t)}_{(b)}$$

where $C(t) \equiv L(t)c(t)$ is aggregate consumption, $A(t)$ is financial assets, $r(t)$ is the rate of return on these assets, $W(t)$ is the wage rate, and $\dot{A}(t) \equiv dA(t)/dt$ is the time rate of change.

(a) uses of income: consumption plus savings [asset accumulation]

(b) sources of income: interest income plus wage income

- In per capita form the budget identity becomes:

$$\dot{a}(t) \equiv [r(t) - n] a(t) + W(t) - c(t) \quad (*)$$

where $a(t) \equiv A(t)/L(t)$ is per capita financial assets.

- **NOTE:** equation (*) is just an *identity*; it does not restrict anything [e.g. household could accumulate debt indefinitely and let $a(t)$ approach $-\infty$]

- The solvency condition is the true *restriction* faced by the household:

$$\lim_{t \rightarrow \infty} a(t) \exp \left[- \int_0^t [r(\tau) - n] d\tau \right] = 0 \quad (\#)$$

Loosely put, the household does not plan to “expire” with positive assets and is not allowed by the capital market to die hopelessly indebted.

- By integrating (*) over the (infinite) life time of the agent and taking into account the solvency condition (#), we obtain the household life-time budget constraint:

$$\underbrace{\int_0^{\infty} c(t)e^{-[R(t)-nt]} dt}_{(a)} = \underbrace{a(0) + h(0)}_{(b)} \quad (\text{HBC})$$

where $a(0)$ is the initial level of financial assets, $R(t) \equiv \int_0^t r(\tau) d\tau$ is a discounting factor, and $h(0)$ is human wealth:

$$h(0) \equiv \underbrace{\int_0^{\infty} W(t)e^{-[R(t)-nt]} dt}_{(c)}$$

- (a) present value of the lifetime consumption path
- (b) total wealth in planning period, i.e. sum of financial and human wealth.
- (c) human wealth is present value of wage [market value of time endowment per capita]

- The consumer chooses a time path for $c(t)$ (for $t \in [0, \infty)$) which maximizes life-time utility, $\Lambda(0)$, subject to the life-time budget restriction. The first-order conditions are the budget restriction and:

$$\underbrace{U' [c(t)] e^{-\rho t}}_{(a)} = \underbrace{\lambda e^{-[R(t) - nt]}}_{(b)} \quad (*)$$

where λ is the marginal utility of wealth, i.e. the Lagrange multiplier associated with the life-time budget restriction.

- (a) the marginal contribution to life-time utility (evaluated from the perspective of “today,” i.e. $t = 0$) of consumption in period t .
- (b) the life-time marginal utility cost of consuming $c(t)$ rather than saving it. [The marginal unit of $c(t)$ costs $\exp(-[R(t) - nt])$ from the perspective of today. This cost is translated into utility terms by multiplying it with the marginal utility of wealth]

- Since λ is constant (i.e. it does not depend on t), differentiation of (*) yields an expression for the optimal *time profile* of consumption:

$$\frac{d}{dt} U' [c(t)] = -\lambda e^{-[R(t)-nt-\rho t]} \left[\frac{dR(t)}{dt} - n - \rho \right] \Leftrightarrow$$

$$U'' [c(t)] \frac{dc(t)}{dt} = -U' [c(t)] [r(t) - n - \rho] \Leftrightarrow$$

$$\theta[c(t)] \left(\frac{1}{c(t)} \frac{dc(t)}{dt} \right) = r(t) - n - \rho,$$

- we have used the fact that $dR(t)/dt = r(t)$
- $\theta[.]$ is the *elasticity of marginal utility* which is positive for all positive consumption levels because of the strict concavity of $U[.]$:

$$\theta[c(t)] \equiv -\frac{U''[c(t)]c(t)}{U'[c(t)]}$$

- The *intertemporal substitution elasticity*, $\sigma[\cdot]$, is the inverse of $\theta[\cdot]$. Hence, (*) can be re-written to yield the consumption Euler equation:

$$\frac{1}{c(t)} \frac{dc(t)}{dt} = \sigma[c(t)] \left[r(t) - n - \rho \right]$$

Intuition:

- if $\sigma[\cdot]$ is low, a large interest gap $(r(t) - n - \rho)$ is needed to induce the household to adopt an upward sloping time profile for consumption. In that case the willingness to substitute consumption across time is low, the elasticity of marginal utility is high, and the marginal utility function has a lot of curvature.
- The opposite holds if $\sigma[\cdot]$ is high. Then, the marginal utility function is almost linear so that a small interest gap can explain a large slope of the consumption profile.

Digression: specific forms of the felicity function

- Macroeconomists use specific functional forms for the felicity function in order to get closed-form solutions and to facilitate the computations. Two such forms are used:

- the *exponential felicity* function [which features $\sigma[\cdot] = \alpha/c(t)$]:

$$U [c(t)] \equiv -\alpha e^{-(1/\alpha)c(t)}, \quad \alpha > 0,$$

- the *iso-elastic* felicity function [which features $\sigma[\cdot] = \sigma$]:

$$U [c(t)] \equiv \begin{cases} \frac{c(t)^{1-1/\sigma} - 1}{1-1/\sigma} & \text{for } \sigma > 0, \quad \sigma \neq 1, \\ \log c(t) & \text{for } \sigma = 1. \end{cases}$$

- The corresponding consumption Euler equations are:

$$\frac{dc(t)}{dt} = \alpha [r(t) - n - \rho], \quad (\text{exponential felicity}),$$
$$\frac{1}{c(t)} \frac{dc(t)}{dt} = \sigma [r(t) - n - \rho], \quad (\text{iso-elastic felicity}).$$

- Most of our discussion will make use of the iso-elastic felicity function.

(B) Representative firm

- perfect competition
- constant returns to scale
- representative firm
- no adjustment costs on investment
- The stock market value of the firm is given by the discounted value of its cash flows:

$$V(0) = \int_0^{\infty} \left[F [K(t), L(t)] - W(t)L(t) - (1 - s_I) I(t) \right] e^{-R(t)} dt$$

where $R(t)$ is the discounting factor given above and $I(t)$ is gross investment:

$$I(t) = \delta K(t) + \dot{K}(t)$$

- The firm maximizes $V(0)$ subject to the capital accumulation constraint.

- Since there are no adjustment costs on investment the firm's decision about factor inputs is essentially a static one, i.e. the familiar marginal productivity conditions for labour and capital hold:

$$F_L [K(t), L(t)] = W(t)$$

$$F_K [K(t), L(t)] = r(t) + \delta$$

- By writing the production function in the intensive form, i.e. $f(k(t)) \equiv F \left[\frac{K(t)}{L(t)}, 1 \right]$, we can rewrite the marginal products of capital and labour as follows:

$$F_K [K(t), L(t)] = f'(k(t))$$

$$F_L [K(t), L(t)] = f(k(t)) - k(t)f'(k(t))$$

(C) Model properties

- We now have the complete Ramsey model (see **Table 14.1**):

$$\dot{c}(t) = \sigma [r(t) - n - \rho] c(t), \quad (\text{T1.1})$$

$$\dot{k}(t) = f(k(t)) - c(t) - (\delta + n)k(t), \quad (\text{T1.2})$$

$$r(t) = f'(k(t)) - \delta. \quad (\text{T1.3})$$

- Eqn (T1.1) is the consumption Euler equation
- Eqn (T1.2) is the FDE for the capital stock
- Eqn (T1.3) shows that the real interest rate is the net marginal product of capital
- In **Figure 14.8** we show the phase diagram of the Ramsey model.

Table 14.1. The Ramsey growth model

$$\dot{c}(t) = \sigma [r(t) - n - \rho] c(t), \quad (\text{T1.1})$$

$$\dot{k}(t) = f(k(t)) - c(t) - (\delta + n)k(t), \quad (\text{T1.2})$$

$$r(t) = f'(k(t)) - \delta. \quad (\text{T1.3})$$

Notes: $c(t)$ is per capita consumption, $k(t)$ is the capital-labour ratio, and $r(t)$ is the interest rate. Capital depreciates at a constant rate δ and the population grows exponentially with rate n .

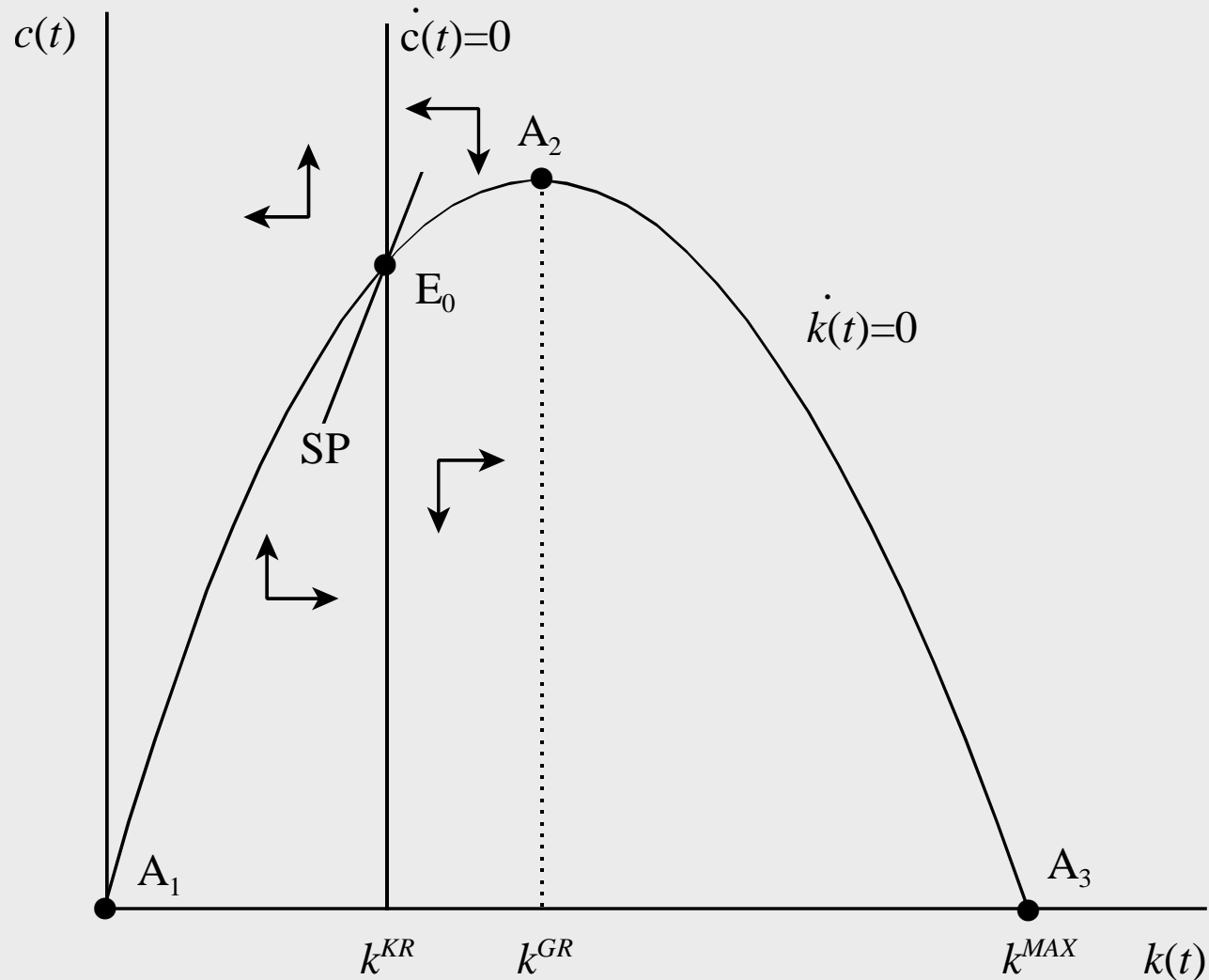


Figure 14.8: Phase Diagram of the Ramsey Model

- Features of the $\dot{k} = 0$ line: equilibrium of capital stock per worker.
 - Vertical in origin [Inada conditions],
 - maximum in golden rule point A_2 :

$$\left(\frac{dc(t)}{dk(t)} \right)_{\dot{k}(t)=0} = 0 : \quad f' [k^{GR}] = \delta + n \quad (*)$$

- maximum attainable k is in point A_3 where k^{MAX} is:

$$\frac{f(k^{MAX})}{k^{MAX}} = \delta + n$$

- capital dynamics:

$$\frac{\partial \dot{k}(t)}{\partial k(t)} = \underbrace{f' - (\delta + n)}_r = r - n \begin{matrix} \geq \\ < \end{matrix} 0 \quad \text{for } k(t) \begin{matrix} \leq \\ > \end{matrix} k^{GR}$$

See the horizontal arrows in Figure 14.8.

- Features of the $\dot{c} = 0$ line: flat per capita consumption profile
 - $\dot{c} = 0$ implies constant interest rate [$r = \rho + n$] and unique capital-labour ratio:

$$f'(k^{KR}) = \delta + n + \rho \quad (\#)$$

[KR stands for “Keynes-Ramsey”]

- for points to the left (right) of the $\dot{c} = 0$ line k is too low (high) and r is too high (low). See the vertical arrows in Figure 14.8.
- Comparing (*) and (#) shows that $f'(k^{KR}) > f'(k^{GR})$ so that $k^{KR} < k^{GR}$ and thus $r^{KR} > r^{GR}$ [no dynamic inefficiency possible!!]. By removing the ad hoc savings function from the Solow-Swan model the possibility of oversaving vanishes.
- The configuration of arrows confirms that the Ramsey model is saddle point stable. The equilibrium is unique (at point E_0) and the saddle path is upward sloping. $k(t)$ is the predetermined variable and $c(t)$ is the non-predetermined or “jumping” variable.

(D) Growth properties

- In the BGP the model features a constant capital labour ratio (k^{KR}). Hence, growth is just like in the Solow-Swan model: all variables grow at the same exogenously given rate [n_L in the present model as we assume $n_A = 0$]
- We can study the convergence speed of the Ramsey model by linearizing it around the steady state:

$$\begin{bmatrix} \dot{c}(t) \\ \dot{k}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \sigma c^* f''(k^*) \\ -1 & \rho \end{bmatrix}}_{\Delta} \begin{bmatrix} c(t) - c^* \\ k(t) - k^* \end{bmatrix},$$

where Δ is the Jacobian matrix. Features:

- $\text{tr}(\Delta) \equiv \lambda_1 + \lambda_2 = \rho > 0$ and $|\Delta| \equiv \lambda_1 \lambda_2 = \sigma c^* f''(k^*) < 0$, where λ_1 and λ_2 are the characteristic roots of Δ ,

- hence, the model is saddle point stability, i.e. λ_1 and λ_2 have opposite signs.
- The absolute value of the stable (negative) characteristic root determines the approximate convergence speed of the economic system. After some manipulation we obtain the following expression:

$$\beta \equiv \left(\frac{\rho}{2}\right) \left[\sqrt{1 - \frac{4\sigma c^* f''(k^*)}{\rho^2}} - 1 \right]$$

$$= \left(\frac{\rho}{2}\right) \left[\sqrt{1 + \frac{4}{\rho^2} \left(\frac{\sigma}{\sigma_{KL}}\right) \left(\frac{c}{k}\right)^* (r^* + \delta) - 1} \right]$$

- by plugging in some realistic numbers for the structural parameters [σ , σ_{KL} , ρ , δ , etcetera] we can compute the speed of convergence implied by the Ramsey model. See **Table 14.2**. Only if the capital share is high and the felicity function is relatively low does the model predict a realistic convergence speed.

Table 14.2. Convergence speed in the Ramsey model

	σ / σ_{KL}			
	0.2	0.5	1	2
$\omega_K = \frac{1}{3}$	4.23	7.38	10.97	16.08
$\omega_K = \frac{1}{2}$	2.41	4.39	6.70	10.00
$\omega_K = \frac{2}{3}$	1.25	2.44	3.88	5.96

(E) Fiscal policy in the Ramsey model

[Heuristic solution concept can be used]

- Suppose the government consumes goods, $g(t)$. The FDE for $k(t)$ becomes:

$$\dot{k}(t) = f(k(t)) - c(t) - g(t) - (\delta + n)k(t),$$

where $g(t) \equiv G(t)/L(t)$ is per capita government consumption.

- Government consumption withdraws resources which are no longer available for private consumption or replacement of the capital stock. As a result, for a given level of per capita public consumption, $g(t) = g$, the $\dot{k}(t) = 0$ line can be drawn as in

Figure 14.11. Features:

- still no dynamic inefficiency
- still unique equilibrium

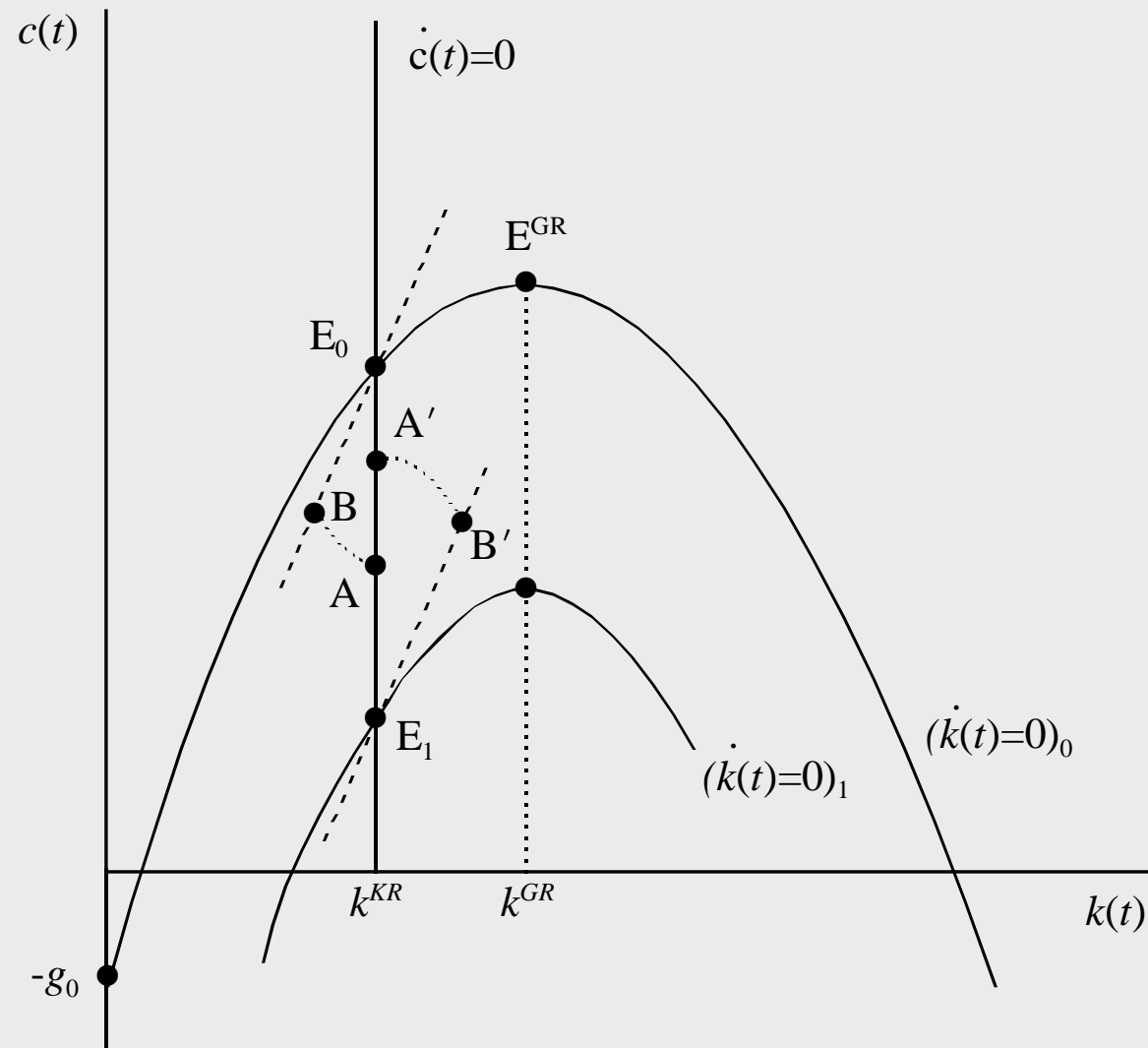


Figure 14.11: Fiscal Policy in the Ramsey Model

- An *unanticipated* and *permanent* increase in the level of government consumption per worker shifts the $\dot{k}(t) = 0$ line down, say to $(\dot{k}(t) = 0)_1$.
 - Since the shock comes as a complete surprise to the representative household, it reacts to the increased level of taxes (needed to finance the additional government consumption) by cutting back private consumption. The representative household feels poorer as a result of the shock and, as consumption is a normal good, reduces it one-for-one:

$$\frac{dc(t)}{dg} = -1, \quad \frac{dy(t)}{dg} = \frac{dk(t)}{dg} = 0,$$

for all $t \in [0, \infty)$.

- There is no transitional dynamics because the shock itself has no long-run effect on the capital stock and there are no anticipation effects. In terms of Figure 14.11 the economy jumps from E_0 to E_1 .

- With a *temporary* increase in g there are non-trivial transition effects.
 - The representative household anticipates the temporarily higher taxes but spreads the negative effect on human wealth out over the entire life-time consumption path. As a result, the impact effect on private consumption is still negative but less than one-for-one:

$$-1 < \frac{dc(0)}{dg} < 0.$$

In terms of Figure 14.11 the economy jumps from E_0 to point A.

- Immediately after the shock the household starts to dissave so that the capital stock falls, the interest rate rises, and (by (T1.1)) the consumption path rises over time. The economy moves from A to B which is reached at the time the government consumption is cut back to its initial level again.
- This cut in g (and the associated taxes) releases resources which allows the capital stock to return to its constant steady-state level. As a result of the temporary boost in government consumption, the policy maker has managed to engineer a temporary decline in output per worker.

- With an *anticipated* and *permanent* increase in g the opposite effect occurs during transition.
 - Consumption falls by less than one-for-one, but since the government consumption has not risen yet it leads to additional saving and a gradual increase in the capital stock, a reduction in the interest rate, and a downward-sloping consumption profile.
 - At impact the economy jumps from E_0 to A' , after which it gradually moves from A' to B' during transition.
 - Point B' is reached at precisely the time the policy is enacted. As g is increased net saving turns into net dissaving and the capital stock starts to fall. The economy moves from point B' to E_1 .
 - Hence, there is a temporary boost to k due to anticipation effects.

(F) Ricardian equivalence in the Ramsey model

- Ricardian equivalence holds in the Ramsey model.
- The government budget *identity* (in per capita form) is given by:

$$\dot{b}(t) = [r(t) - n] b(t) + g(t) - \tau(t) \quad (*)$$

- Like the representative household, the government must also remain solvent so that it faces an intertemporal solvency condition of the following form:

$$\lim_{t \rightarrow \infty} b(t) e^{-[R(t) - nt]} = 0 \quad (\#)$$

By combining (*) and (#), we obtain the government budget *restriction*:

$$b(0) = \int_0^{\infty} [\tau(t) - g(t)] e^{-[R(t) - nt]} dt \quad (\text{GBC})$$

- To the extent that there is a pre-existing government debt ($b(0) > 0$), solvency requires that this debt must be equal to the present value of future primary surpluses. In principle, there are infinitely many paths for $\tau(t)$ and $g(t)$ (and hence for the primary deficit), for which the GBC is satisfied.
- The budget *identity* of the representative agent is:

$$\dot{a}(t) \equiv [r(t) - n] a(t) + W(t) - \tau(t) - c(t)$$

- The household solvency condition is:

$$\lim_{t \rightarrow \infty} a(t) \exp \left[- \int_0^t [r(\tau) - n] d\tau \right] = 0$$

- The household budget *restriction* is then:

$$\int_0^{\infty} c(t) e^{-[R(t)-nt]} dt = a(0) + h(0), \quad (\text{HBC})$$

$$h(0) \equiv \int_0^{\infty} [W(t) - \tau(t)] e^{-[R(t)-nt]} dt$$

- By using the GBC, the expression for human wealth can be rewritten as:

$$h(0) = \int_0^{\infty} [W(t) - g(t)] e^{-[R(t)-nt]} dt - b(0) \quad (*)$$

The path of lump-sum taxes completely vanishes from the expression for human wealth. Since $b(0)$ and the path for $g(t)$ are given, the particular path for lump-sum taxes does not affect the total amount of resources available to the representative agent. As a result, the agent's real consumption plans are not affected either.

- By using (*) in the HBC, the household budget restriction can be written as:

$$\int_0^{\infty} c(t)e^{-[R(t)-nt]}dt = [a(0) - b(0)] + \int_0^{\infty} [W(t) - g(t)] e^{-[R(t)-nt]}dt$$

Under Ricardian equivalence, government debt should not be seen as household wealth, i.e. $b(0)$ must be deducted from total financial wealth in order to reveal the household's true financial asset position.

(G) Conclusion regarding the Ramsey model

- It yields very similar growth predictions as the Solow-Swan model does
- Unlike the Solow-Swan model it features Ricardian equivalence and rules out oversaving.
- It is attractive because it features intertemporal optimization by households rather than ad hoc consumption-saving rules
- It forms the basis of much of modern macroeconomics [endogenous growth theory, RBC theory]

- It can easily be extended to the open economy. Two complications in a small open economy (SOE) setting:
 - world interest rate, r^* , constant, so physical capital stock jumps across borders unless we are willing to assume adjustment costs on investment
 - consumption steady state only defined if r^* is equal to $\rho + n$. [knife-edge case] If $r^* > \rho + n$ [patient country] then country ends up owning all assets; if $r^* < \rho + n$ [impatient country] then country slowly disappears.
 - if $r^* = \rho + n$ then there is a zero root in consumption [as $\dot{c}(t)/c(t) = 0$ is the Euler equation]: flat consumption profile. National solvency condition determines level of consumption. There is *hysteresis* in consumption and the stock of net foreign assets.
- but empirically the Ramsey consumption theory does not work very well [and the ad hoc rules do work well!!]

Table 14.3. The Ramsey model for the open economy

$$\frac{\dot{c}(t)}{c(t)} = 0 \tag{T3.1}$$

$$q(t)\Phi' \left(\frac{i(t)}{k(t)} \right) = 1 - s_I \tag{T3.2}$$

$$\dot{q}(t) = \left[\rho + n + \delta - \Phi \left(\frac{i(t)}{k(t)} \right) \right] q(t) - f'(k(t)) + \frac{(1 - s_I) i(t)}{k(t)} \tag{T3.3}$$

$$\frac{\dot{k}(t)}{k(t)} = \Phi \left(\frac{i(t)}{k(t)} \right) - n - \delta \tag{T3.4}$$

$$\dot{a}_F(t) = \rho a_F(t) + f(k(t)) - c(t) - i(t) \tag{T3.5}$$

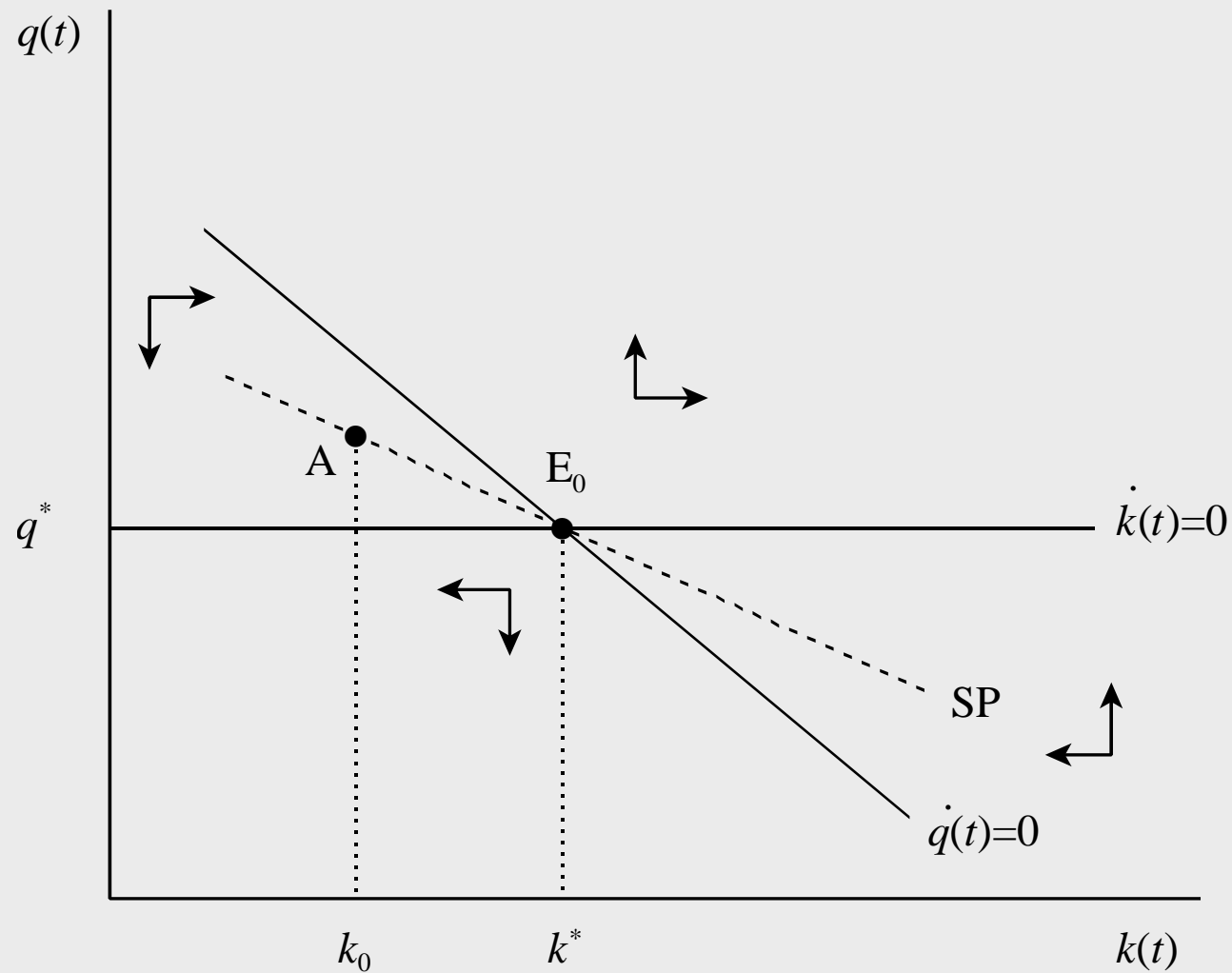


Figure 14.9: Investment in the Open Economy

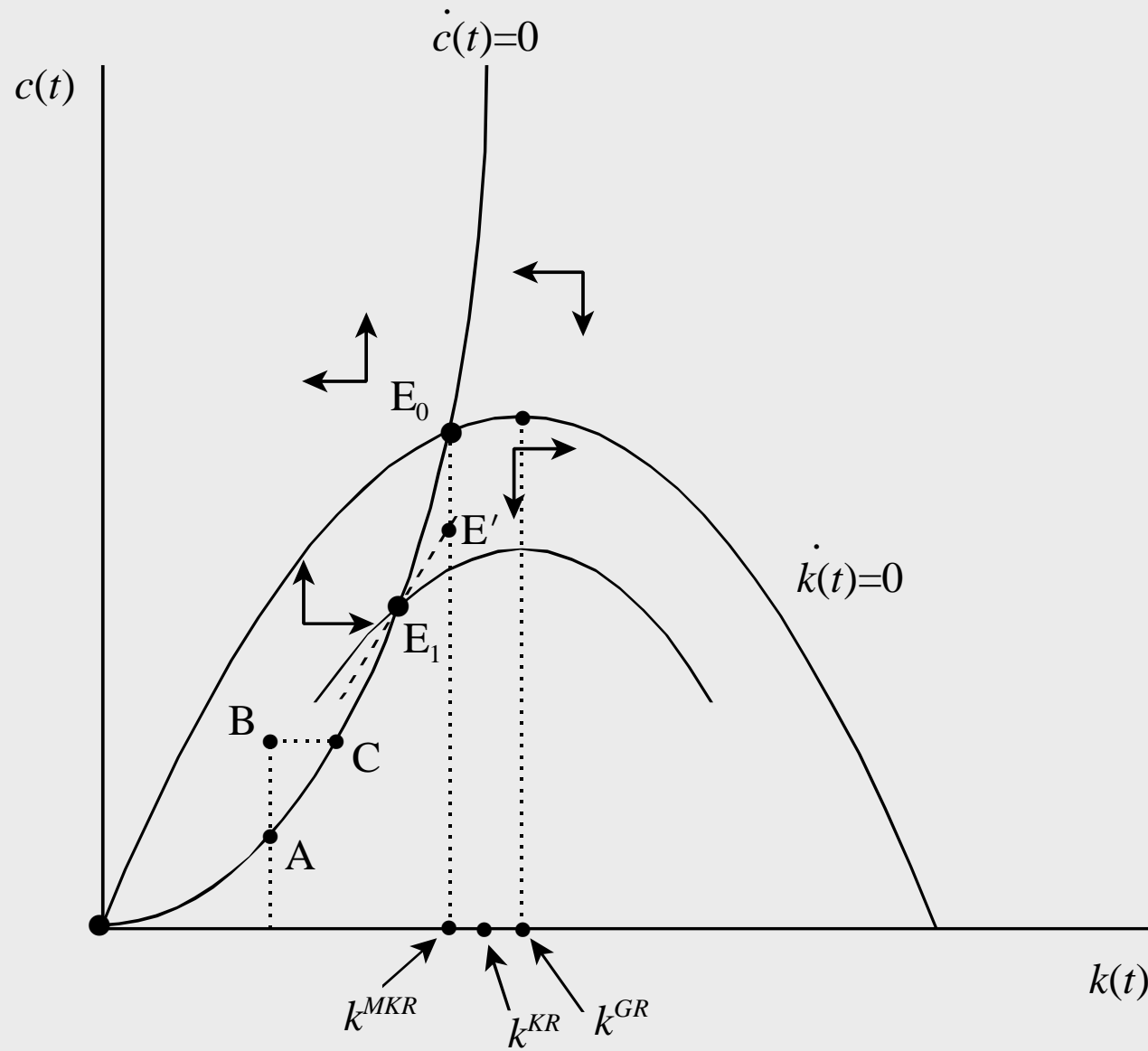


Figure 14.10: An Investment Subsidy With High Mobility of Physical Capital

Table 14.4. The Weil model of overlapping generations

$$\dot{c}(t) = [r(t) - \rho] c(t) - n\rho k(t), \quad (\text{T4.1})$$

$$\dot{k}(t) = f(k(t)) - c(t) - g(t) - (\delta + n)k(t) \quad (\text{T4.2})$$

$$r(t) = f'(k(t)) - \delta \quad (\text{T4.3})$$

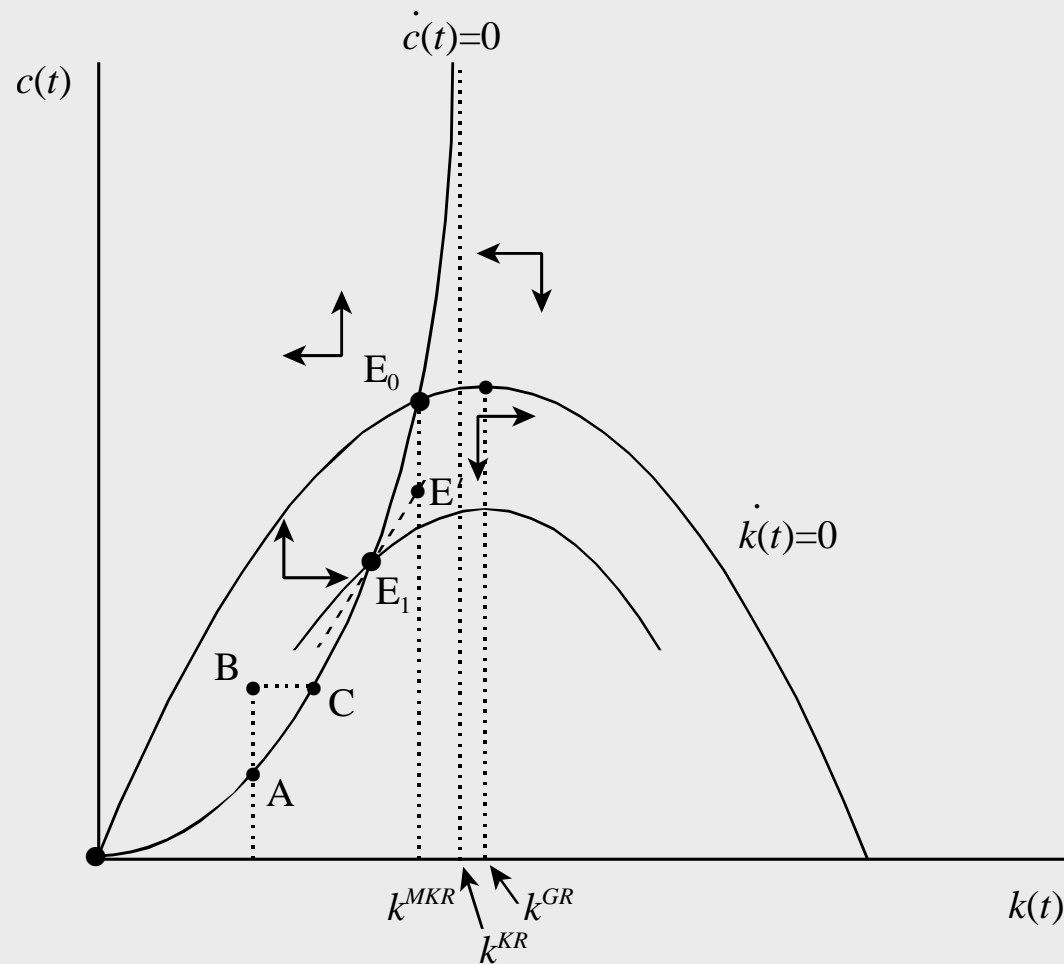


Figure 14.12: Fiscal Policy in the Overlapping Generations Model

Theories of endogenous growth

- *Key notion*: can we devise a growth theory in which the steady-state growth rate is endogenous, i.e. depends not just on things like the population growth rate and the rate of Harrod-neutral technological change?
- Following the influential work of Paul Romer in the mid 1980s a very active research field has developed.
- We will give a selective overview of this huge body of literature. Three groups can be distinguished:
 - (A) “Capital fundamentalist” models
 - (B) Human capital formation
 - (C) Endogenous technology

(A) “Capital fundamentalist” models of endogenous growth

- *Key idea*: there are perfectly reasonable production functions which do not satisfy the Inada conditions.
- Case I: Take the CES production function:

$$F [K(t), N(t)] \equiv \left[\alpha K(t)^{(\sigma_{KL}-1)/\sigma_{KL}} + (1 - \alpha) N(t)^{(\sigma_{KL}-1)/\sigma_{KL}} \right]^{\sigma_{KL}/(\sigma_{KL}-1)} \Leftrightarrow$$

$$f [k(t)] \equiv \left[1 - \alpha + \alpha k(t)^{(\sigma_{KL}-1)/\sigma_{KL}} \right]^{\sigma_{KL}/(\sigma_{KL}-1)}$$

- The average product of capital (APK) equals:

$$\frac{f(k(t))}{k(t)} = \left[(1 - \alpha) k(t)^{(1-\sigma_{KL})/\sigma_{KL}} + \alpha \right]^{\sigma_{KL}/(\sigma_{KL}-1)}$$

Two cases must be distinguished:

(a) difficult substitution between K and N [$0 < \sigma_{KL} < 1$]

(b) easy substitution between K and N [$\sigma_{KL} > 1$]

(a) With difficult substitution we have:

$$\lim_{k(t) \rightarrow 0} \frac{f(k(t))}{k(t)} = \alpha^{\sigma_{KL}/(\sigma_{KL}-1)} > 0$$

$$\lim_{k(t) \rightarrow \infty} \frac{f(k(t))}{k(t)} = \lim_{k(t) \rightarrow \infty} \frac{f'(k(t))}{1} = 0$$

The APK is finite for $k(t) \rightarrow 0$. Hence, it may not be even high enough to sustain a non-trivial steady state [see **Figure 14.13**]

(b) With easy substitution we have:

$$\lim_{k(t) \rightarrow 0} \frac{f(k(t))}{k(t)} = \lim_{k(t) \rightarrow 0} \frac{f'(k(t))}{1} = \infty$$

$$\lim_{k(t) \rightarrow \infty} \frac{f(k(t))}{k(t)} = \alpha^{\sigma_{KL}/(\sigma_{KL}-1)} > 0$$

Hence, there is a lower bound on the APK as $k(t) \rightarrow \infty$. Hence, there may be perpetual growth in $k(t)$ [see **Figure 14.14**]. The asymptotic growth rate is:

$$\gamma^* = s\alpha^{\sigma_{KL}/(\sigma_{KL}-1)} - (\delta + n) > 0$$

We call this an *endogenous* growth rate because the savings rate, s , affects it!

- Even though there are diminishing returns to capital, K and N substitute easily. Hence, labour does not become an effective constraint. Scarce labour is substituted by capital indefinitely.
- But, the share of labour goes to zero. This is contra SF3 and SF5.

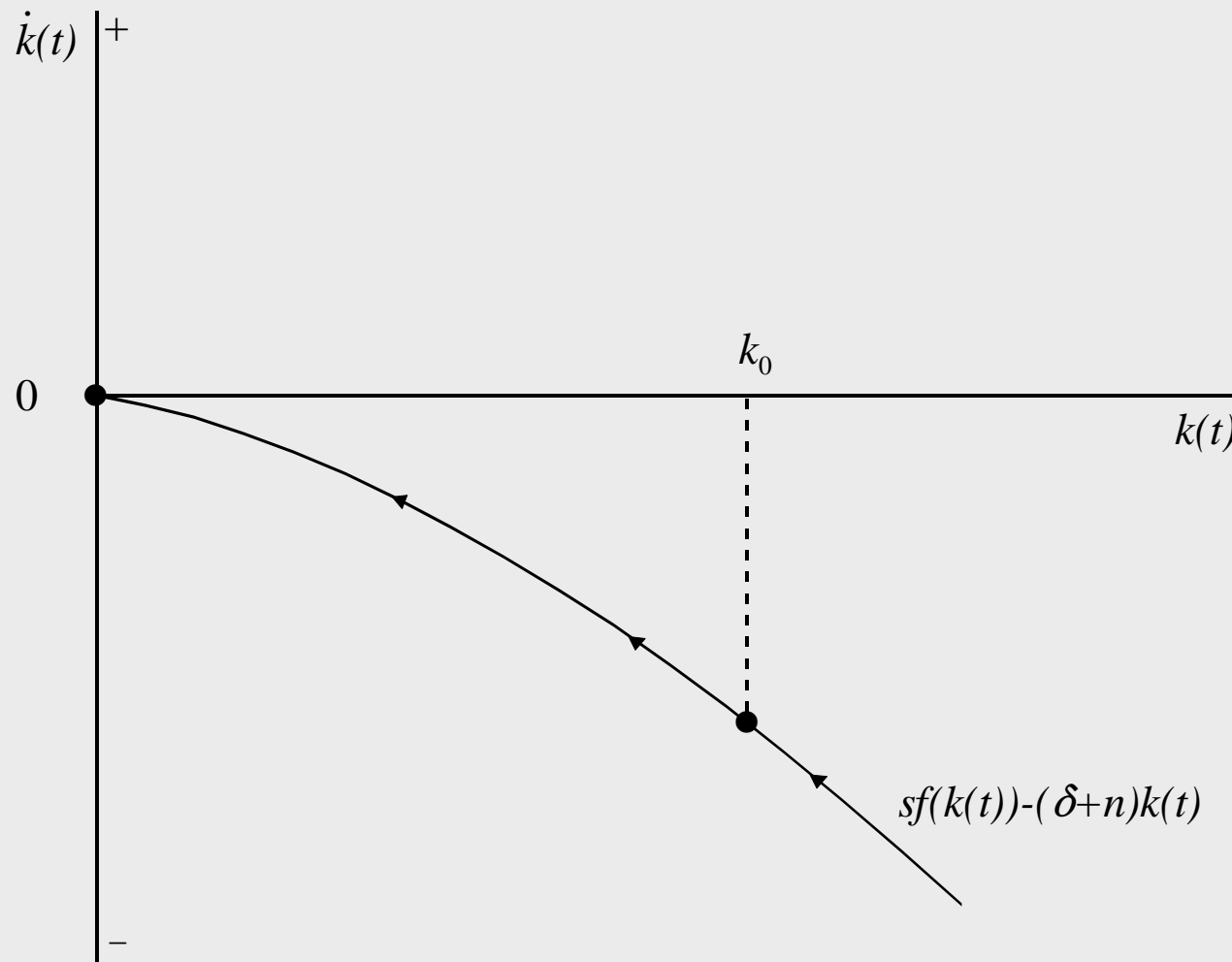


Figure 14.13: Difficult Substitution

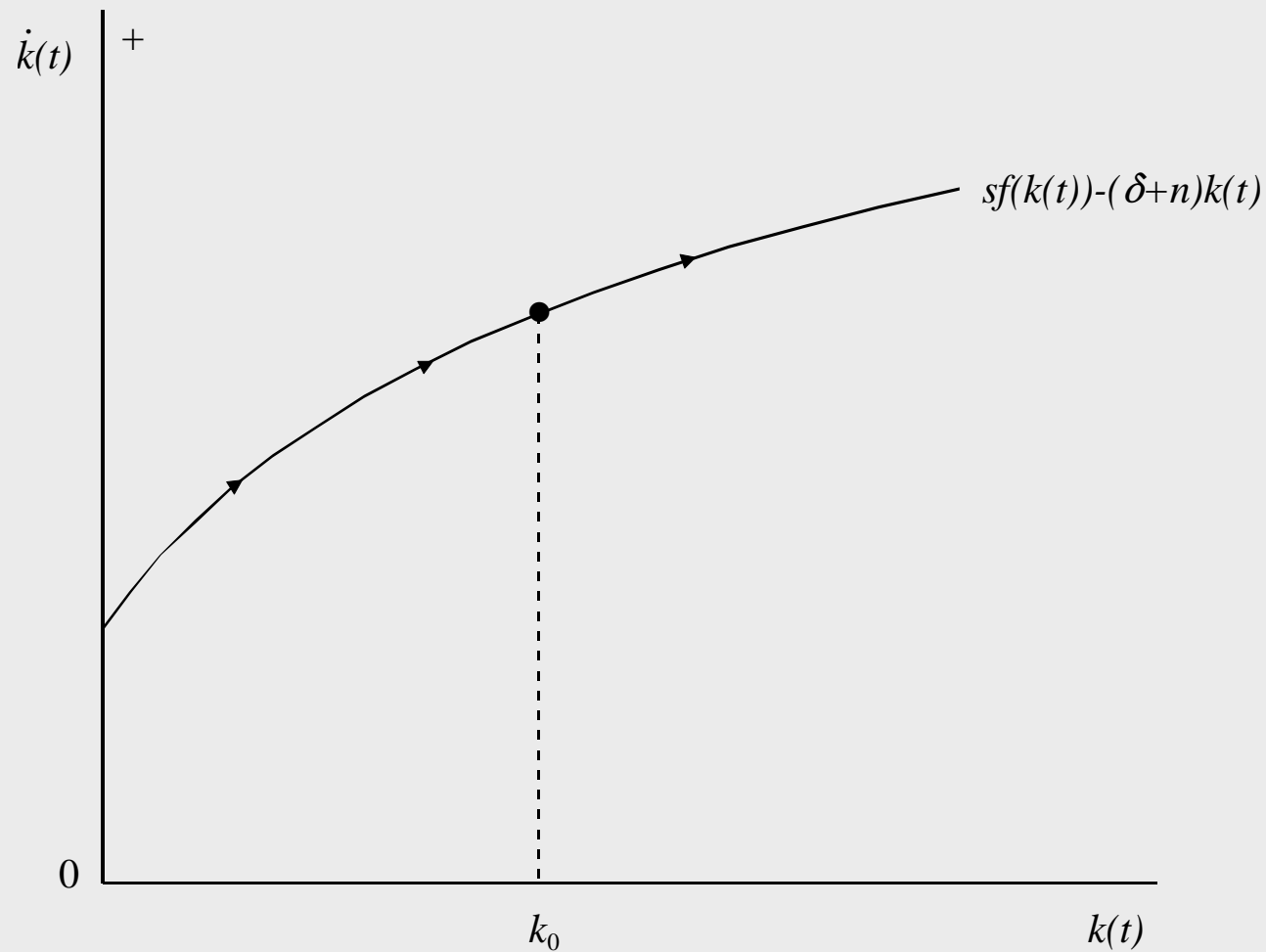


Figure 14.14: Easy Substitution

- Case II: An even more radical model is the so-called *AK model*. Technology is:

$$Y(t) = AK(t)$$

The MPK is constant and labour is eliminated from the model altogether!

- The growth rate in the economy is [see book]:

$$\gamma^* = \frac{\dot{C}(t)}{C(t)} = \frac{\dot{Y}(t)}{Y(t)} = \frac{\dot{K}(t)}{K(t)} = \frac{\dot{I}(t)}{I(t)} = \sigma \left[\frac{A}{1 - s_I} - \delta - \rho \right],$$

where σ is the intertemporal substitution elasticity, ρ is the rate of time preference, and s_I is an investment subsidy.

- Endogenous growth: the policy maker can affect it by setting s_I
- Growth path is efficient [no source of market failure]
- No transitional dynamics

- Case III: Barro suggest a model in which productive government spending affects productivity [and growth]. Technology is:

$$Y(t) = AK(t)^\alpha G(t)^{1-\alpha} = AK(t) \left(\frac{G(t)}{K(t)} \right)^{1-\alpha}, \quad 0 < \alpha < 1,$$

where $G(t)$ is government spending [flow].

- diminishing returns to private capital, but ...
- ... if the government maintains constant G/K ratio then model is like AK model.
- The government budget constraint is:

$$G(t) = t_Y Y(t)$$

– so that the growth rate is:

$$\begin{aligned} \gamma^* &= \frac{\dot{C}(t)}{C(t)} = \frac{\dot{Y}(t)}{Y(t)} = \frac{\dot{K}(t)}{K(t)} = \frac{\dot{I}(t)}{I(t)} \\ &= \sigma \left[\alpha(1 - t_Y) A \left(\frac{G(t)}{K(t)} \right)^{1-\alpha} - \delta - \rho \right] \\ &= \sigma \left[\alpha(1 - t_Y) A^{1/\alpha} t_Y^{(1-\alpha)/\alpha} - \delta - \rho \right] \end{aligned}$$

– in **Figure 14.15** we plot the growth rate as a function of the output tax. The optimal [growth and welfare maximizing] tax is:

$$\frac{d\gamma^*}{dt_Y} = 0 \quad \Leftrightarrow \quad t_Y = \frac{G(t)}{Y(t)} = 1 - \alpha.$$

Table 14.5. The basic AK growth model

$$\dot{C}(t) = \sigma \left[r(t) - \rho - \frac{\dot{t}_C(t)}{1 + t_C(t)} \right] C(t) \quad (\text{T5.1})$$

$$\dot{K}(t) = (A - \delta)K(t) - C(t) \quad (\text{T5.2})$$

$$r(t) = \frac{A}{1 - s_I(t)} - \frac{\dot{s}_I(t)}{1 - s_I(t)} - \delta \quad (\text{T5.3})$$

Notes: $C(t)$ is consumption, $K(t)$ is the capital stock, $r(t)$ is the interest rate, $t_C(t)$ is the consumption tax, $s_I(t)$ is an investment subsidy, ρ is the pure rate of time preference, and δ is the depreciation rate of capital.

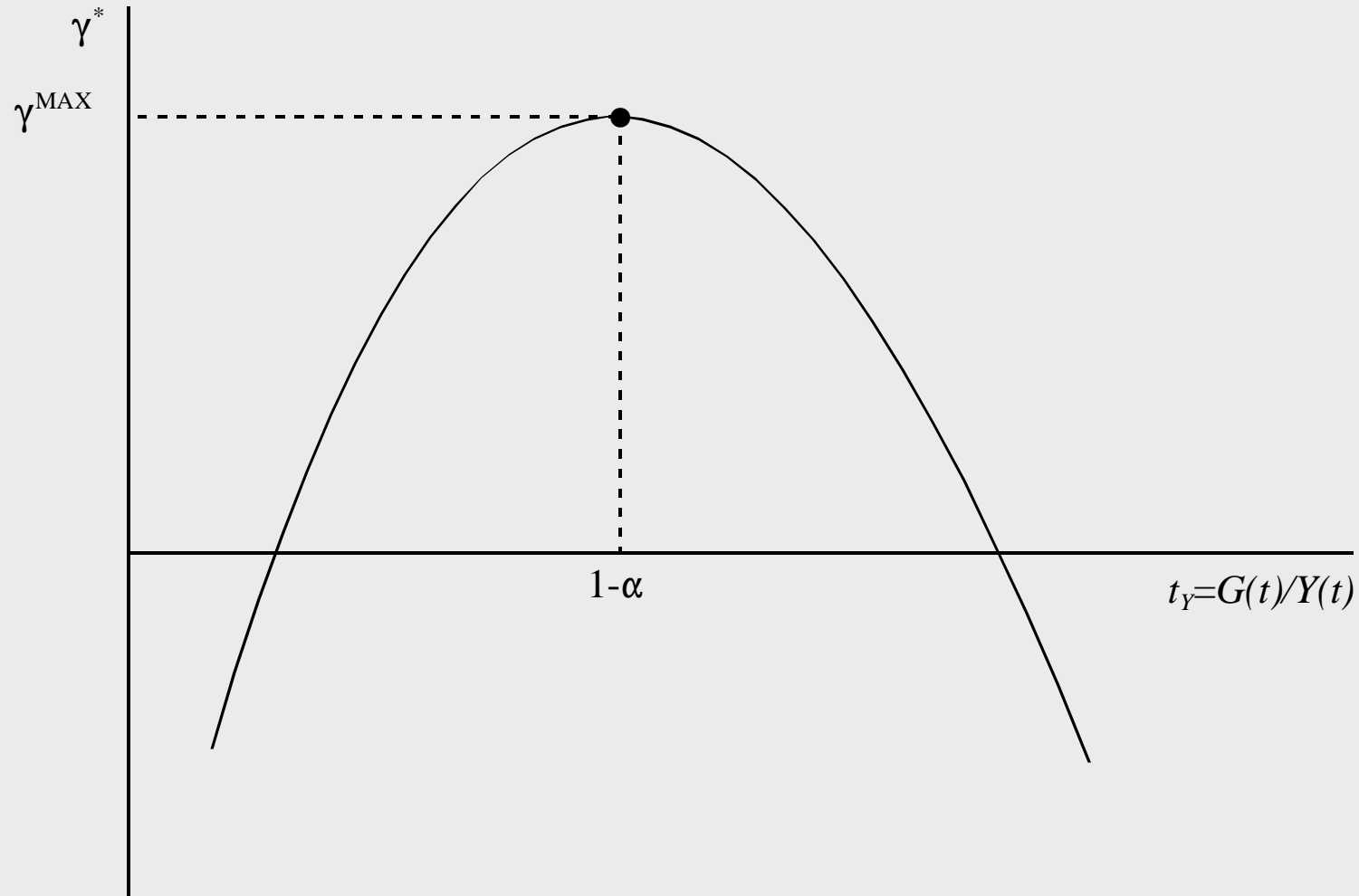


Figure 14.15: Productive Government Spending and Growth

(B) Human capital accumulation as the engine of growth

- *Key idea:* [Uzawa] all technical knowledge is embodied in labour. Educational sector uses labour to augment the state of knowledge in the economy. Uzawa assumed $\dot{A}_L/A_L = \Psi(L_E/L)$, where A_L is labour-augmenting technical progress and L_E is labour used in the educational sector [$\Psi' > 0 > \Psi''$].
- Basic idea was taken over by Lucas (1988). He interprets A_L as human capital (“skills”) and calls it H . Rational agents accumulate human capital by dedicating some of their time on education [hence, the name of the model: “learning or doing”]
- Model elements:
 - human capital accumulation function:

$$\frac{\dot{H}(t)}{H(t)} = \psi_E \left(\frac{L_E(t)}{L(t)} \right)$$

where ψ_E is a positive constant

- lifetime utility of the representative household:

$$\Lambda(0) = \int_0^{\infty} \left[\frac{C(t)^{1-1/\sigma} - 1}{1 - 1/\sigma} \right] e^{-\rho t} dt$$

- time constraint of the household:

$$L_E(t) + L_P(t) = 1$$

where L_P is time spent working [“doing”] rather than going to school

- aggregate production function:

$$Y(t) = F[K(t), N(t)] = N(t)^{1-\alpha_K} K(t)^{\alpha_K}$$

$$N(t) \equiv H(t)L_P(t)$$

- Factors receive their respective marginal products:

$$R^K(t) = F_K[K(t), N(t)] = \alpha_K \left(\frac{K(t)}{N(t)} \right)^{\alpha_K - 1}$$

$$W(t) = H(t)F_N[K(t), N(t)] = (1 - \alpha_K)H(t) \left(\frac{K(t)}{N(t)} \right)^{\alpha_K} \quad (*)$$

Equation (*) shows that the wage rises with the skill level. Hence the household has an incentive to accumulate human capital.

- Household has two stocks it can accumulate, namely physical and human capital. Since the two stocks have the same risk characteristics, they must bring the same yield. A simple arbitrage equation governs the accumulation decision:

$$\rho = \frac{\dot{\mu}_K(t) + D_K(t)}{\mu_K(t)} = \frac{\dot{\mu}_H(t) + D_H(t)}{\mu_H(t)} \quad (\#)$$

- μ_K and μ_H are the imputed shadow prices of, respectively, physical and human capital [if a Martian were to drop a little capital in the household's lap it would experience a lifetime utility gain of μ_K]
- $D_K \equiv \mu_K(F_K - \delta)$ and $D_H \equiv \psi_E \mu_H$ are the “dividends” on, respectively, physical and human capital
- eqn (#) says that the rate of return (RoR) on both assets should be equal. The RoR consists of dividend plus capital gain expressed in terms of initial value.
- In the BGP the household devotes a constant fraction of its time on education. The growth rate along the BGP satisfies [see book for details on the solution method]:

$$\gamma_K = \gamma_Y = \gamma_C = \gamma_H = \sigma(\psi_E - \rho)$$

(C) R&D as the engine of growth

- *Key idea*: purposeful conduct of R&D activities is the source of growth.
 - In the absence of human and physical capital, households can nevertheless save by accumulating *patents*.
 - Patents are blueprint for the production of “slightly unique” products.
 - The patent holder has a little bit of monopoly power which can be exploited.
 - Hence, in this literature we leave the competitive framework and enter the realm of monopolistic competition. [Schumpeterian models of “creative destruction” can be built along the lines of the present model]

Overview of the model

- Three productive sectors
- *Final goods sector* [CRTS, perfectly competitive, external effect “returns to specialization”]: produces a homogenous good using differentiated inputs in the production process.

– **Technology:**

$$Y(t) \equiv N(t)^\eta \left[N(t)^{-1} \int_0^{N(t)} X_j(t)^{1/\mu} dj \right]^\mu, \quad \mu > 1, \quad \eta \geq 1$$

where X_j is intermediate input j , N is the existing number of varieties, and μ and η are parameters. if $\eta > 1$ there are returns to specialization as in Adam Smith’s famous pin factory. If intermediate inputs are more finely differentiated then firms can use a more roundabout production process.

– Pricing decision:

$$P_Y(t) \equiv N(t)^{-\eta} \left[N(t)^{\mu/(1-\mu)} \int_0^{N(t)} P_j(t)^{1/(1-\mu)} dj \right]^{1-\mu}$$

– Derived demand for input j :

$$\frac{X_j(t)}{Y(t)} = N(t)^{(\eta-\mu)/(\mu-1)} \left(\frac{P_j(t)}{P_Y(t)} \right)^{\mu/(1-\mu)}, \quad j \in [0, N(t)]$$

so $\mu/(1 - \mu)$ is the demand elasticity.

- *R&D sector* [CRTS, perfectly competitive, external effect “standing on the shoulder of giants”]: produces blueprints for new intermediate inputs, using labour as an input.

- **Technology:**

$$\dot{N}(t) = (1/k_R)N(t)L_R(t)$$

Labour engaged in the R&D sector becomes more productive as more patents already exist. Today’s engineers “stand on the shoulders of giants.”

- **Pricing decision:**

$$P_N(t) = \frac{k_R W(t)}{N(t)}$$

- *Intermediate goods sector* [many small monopolistically competitive firms]: each firm [patent holder] uses labour to produce its own slightly unique variety of the intermediate input.

- **Technology:**

$$X_j(t) = (1/k_X)L_j(t)$$

constant marginal production costs.

- **Pricing decision:**

$$P_j(t) = \mu W(t)k_X,$$

where μ is the gross monopoly markup.

- Households

- lifetime utility function:

$$\Lambda(0) = \int_0^{\infty} \left[\frac{C(t)^{1-1/\sigma} - 1}{1 - 1/\sigma} \right] e^{-\rho t} dt$$

- household budget identity:

$$P_Y(t)C(t) + P_N(t)\dot{N}(t) = W(t)L + N(t)\bar{\Pi}(t)$$

- optimality conditions:

$$\frac{\dot{C}(t)}{C(t)} = \sigma [r(t) - \rho]$$

$$r(t) = \frac{\bar{\Pi}(t) + \dot{P}_N(t)}{P_N(t)}$$

where $r(t)$ is the rate of return on blueprints.

- *Growth aspects*: there is no transitional dynamics [no capital]. The growth rates are:

$$\gamma_N = \frac{\sigma(\mu - 1)(L/k_R) - \sigma\rho}{\sigma\mu + (1 - \sigma)(\eta - 1)} > 0$$

$$\gamma_C = \gamma_Y = (\eta - 1)\gamma_N$$

The innovation rate increases with the monopoly markup (μ), and the size of the labour force (L), and decreases with the rate of time preference (ρ). Consumption and aggregate output grow only if the returns to specialization are operative (so that $\eta > 1$)

- *Efficiency aspects*: now no longer obvious that market equilibrium is efficient as there are both external effects and non-competitive behaviour. The quick-and-dirty intuition would seem to suggest that there is too little innovation [under-investment in R&D] because the innovator does not capture all the beneficial effects of his act. It turns out that:
 - if $\eta = \mu$ [knife-edge case]: q&d intuition is OK!
 - if $\eta \approx 1$ [weak specialization effect]: there may be too much innovation!

- *Problematic aspect*: the growth rate depends on the scale of the economy (L in this case). Hence, large countries should grow faster than small countries. This is not observed in reality. Jones removes the scale effect by replacing the R&D technology by:

$$\dot{N}(t) = (1/k_R)L_R(t)N(t)^{\phi_1} [\bar{L}_R(t)]^{\phi_2-1},$$

where \bar{L}_R is average R&D labour per R&D firm.

- We had $\phi_1 = 1$ but now assume $0 < \phi_1 < 1$ [the giants don't grow forever]
- We had $\phi_2 = 1$ but now assume $0 < \phi_2 \leq 1$ [duplication externality: individual R&D firms think the production function is linear, but in actuality it features diminishing returns to labour]

- Assuming that the population grows at an exponential rate, n_L , the growth rates are now [see book for details]:

$$\gamma_N = \frac{\phi_2 n_L}{1 - \phi_1}$$

$$\gamma_y = \gamma_Y - n_L = (\mu - 1)\gamma_N$$

$$\gamma_c = \gamma_C - n_L = \gamma_y$$

- We reach the striking conclusion that by eliminating the scale effect we are back in the realm of exogenous growth and the Solow model!!!

Punchlines

- Kaldor's stylized facts about economic growth
- Solow-Swan growth models: long-run growth rate exogenous
- Ramsey: still exogenous growth but refined savings behaviour
- “Endogenous growth models.” Key mechanisms:
 - capital fundamentalism
 - human capital accumulation
 - endogenous technology