Foundations of Modern Macroeconomics

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Chapter 12: Money

Aims of this lecture

- principle functions of money
- modelling money in tractable ways
- optimal quantity of money
- money and the government budget constraint
  - seigniorage
  - inflation tax
- punchlines

Functions of Money

- money "is" what money "does"
  - medium of exchange
  - medium of account
  - store of value
- In **Figure 12.1** there are four agents
  - each agent produces unique non-storable good; consumes all goods
  - central market place at A
  - barter: six relative prices must be determined ($p_{ij} = 1/p_{ji}$)
  - without some kind of friction no use for money

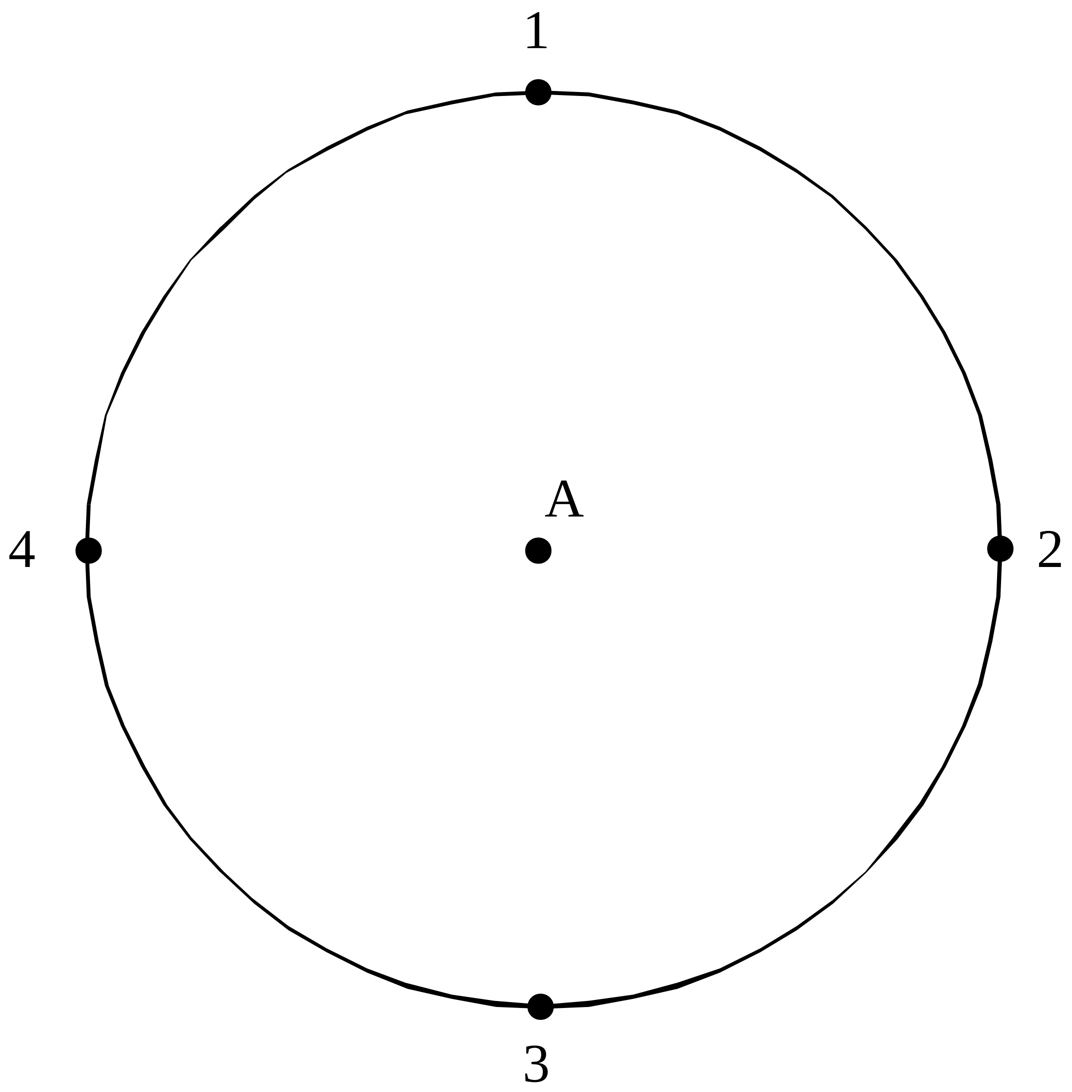

Figure 12.1: The Barter Economy

- Change story a little:
  - agents meet randomly (as in labour market search model of Chapter 9)
  - in barter setting: “double coincidence of wants” not always satisfied
  - if storable “money” is available: trade friction reduced
- **Role 1**. Medium-of-exchange test: Something serves the role of medium of exchange if its existence ensures that agents can attain a higher level of welfare.
- What serves?
  - rare seashells
  - gold, silver, copper
  - cigarettes
  - printed paper with holograms [best; cheap to produce]

- **Role 2**. Medium of account:
  - barter: need to know many different relative prices [for N goods, $N(N-1)/2$ relative prices]
  - money: every price in terms of money [N absolute prices]
- **Role 3**. Store of value:
  - money as savings instrument
  - better yield on bonds, shares, real estate
- Conclusion: medium-of-exchange role **the** distinguishing feature of money

Modelling Money as Medium of Exchange

- money as medium of exchange in general equilibrium model
- individual agent:
  - lives two periods: period 1 (now) and period 2 (the future)
  - initially holds stocks of bonds (B_0) and money (M_0)
  - has real endowment incomes Y_1 and Y_2
  - consumes in the two periods, C_1 and C_2
  - faces goods prices P_1 and P_2
  - faces nominal interest rates on bonds R_0 and R_1

- Budget identities:

$$P_1Y_1 + M_0 + (1+R_0)B_0 = P_1C_1 + M_1 + B_1$$
$$P_2Y_2 + M_1 + (1+R_1)B_1 = P_2C_2 + M_2 + B_2$$

- Terminal condition: $M_2 = B_2 = 0$
- No lending/borrowing constraints
- Consolidated budget constraint:

$$[A \equiv]\ Y_1 + \frac{Y_2}{1+r_1} + \left(\frac{P_0}{P_1}\right) m_0 + (1+r_0)b_0 = C_1 + \frac{C_2}{1+r_1} + \frac{R_1 m_1}{1+R_1} \quad \text{(BC)}$$

  – $m_t \equiv M_t/P_t$ is real money balances
  – $b_t \equiv B_t/P_t$ is real bonds
  – $r_t \equiv \frac{P_t(1+R_t)}{P_{t+1}} - 1$ is real interest rate

$\longrightarrow$ nominal interest rate features in term involving m_1!

- Lifetime utility function:

$$V = U(C_1) + \left(\frac{1}{1+\rho}\right) U(C_2) \qquad \text{(UF)}$$

  - $\rho > 0$ is rate of pure time preference
  - $U(\cdot)$ is felicity function: $U''(\cdot) < 0 < U'(\cdot)$

- Agent chooses C_1, C_2, and m_1
  - to maximize (UF) subject to (BC) and the non-negativity constraint:

$$m_1 \geq 0 \qquad \text{(NNC)}$$

  - taking as given: P_t, Y_t, r_t, R_t, m_0, and b_0

- Lagrangian:

$$\mathcal{L} \equiv U(C_1) + \left(\frac{1}{1+\rho}\right) U(C_2) + \lambda \left[A - C_1 - \frac{C_2}{1+r_1} - \frac{R_1 m_1}{1+R_1}\right]$$

- First-order (Kuhn-Tucker) conditions:

$$\frac{\partial \mathcal{L}}{\partial C_1} = U'(C_1) - \lambda = 0 \quad \text{(F1)}$$

$$\frac{\partial \mathcal{L}}{\partial C_2} = \left(\frac{1}{1+\rho}\right) U'(C_2) - \frac{\lambda}{1+r_1} = 0 \quad \text{(F2)}$$

$$\frac{\partial \mathcal{L}}{\partial m_1} \equiv \lambda \left(\frac{-R_t}{1+R_1}\right) \leq 0, \quad m_1 \geq 0, \quad m_1 \frac{\partial \mathcal{L}}{\partial m_1} = 0 \quad \text{(F3)}$$

  - (F1) and (F2). Combined they yield the consumption Euler equation
  - (F3) is new element

- Optimal money holding:
  - (normal) **case 1**: positive nominal interest rate, $R_1 > 0$
    - $\lambda \left(\frac{-R_t}{1+R_1} \right) < 0$ (strict inequality) so that $\cdots$
    - $m_1 \frac{\partial \mathcal{L}}{\partial m_1} = 0$ (complementary slackness condition) implies $\cdots$
    - $m_1 = 0$. Optimal to hold no money at all! Money serves no useful purpose to agent.
  - (strange) **case 2**: negative nominal interest rate, $R_1 < 0$
    - $\lambda \left(\frac{-R_t}{1+R_1} \right) > 0$ so optimal to hold as much money as possible, i.e. $\cdots$
    - $m_1 \longrightarrow +\infty$
  - focus on normal case
- How can we change model to get useful role for money?

Shopping Cost Model of Money

- leisure is valued by households
- money saves leisure time (“shopping around”)
- Lifetime utility now:

$$V = U(C_1, 1 - \bar{N} - S_1) + \left(\frac{1}{1+\rho}\right) U(C_2, 1 - \bar{N} - S_2) \qquad \text{(UF)}$$

  - $L_t \equiv 1 - \bar{N} - S_t$ is leisure in period t
  - $\bar{N}$ is exogenous labour supply
  - S_t is time spent shopping
  - $U_L \equiv \frac{\partial U}{\partial L_t} > 0$ is marginal utility of leisure [$U_{LL} \equiv \frac{\partial^2 U}{\partial L_t^2} < 0$].

- Budget constraint:

$$[A \equiv] \ \frac{W_1 \bar{N}}{P_1} + \frac{W_2 \bar{N}}{(1+r_1)\, P_2} + \left(\frac{P_0}{P_1}\right) m_0 + (1+r_0) b_0 = C_1 + \frac{C_2}{1+r_1} + \frac{R_1 m_1}{1+R_1} \quad \text{(BC)}$$

where $Y_t \equiv W_t \bar{N} / P_t$ has been used.

- Shopping technology:

$$1 - \bar{N} - S_t = \psi(\underset{+}{m_{t-1}}, \underset{-}{C_t}) \quad \text{(ST)}$$

  – money balances reduce shopping time $[\psi_m \equiv \frac{\partial \psi}{\partial m_{t-1}} > 0]$
  – consumption costs shopping time $[\psi_C \equiv \frac{\partial \psi}{\partial C_t} < 0]$
  – further assumptions: $\psi_{mm} < 0$, $\psi_{CC} > 0$, and $0 < \psi(m_{t-1}, \infty) < \psi m_{t-1}, 0) < 1 - \bar{N}$
  – **Figures A and B** shows various aspects of the $\psi(\cdot)$ function

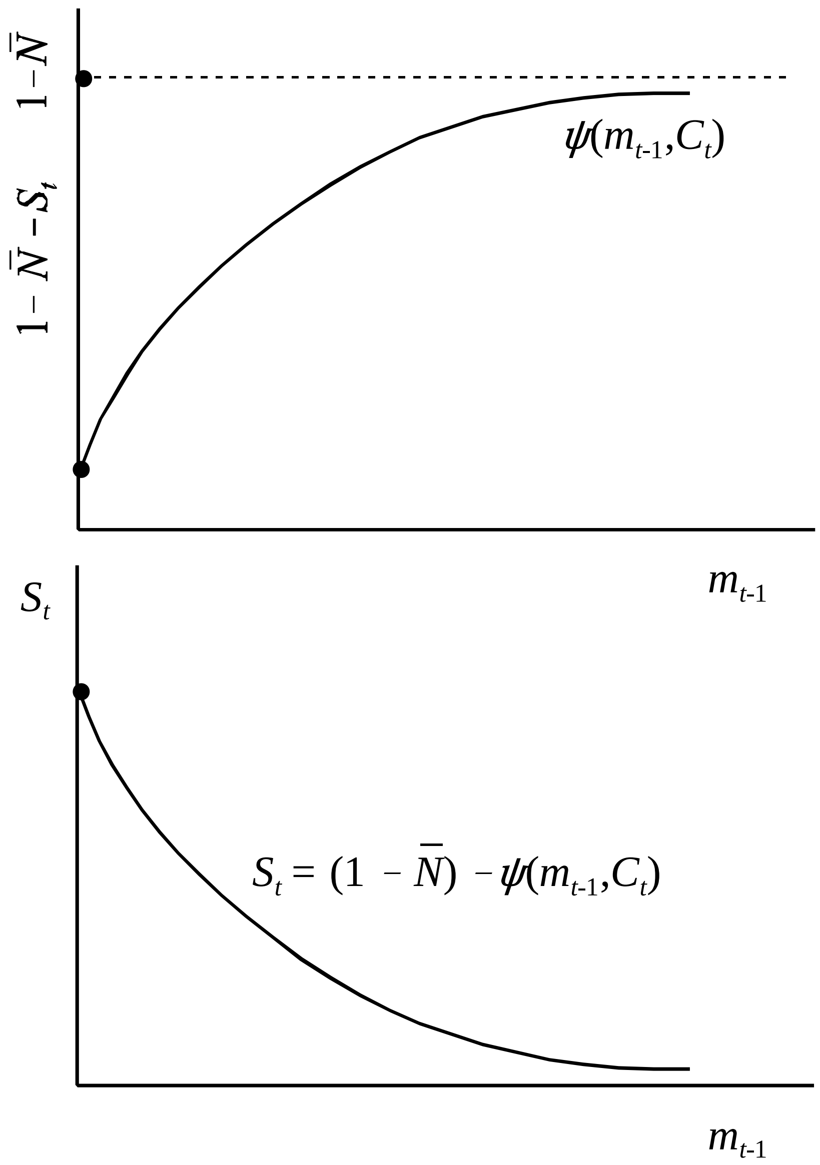

Figure A: Shopping Cost Function and Money Balances

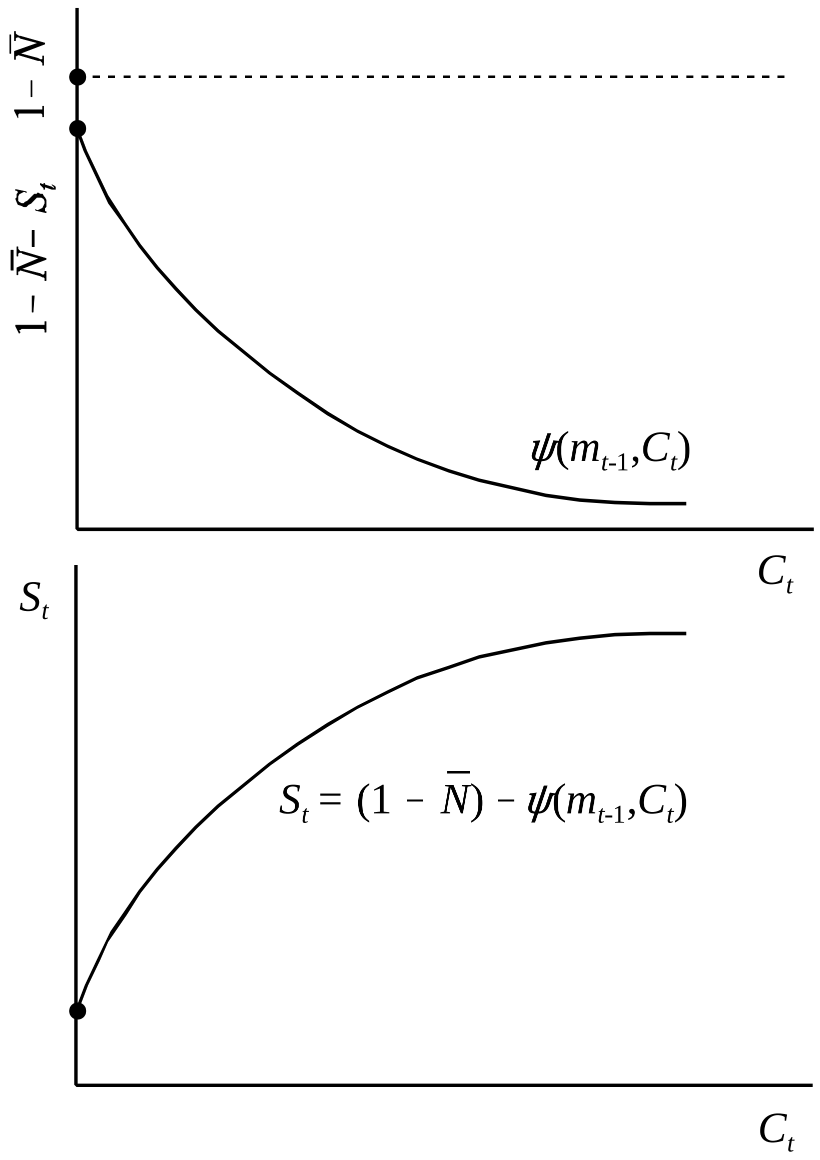

Figure B: Shopping Cost Function and Consumption

- The household chooses C_t, S_t (for $t = 1, 2$), and m_1 (m_0 being predetermined) in order to maximize (UF) subject to (BC), (ST), and the non-negativity constraint on money balances ($m_1 \geq 0$).

- The Lagrangian expression is:

$$\mathcal{L} \equiv U(C_1, 1 - \bar{N} - S_1) + \left(\frac{1}{1+\rho}\right) U(C_2, 1 - \bar{N} - S_2)$$
$$+ \lambda \left[A - C_1 - \frac{C_2}{1+r_1} - \frac{R_1 m_1}{1+R_1}\right] - \sum_{t=1}^{2} \lambda_t \left[1 - \bar{N} - S_t - \psi(m_{t-1}, C_t)\right]$$

where λ_t are the Lagrangian multipliers associated with the shopping technology in the two periods.

- First-order conditions are:

$$\frac{\partial \mathcal{L}}{\partial C_1} = U_C(C_1, 1-\bar{N}-S_1) - \lambda + \lambda_1 \psi_C(m_0, C_1) = 0 \tag{F1}$$

$$\frac{\partial \mathcal{L}}{\partial C_2} = \left(\frac{1}{1+\rho}\right) U_C(C_2, 1-\bar{N}-S_2) - \frac{\lambda}{1+r_1} + \lambda_2 \psi_C(m_1, C_2) = 0 \tag{F2}$$

$$\frac{\partial \mathcal{L}}{\partial S_1} = -U_L(C_1, 1-\bar{N}-S_1) + \lambda_1 = 0 \tag{F3}$$

$$\frac{\partial \mathcal{L}}{\partial S_2} = -\left(\frac{1}{1+\rho}\right) U_L(C_2, 1-\bar{N}-S_2) + \lambda_2 = 0 \tag{F4}$$

$$\frac{\partial \mathcal{L}}{\partial m_1} \equiv \lambda \left(\frac{-R_1}{1+R_1}\right) + \lambda_2 \psi_m(m_1, C_2) \leq 0, \quad m_1 \geq 0, \quad m_1 \frac{\partial \mathcal{L}}{\partial m_1} = 0 \tag{F5}$$

  - $U_C(\cdot)$ and $U_L(\cdot)$ denote the marginal utility of consumption and leisure, respectively.

- Note (F5): using (F4) we get:

$$\lambda_2 \psi_m = \frac{U_L(C_2, 1 - \bar{N} - S_2)\psi_m(m_1, C_2)}{1+\rho}$$

which representing marginal utility of money balances.

* If $\lambda_2 \psi_m$ is low, still corner solution ($m_1 = 0$)
* If $\lambda_2 \psi_m$ is high, money will be held ($m_1 > 0$)

- Focus on second case. The following optimality conditions are obtained:

$$\lambda = U_C(C_1, 1-\bar{N}-S_1) + U_L(C_1, 1-\bar{N}-S_1)\psi_C(m_0, C_1) \quad \text{(a)}$$

$$= \left(\frac{1+r_1}{1+\rho}\right)\left[U_C(C_2, 1-\bar{N}-S_2) + U_L(C_2, 1-\bar{N}-S_2)\psi_C(m_1, C_2)\right] \quad \text{(b)}$$

$$= \frac{U_L(C_2, 1-\bar{N}-S_2)\psi_m(m_1, C_2)(1+R_1)}{(1+\rho)R_1} \quad \text{(c)}$$

(a) equate λ (marginal utility of wealth) to *net* marginal utility of consumption [direct marginal utility of consumption minus the disutility caused by the additional shopping costs which must be incurred]

(b) same as (a) for future period

(c) equate λ to the marginal utility of money balances ($U_L(\cdot)\psi_m(\cdot)$)

Money in Utility Function

- Shopping Cost Model (SCM) equivalent to Putting Money in Utility Function (MIU)
- Substitute (ST) into felicity:

$$\begin{aligned} U(C_t, 1-\bar{N}-S_t) &= U(C_t, \psi(\underset{+}{m_{t-1}}, \underset{-}{C_t})) \\ &\equiv \bar{U}\left(\underset{+}{C_t}, \underset{+}{m_{t-1}}\right) \end{aligned}$$

- Alternative approach: *Cash-in-Advance Constraint* (Clower constraint): money buys goods, and goods buy money, but goods do not buy goods (no barter):

$$P_t C_t \leq M_{t-1} \quad \Leftrightarrow \quad C_t \leq (P_{t-1}/P_t) m_{t-1} \qquad \text{(CiA)}$$

- How does this approach work?
  - assume (CiA) holds strictly in first period: m_0 is predetermined, so the same then holds for consumption in the first period ($C_1 = P_0 m_0 / P_1$)

– household chooses C_2 and m_1 in order to maximize

$$V = U(C_1) + \left(\frac{1}{1+\rho}\right) U(C_2)$$

subject to

$$[A \equiv] \; Y_1 + \frac{Y_2}{1+r_1} + \left(\frac{P_0}{P_1}\right) m_0 + (1+r_0)b_0 = C_1 + \frac{C_2}{1+r_1} + \frac{R_1 m_1}{1+R_1}$$

and (CiA). The Lagrangian is:

$$\mathcal{L} \equiv U((P_0/P_1)m_0) + \left(\frac{1}{1+\rho}\right) U(C_2) + \lambda_2 \left[(P_1/P_2)m_1 - C_2\right]$$
$$+ \lambda \left[A - (P_0/P_1)m_0 - \frac{C_2}{1+r_1} - \frac{R_1 m_1}{1+R_1}\right]$$

where λ_2 is the Lagrangian multiplier associated with the Clower constraint.

– first-order conditions:

$$\frac{\partial \mathcal{L}}{\partial C_2} \equiv \left(\frac{1}{1+\rho}\right) U'(C_2) - \frac{\lambda}{1+r_1} - \lambda_2 \leq 0, \quad C_2 \geq 0, \quad C_2 \frac{\partial \mathcal{L}}{\partial C_2} = 0 \tag{F1}$$

$$\frac{\partial \mathcal{L}}{\partial m_1} \equiv -\lambda \left(\frac{R_1}{1+R_1}\right) + \lambda_2 \frac{P_1}{P_2} \leq 0, \quad m_1 \geq 0, \quad m_1 \frac{\partial \mathcal{L}}{\partial m_1} = 0 \tag{F2}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_2} \equiv (P_1/P_2) m_1 - C_2 \geq 0, \quad \lambda_2 \geq 0, \quad \lambda_2 \frac{\partial \mathcal{L}}{\partial \lambda_2} = 0 \tag{F3}$$

– since $\lambda > 0$, we find from (F1) that the marginal utility of consumption is bounded.

– since $\lim_{C_t \to \infty} U'(C_t) = 0$ by assumption, this implies that $C_2 > 0$ and (by the first inequality in (F3)) that $m_1 > 0$

– hence, money is essential, not because it is valued intrinsically, but rather because households wish to consume in the second period

– no *excess* cash balances will be held. Since $m_1 > 0$, the first expression in (F2) holds with equality, which ensures that the shadow price of cash balances is strictly positive:

$$\lambda_2 = \lambda \frac{P_2}{P_1} \left(\frac{R_1}{1+R_1} \right) > 0.$$

This implies that the first expression in (F3) holds with an equality, i.e. $C_2 = (P_1/P_2)m_1$ (generalizes to multi-period setting)

- CiA approach also equivalent to a utility-of-money approach:
  - Clower constraint always holds with equality ($C_t = (P_{t-1}/P_t)m_{t-1}$), the same results are obtained if the indirect felicity function $\tilde{U}(C_t, m_{t-1}) \equiv \min[C_t, m_{t-1}]$ is maximized subject to the budget constraint only
  - in this indirect felicity function the substitution elasticity between consumption and money balances is zero (key difference with shopping model and the Baumol-Tobin model).

Money as Store of Value

- Bewley: assume money is only asset available to household
- household wants to hold asset to transfer resources through time (money as store of value)
- household budget identities become (with $B_t = 0$):

$$Y_1 + \frac{m_0}{1+\pi_0} = C_1 + m_1 \tag{BI1}$$

$$Y_2 + \frac{m_1}{1+\pi_1} = C_2, \quad m_1 \geq 0 \tag{BI2}$$

where $\pi_t \equiv P_{t+1}/P_t - 1$ is the inflation rate

- How does Bewley model work?

– agent chooses (C_1, C_2, m_1) to maximize lifetime utility

$$V = U(C_1) + \left(\frac{1}{1+\rho}\right) U(C_2)$$

subject to (BI1)-(BI2).

– Lagrangian:

$$\mathcal{L} \equiv U(C_1) + \left(\frac{1}{1+\rho}\right) U(C_2) + \lambda_1 \left[Y_1 + \frac{m_0}{1+\pi_0} - C_1 - m_1\right] + \lambda_2 \left[Y_2 + \frac{m_1}{1+\pi_1} - C_2\right]$$

– first-order conditions:

$$\frac{\partial \mathcal{L}}{\partial C_1} = U'(C_1) - \lambda_1 = 0 \tag{F1}$$

$$\frac{\partial \mathcal{L}}{\partial C_2} = \left(\frac{1}{1+\rho}\right) U'(C_2) - \lambda_2 = 0 \tag{F2}$$

$$\frac{\partial \mathcal{L}}{\partial m_1} \equiv -\lambda_1 + \frac{\lambda_2}{1+\pi_1} \leq 0, \quad m_1 \geq 0, \quad m_1 \frac{\partial \mathcal{L}}{\partial m_1} = 0 \tag{F3}$$

– combining (F1)-(F3) yields:

$$\frac{\partial \mathcal{L}}{\partial m_1} \equiv \frac{U'(C_2)}{1+\rho} \left[\frac{1}{1+\pi_1} - \frac{(1+\rho)U'(C_1)}{U'(C_2)}\right] \leq 0,$$
$$m_1 \geq 0, \quad m_1 \frac{\partial \mathcal{L}}{\partial m_1} = 0. \tag{F3$'$}$$

– intuition behind (F3′) can be illustrated with the aid of **Figure 12.2**.

  * consolidated budget equation is AB (slope $dC_2/dC_1 = -1/(1+\pi_1)$)
  * indifference curve, V_0, has a slope of $dC_2/dC_1 = -(1+\rho)U'(C_1)/U'(C_2)$ and a tangency at point E^C.
  * *case 1*: income endowment point at E$_0^Y$. Money is of no use as a store of value to the agent. In economic terms, the agent would like to be a net supplier of money in order to attain the consumption point E^C but this is impossible. Graphically, the indifference curve through E$_0^Y$ (the dashed curve) is steeper than the budget line, the choice set is only AE$_0^Y$D, and the best the agent can do is to consume his endowments in the two periods. In mathematical terms, the slope configuration implies that $\partial\mathcal{L}/\partial m_1 < 0$ (lifetime utility rises by supplying money) and complementary slackness results in $m_1 = 0$.

∗ *case 2*: income endowment point at E_1^Y. Agent saves in the first period by holding money and the first expression in (F3′) holds with equality so that the Euler equation becomes:

$$\frac{U'(C_2)}{U'(C_1)} = (1+\rho)(1+\pi_1). \tag{1}$$

money held because it provides a means by which intertemporal consumption smoothing can be achieved

– Generalization of Bewley approach: indivisible bonds. Poor agents save with money, rich agents use bonds.

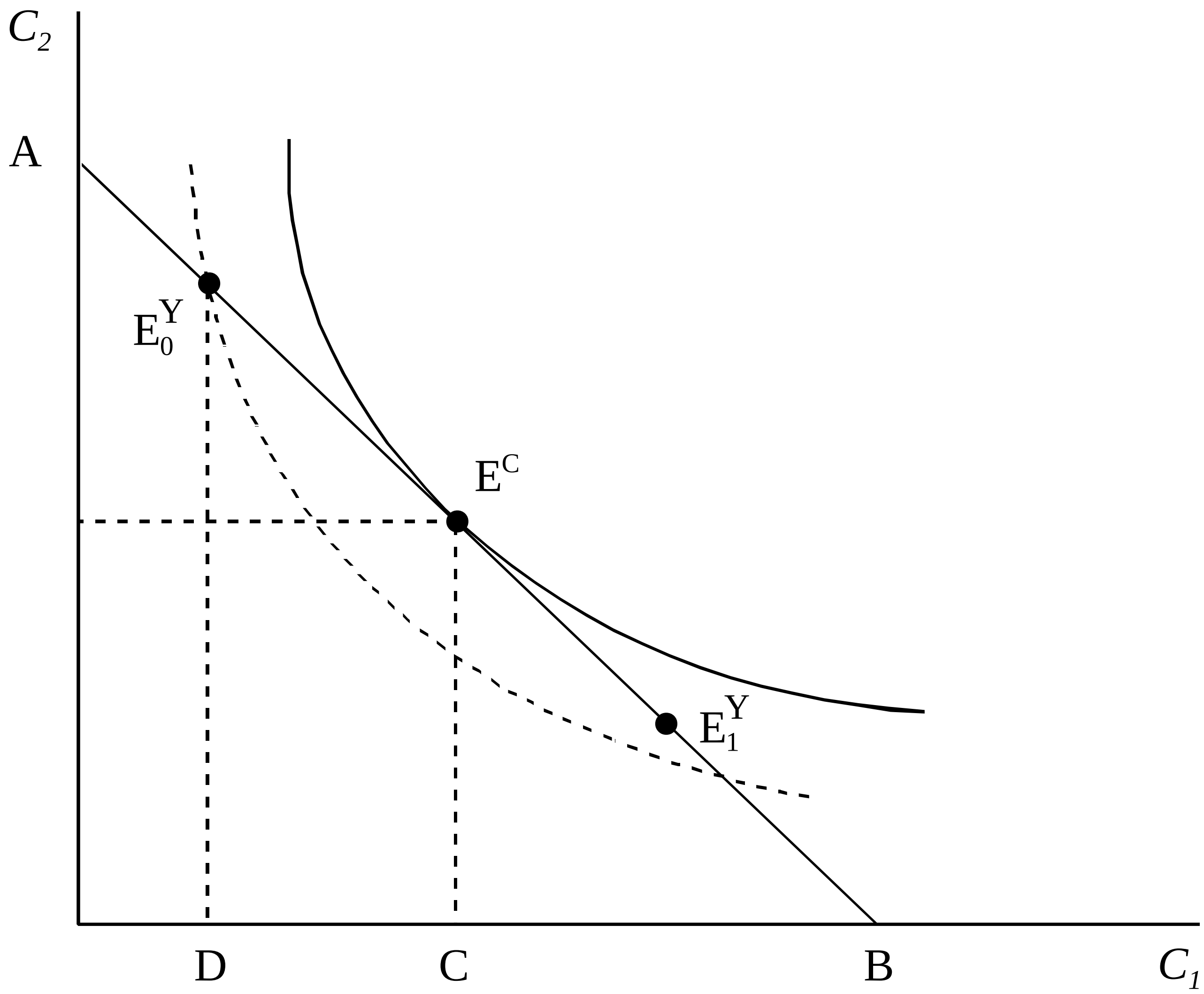

Figure 12.2: Money as a Store of Value

Overlapping Generations and Money

- Samuelson (1958): intergenerational friction central (overlapping generations)
- Model elements:
  - in period t there are $N/2$ young agents and $N/2$ old agents (normalize $N = 1$)
  - agents live two periods
  - endowment income Y (when young) and 0 (when old)
  - endowment potantially storable: storing Y units yields in next period $Y/\left(1+\delta\right)$ of units ($\delta > -1$)
  - cases: (a) $\delta \to \infty$ (no storage); (b) $\infty \gg \delta > 0$ (some spoilage); (c) $\delta = 0$ (no spoilage); (d) $0 < \delta < 1$ (positive productivity)
  - savings instruments: store part of endowment income or trade for fiat (= intrinsically useless) money

- Budget identity young agent during youth:

$$Y - C_t^Y = K_t + \frac{M_t}{P_t} \qquad \text{(BIyy)}$$

  - C_t^Y is consumption when young
  - K_t is stored output
  - M_t is end-of-period stock of fiat money
  - P_t is money price of output

- Budget identity old agent:

$$C_t^O = \frac{K_{t-1}}{1+\delta} + T_t + \frac{P_{t-1}}{P_t} m_{t-1} \qquad \text{(BIo)}$$

  - C_t^O is consumption when old
  - $m_{t-1} \equiv M_{t-1}/P_{t-1}$ is real money balances
  - T_t is government transfers

- Budget identity young agent when old:

$$C_{t+1}^{O} = \frac{K_t}{1+\delta} + T_{t+1} + \frac{P_t}{P_{t+1}} m_t \qquad \text{(BIyo)}$$

- Lifetime utility function of the young agent in period t:

$$V_t^{Y} = U(C_t^{Y}) + \left(\frac{1}{1+\rho}\right) U(C_{t+1}^{O}), \quad \rho > 0 \qquad \text{(UF)}$$

- Agent chooses C_t^Y, C_{t+1}^O, K_t, and m_t to maximize (UF) subject to (BIyy) and (BIyo) and non-negativity conditions on money holdings and stored output ($M_t \geq 0$ and $K_t \geq 0$, respectively)

- Lagrangian:

$$\mathcal{L} \equiv U(C_t^Y) + \left(\frac{1}{1+\rho}\right) U(C_{t+1}^O) + \lambda_{1,t}\left[Y - C_t^Y - K_t - m_t\right] + \lambda_{2,t}\left[\frac{K_t}{1+\delta} + T_{t+1} + \frac{P_t}{P_{t+1}} m_t - C_{t+1}^O\right]$$

- First-order conditions:

$$\frac{\partial \mathcal{L}}{\partial C_t^Y} = U'(C_t^Y) - \lambda_{1,t} = 0 \quad \text{(F1)}$$

$$\frac{\partial \mathcal{L}}{\partial C_{t+1}^O} = \left(\frac{1}{1+\rho}\right) U'(C_{t+1}^O) - \lambda_{2,t} = 0 \quad \text{(F2)}$$

$$\frac{\partial \mathcal{L}}{\partial m_t} \equiv -\lambda_{1,t} + \lambda_{2,t}/(1+\pi_t) \leq 0, \quad m_t \geq 0, \quad m_t \frac{\partial \mathcal{L}}{\partial m_t} = 0 \quad \text{(F3)}$$

$$\frac{\partial \mathcal{L}}{\partial K_t} \equiv -\lambda_{1,t} + \lambda_{2,t}/(1+\delta) \leq 0, \quad K_t \geq 0, \quad K_t \frac{\partial \mathcal{L}}{\partial K_t} = 0 \quad \text{(F4)}$$

– $\pi_t \equiv P_{t+1}/P_t - 1$ is the inflation rate

– in (F1)-(F2) we have incorporated interior solution (due to properties $U(\cdot)$)

– provided $\pi_t \neq \delta$, the young agent will choose a single type of asset to serve as a store of value, depending on which one has the highest yield (corner solution)

  * case 1: if $\pi_t < \delta$ then $K_t = 0$ and $m_t > 0$
  * case 2: if $\pi_t > \delta$ then $K_t > 0$ and $m_t = 0$
  * see **Figure 12.3** for case 1.

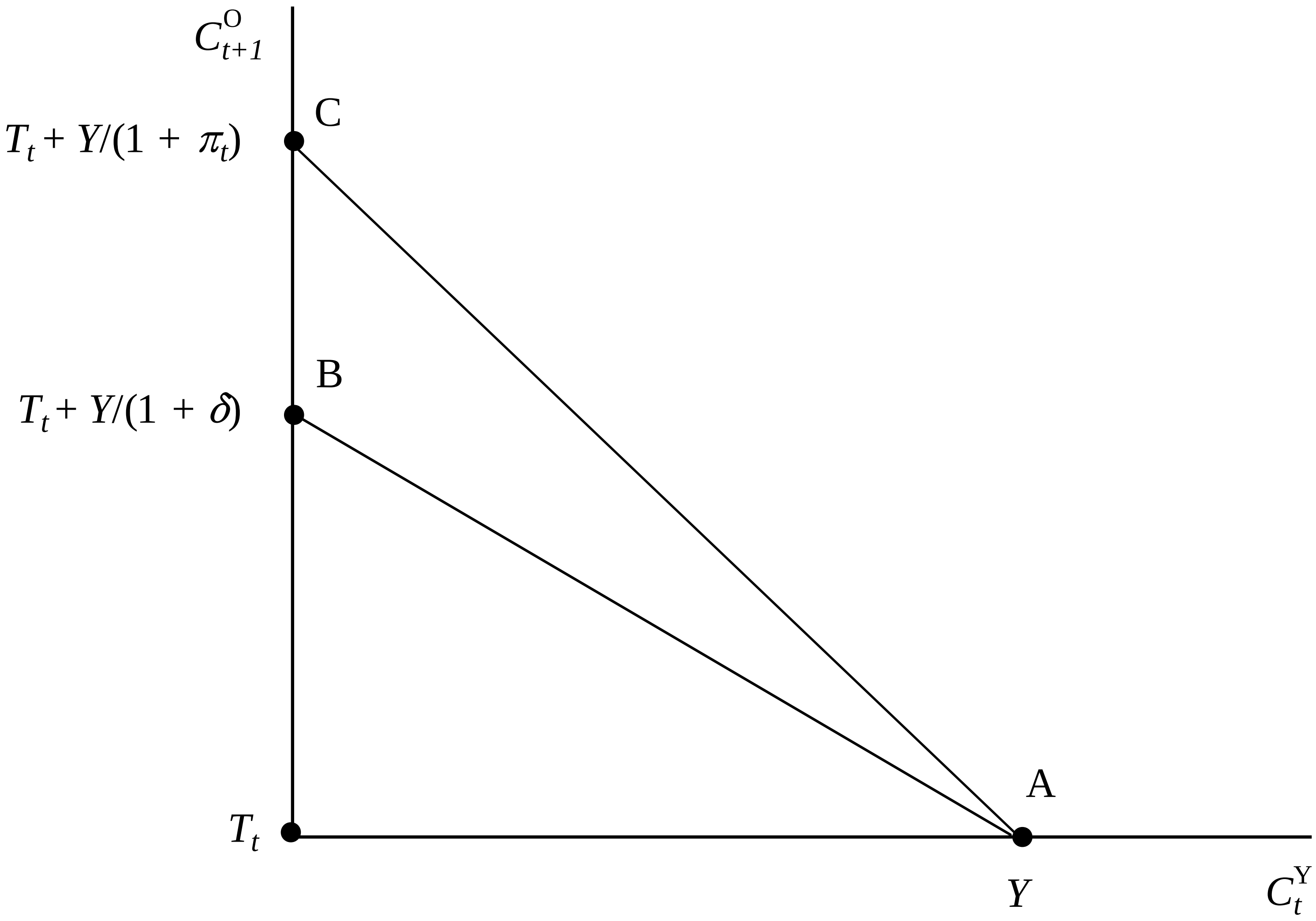

Figure 12.3: Choice Set with Storage and Money

- Old agents maximize remaining lifetime utility, $U(C_t^O)$ subject to the budget identity (BIo)., i.e. they choose:

$$C_t^O = \frac{K_{t-1}}{1+\delta} + T_t + \frac{P_{t-1}}{P_t} m_{t-1}$$

- Money supply rule:

$$M_t = (1+\mu) M_{t-1}, \qquad \text{(MSR)}$$

with $\mu > -1$ representing the constant rate of nominal money growth.

- Government budget restriction is $M_t - M_{t-1} = P_t T_t$ so that the transfer in period $t+1$ is:

$$T_{t+1} = \frac{M_{t+1} - M_t}{P_{t+1}} = \frac{\mu M_t}{P_t} \frac{P_t}{P_{t+1}} = \frac{\mu m_t}{1+\pi_t}. \qquad \text{(TR)}$$

- Equilibrium in the model requires both money and goods markets to be in equilibrium in all periods. By Walras' Law, however, the goods market is in equilibrium provided the money market is, i.e. provided demand and supply are equated in the money market:

$$m(T_{t+1}, \pi_t) = \frac{M_t}{P_t} \qquad \text{(MME)}$$

  - $m(\cdot)$ is the demand for money by the young in period t (implied by FONCs (F1)-(F4))
  - if felicity is logarithmic, $U(x) \equiv \log x$, we have:

$$m_t = \begin{cases} m(T_{t+1}, \pi_t) = \frac{Y - (1+\rho)(1+\pi_t)T_{t+1}}{2+\rho} & \text{if } \pi_t < \delta \\ 0 & \text{if } \pi_t > \delta. \end{cases} \qquad \text{(MD)}$$

- Model solution:
  - model consists of (TR) and (MME)
  - equilibrium concept: find sequence of price levels (P_t, P_{t+1}, P_{t+2}, etc.) such that the equilibrium conditions (TR) and (MME) holds for all periods
  - for logarithmic felicity function we substitute (TR) into the first line of (MD) and solve for the equilibrium level of real money balances:

$$
\begin{aligned}
m_t &= \frac{Y - (1+\rho)(1+\pi_t)\frac{\mu m_t}{1+\pi_t}}{2+\rho} \\
&= \frac{Y}{2+\rho+\mu(1+\rho)} \qquad \Leftrightarrow \\
P_t &= \left(\frac{2+\rho+\mu(1+\rho)}{Y}\right) M_t \qquad \text{(ME)}
\end{aligned}
$$

- Remarks about monetary equilibrium expression (ME):
  - only valid if $\pi_t < \delta$
  - real money balances are constant so that the price level is proportional to the nominal money supply and the inflation rate is equal to the rate of growth of the money supply ($\pi_t = \mu$)
  - so provided the money growth rate μ is less than the depreciation rate δ, intrinsically useless fiat money will be held by agents in a general equilibrium setting. Intuitively, money is the best available financial instrument to serve as a store of value as it outperforms the storage technology in that case.

– if the storage technology yields net productivity ($\delta < 0$) then the monetary equilibrium will only obtain if the money growth rate is negative ($\mu < \delta < 0$), i.e. if there is a constant rate of *deflation* of the price level. In contrast, if goods are perishable ($\delta \to \infty$) then the monetary equilibrium will always hold since money represents the only store of value in that case

- If $\mu > \delta$ then the storage technology outperforms money as a store of value and consequently the demand for real money balances will be zero (see the second line in (MD)). In this non-monetary equilibrium fiat money exists ($M_t > 0$) and is distributed to agents in the economy, but is not used by these agents as a store of value. Money is valueless and the nominal price level is infinite, i.e. $1/P_t = 0$ for all t.

Uncertainty and Money Demand

- friction: bonds have higher yield but are more risky than money
- capital risk: yield on investment uncertain
- abstract from income risk
- assume that money is a perfectly safe asset. "Yield" on money is defined as:

$$r_t^M \equiv \frac{1}{1+\pi_t} - 1$$

  - $\pi_t \equiv \frac{P_{t+1}}{P_t} - 1$ is the inflation rate (assumed known in advance)
  - e.g., if $\pi_t = 0$ then $r_t^M = 0$

- household budget identities in real terms:

$$Y_1^* + \left(1 + r_0^M\right) m_0 + (1 + r_0) b_0 = C_1 + m_1 + b_1 \qquad \text{(BI1)}$$

$$(1 + r_1^M) m_1 + (1 + \tilde{r}_1) b_1 = \tilde{C}_2 \qquad \text{(BI2)}$$

  - already incorporated $m_2 = b_2 = 0$
  - $Y_1^* \equiv Y_1 + Y_2/(1 + r_1^M)$ is present value of present and future endowment income, capitalized at the risk-free rate
  - the tilde above r_1 denotes that the yield on bonds is a *stochastic variable*, the realization of which (r_1) will only be known to the agent at the end of the first period, i.e. after consumption and savings plans have been made (C_1, m_1, b_1)
  - consumption in the second period is also a stochastic variable, i.e. $\tilde{C}_2$ appears in (BI2)

- investing in bonds represents a *temporal uncertain prospect*, i.e. time must elapse before the uncertainty is removed.

- rewrite the budget identities (BI1)-(BI2) as:

$$Y_1^* + A_1 = C_1 + \frac{\tilde{A}_2}{(1+r_1^M)\omega_1 + (1+\tilde{r}_1)(1-\omega_1)} \quad \text{(BI1}'\text{)}$$

$$\tilde{A}_2 = \tilde{C}_2 \quad \text{(BI2}'\text{)}$$

  - $A_t \equiv (1+r_{t-1}^M)m_{t-1} + (1+r_{t-1})b_{t-1}$ represents total assets inclusive of interest receipts available at the beginning of period t
  - $\omega_1 \equiv m_1/(m_1+b_1)$ represents the portfolio share of money

- In the first period the agent chooses consumption C_1 and the portfolio share ω_1, not knowing how high will be the value of his assets at the beginning of the second period because the yield on the risky investment is uncertain.

- Since $\tilde{r}_1$ (and thus $\tilde{C}_2$ and $\tilde{A}_2$) is stochastic, the agent must somehow evaluate the utility value of the uncertain prospect $\tilde{C}_2$. Utility is stochastic:

$$\tilde{V} = U(C_1) + \left(\frac{1}{1+\rho}\right) U(\tilde{C}_2) \qquad \text{(UF)}$$

- The theory of *expected utility* (Von Neumann and Morgenstern) postulates that agent will evaluate the expected utility in order to make his optimal decision, i.e. instead of using $\tilde{V}$ in (UF) as the welfare indicator the agent uses the expected value of $\tilde{V}$, denoted by $E(\tilde{V})$:

$$E(\tilde{V}) \equiv \int_{-1}^{\infty} \left[U(C_1) + \left(\frac{1}{1+\rho}\right) U(\tilde{C}_2) \right] f(\tilde{r}_1) d\tilde{r}_1$$
$$= U(C_1) + \left(\frac{1}{1+\rho}\right) \int_{-1}^{\infty} U\left[S_1 \left[(1+r_1^M)\omega_1 + (1+\tilde{r}_1)(1-\omega_1) \right] \right] f(\tilde{r}_1) d\tilde{r}_1$$

- $S_1 \equiv A_1 + Y_1^* - C_1$
- density function is given by $f(\tilde{r}_1)$
- assume that $\tilde{r}_1$ is restricted to lie in the interval $[-1, \infty)$
- assume that the parameters of the model and the stochastic process for $\tilde{r}_1$ are such that we can ignore the non-negativity constraint for money holdings. Since there is no sign restriction on bond holdings, this means that we only need to study an internal optimum.

- The agent chooses C_1 (and thus S_1) and ω_1 in order to maximize his expected utility, $E(\tilde{V})$
  - FONC for ω_1:

$$
\begin{aligned}
0 &= \frac{1}{1+\rho}\int_{-1}^{\infty} U'(\tilde{C}_2)(A_1+Y_1^*-C_1)(r_1^M-\tilde{r}_1)f(\tilde{r}_1)d\tilde{r}_1 \Leftrightarrow \\
0 &= E\left[U'(\tilde{C}_2)(A_1+Y_1^*-C_1)(r_1^M-\tilde{r}_1)\right] \qquad \text{(F1)}
\end{aligned}
$$

    - $*$ determines optimal composition of the investment portfolio in terms of money (with certain yield r_1^M) and bonds (carrying stochastic yield $\tilde{r}_1$)
    - $*$ expected marginal utility per dollar invested should be equated for the two assets

– FONC for C_1:

$$U'(C_1) = \frac{1}{1+\rho}\int_{-1}^{\infty} U'(\tilde{C}_2)\left[(1+r_1^M)\omega_1 + (1+\tilde{r}_1)(1-\omega_1)\right] f(\tilde{r}_1)d\tilde{r}_1 \Leftrightarrow$$

$$U'(C_1) = \frac{1}{1+\rho}E\left(U'(\tilde{C}_2)\left[(1+r_1^M) + (1+\tilde{r}_1)(1-\omega_1)\right]\right) \quad \text{(F2)}$$

∗ the stochastic Euler equation, determining the optimal time profile of consumption, generalized for the existence of capital uncertainty

- Assume that the agent has an iso-elastic felicity function, $U(C_t)$:

$$U(C_t) = \begin{cases} (1/\gamma)\left[C_t^{\gamma} - 1\right] & \text{if } \gamma \neq 0 \\ \log C_t & \text{if } \gamma = 0, \end{cases} \quad \text{(FF)}$$

– $\gamma < 1$ represents the *degree of risk aversion*

– The FONC for ω_1 (F1) becomes:

$$
\begin{aligned}
0 &= E\left[\tilde{C}_2^{\gamma-1}(A_1+Y_1^*-C_1)(r_1^M-\tilde{r}_1)\right] \\
&= E\left[(A_1+Y_1^*-C_1)^{\gamma}\left[(1+r_1^M)\omega_1+(1+\tilde{r}_1)(1-\omega_1)\right]^{\gamma-1}(r_1^M-\tilde{r}_1)\right] \\
&= E\left[\left[(1+r_1^M)\omega_1+(1+\tilde{r}_1)(1-\omega_1)\right]^{\gamma-1}(r_1^M-\tilde{r}_1)\right] \qquad \text{(F1iso)}
\end{aligned}
$$

– equation (F1iso) implicitly determines the optimal portfolio share, ω_1^*, as a function of r_1^M, γ, and parameters characterizing the probability distribution of $\tilde{r}_1$. Note that ω_1^* maximizes the *subjective mean return on the portfolio*, r^*, which is defined as:

$$
\begin{aligned}
(1+r^*)^{\gamma} &\equiv \max_{\omega_1} E\left[(1+r_1^M)\omega_1+(1+\tilde{r}_1)(1-\omega_1)\right]^{\gamma} \\
&= E\left[(1+r_1^M)\omega_1^*+(1+\tilde{r}_1)(1-\omega_1^*)\right]^{\gamma} \qquad \text{(SMR)}
\end{aligned}
$$

– The FONC for C_1 (F2) becomes:

$$
\begin{aligned}
C_1^{\gamma-1} &= \frac{1}{1+\rho} E\left[\tilde{C}_2^{\gamma-1}\left[(1+r_1^M)\omega_1 + (1+\tilde{r}_1)(1-\omega_1)\right]\right] \\
&= \frac{1}{1+\rho}(A_1 + Y_1^* - C_1)^{\gamma-1} E\left[(1+r_1^M)\omega_1 + (1+\tilde{r}_1)(1-\omega_1)\right]^{\gamma} \\
&= \frac{1}{1+\rho}(A_1 + Y_1^* - C_1)^{\gamma-1}(1+r^*)^{\gamma} \qquad \Rightarrow \\
C_1 &= c\left[A_1 + Y_1^*\right] \qquad \text{(F2iso)}
\end{aligned}
$$

where c is the marginal propensity to consume out of total wealth:

$$
c \equiv \frac{(1+r^*)^{\gamma/(\gamma-1)}}{(1+\rho)^{1/(\gamma-1)} + (1+r^*)^{\gamma/(\gamma-1)}} \qquad \text{(MPC)}
$$

– Striking result: the optimal consumption plan for the first period looks very much like the solution that would be obtained under certainty!

  * in the absence of uncertainty about the bond yield, maximization of lifetime utility would give rise to the same expression but with r^* replaced by $\max[\bar{r}_1, r_1^M]$, where $\bar{r}_1$ is the certain return on bonds
  * in the case of a logarithmic felicity function ($\gamma = 0$), r^* drops out of (MPC) altogether and the capital risk does not affect present consumption at all

– With iso-elastic felicity functions, there thus exists a *separability property* between the savings problem (choosing when to consume) and the portfolio problem (choosing what to use as a savings instrument).

Portfolio Decision in Isolation

- step back from two-period model and study portfolio decision, holding constant the amount to be invested
- model developed by Tobin (1958) and by Arrow (1965)
- investor chooses the portfolio share of money ω in order to maximize expected utility:

$$EU(\tilde{Z}) \tag{EU}$$

where $\tilde{Z}$ is end-of-period wealth:

$$\tilde{Z} \equiv S\left[(1+r^M)\omega + (1+\tilde{r})(1-\omega)\right] \tag{EOPW}$$

  – S is the amount to be invested (exogenous)
  – r^M is the risk-free rate (exogenous)

- First-order condition:

$$EU'(\tilde{Z})(r^M - \tilde{r}) = 0 \tag{F1}$$

- To develop intuition behind (F1) we to the *mean-variance model*
  - Step 1. expand the utility function, $U(\tilde{Z})$, by means of a Taylor approximation around the expected value (or mean) of $\tilde{Z}$, denoted by $E(\tilde{Z})$:

$$\begin{aligned} U(\tilde{Z}) &\approx U(E(\tilde{Z})) + U'(E(\tilde{Z}))\left[\tilde{Z} - E(\tilde{Z})\right] + \tfrac{1}{2}U''(E(\tilde{Z}))\left[\tilde{Z} - E(\tilde{Z})\right]^2 \\ &\quad + \tfrac{1}{6}U'''(E(\tilde{Z}))\left[\tilde{Z} - E(\tilde{Z})\right]^3 + \cdots \end{aligned} \tag{A}$$

  - Step 2. take expectations on both sides of (A):

$$\begin{aligned} EU(\tilde{Z}) &\approx EU(E(\tilde{Z})) + EU'(E(\tilde{Z}))\left[\tilde{Z} - E(\tilde{Z})\right] + \tfrac{1}{2}EU''(E(\tilde{Z}))\left[\tilde{Z} - E(\tilde{Z})\right]^2 + \cdots \\ &= U(E(\tilde{Z})) + \tfrac{1}{2}U''(E(\tilde{Z}))E\left[\tilde{Z} - E(\tilde{Z})\right]^2 + \cdots \end{aligned} \tag{B}$$

– Step 3. ignore all higher than second-order terms in (B) so that preferences of the investor are (assumed to be) fully described by only the mean and the variance of end-of-period wealth (hence its name):

$$EU(\tilde{Z}) = U(E(\tilde{Z})) - \eta E\left[\tilde{Z} - E(\tilde{Z})\right]^2 \quad \text{(AEU)}$$

– $\eta \equiv -\frac{1}{2}U''(E(\tilde{Z}))$. The sign of η fully characterizes the investor's attitude towards risk:

  * if $\eta = 0$, the variance term drops out of (AEU): risk neutrality. In **Figure 12.4** the , the underlying utility function, $U(\tilde{Z})$, is simply a straight line from the origin ($U'(\tilde{Z}) > 0$ and $U''(\tilde{Z}) = 0$ in this case).
  * if $\eta > 0$, then risk is seen as a bad thing: risk aversion. In Figure 12.4 the felicity function is concave. If outcomes $\tilde{Z} = Z_0 - h$ and $\tilde{Z} = Z_0 + h$ have equal probability $\frac{1}{2}$ then the *risk premium* is $\pi_R > 0$
  * if $\eta < 0$, then risk is seen as a good (a "thrill"): risk loving.

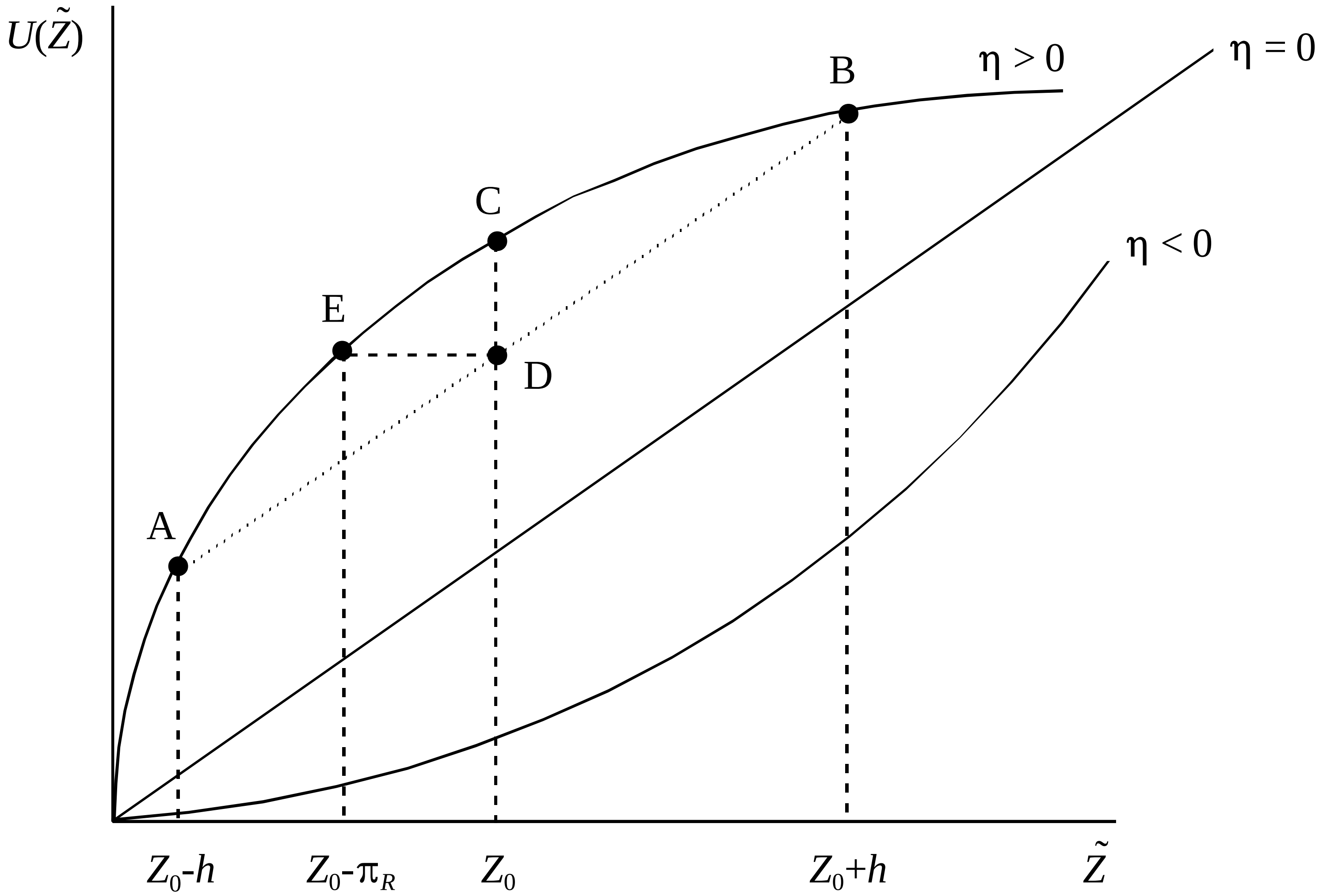

Figure 12.4: Attitude Towards Risk and the Felicity Function

- Postulate particular probability distribution for $\tilde{r}$. A particularly simple and convenient distribution to choose in this context is the normal distribution:

$$\tilde{r} \sim N(\bar{r}, \sigma_R^2) \tag{ND}$$

  - "$\sim$" means "is distributed as,"
  - "N" stands for "normal or Gaussian distribution"
  - $\bar{r} \equiv E\tilde{r}$ is the mean of the distribution
  - $\sigma_R^2 \equiv E\left(\tilde{r} - \bar{r}\right)^2$ is the variance of the distribution
  - normal distribution fully characterized by only two parameters, $\bar{r}$ and σ_R^2. All higher-order uneven terms, such as $E(\tilde{r} - \bar{r})^i$ (for $i = 3, 5, 7, \cdots$) are equal to zero as the distribution is symmetric around its mean. Furthermore, the higher-order even terms, such as $E(\tilde{r} - \bar{r})^i$ (for $i = 4, 6, 8, \cdots$) can be expressed in terms of $\bar{r}$ and σ_R^2

- Advantages of using normal distribution:
  - (B) can always be written as in (AEU) even without ignoring the higher-order terms, i.e. preferences are fully described by only two parameters.
  - enables us to conduct simple comparative static experiments pertaining to $\bar{r}$ and σ_R^2 and the optimal portfolio choice
- Using definition of $\tilde{Z}$ from (EOPW) we find:

$$\tilde{Z} \sim N(\bar{Z}, \sigma_Z^2)$$

$$\bar{Z} \equiv E(\tilde{Z}) = S\left[(1+r^M)\omega + (1-\omega)(1+\bar{r})\right] \quad \text{(A)}$$

$$\sigma_Z^2 \equiv E\left[\tilde{Z} - E(\tilde{Z})\right]^2 = S^2(1-\omega)^2\sigma_R^2 \quad \text{(B)}$$

so that (AEU) can be written as:

$$EU(\tilde{Z}) = U(\bar{Z}) - \eta\sigma_Z^2 \quad \text{(AEU}'\text{)}$$

- By choice of ω, the investor can influence $\bar{Z}$ and σ_Z^2. See **Figure 12.5**.
  - based on risk-averse investor. Indifference curve slopes up (for $\sigma_Z > 0$):

$$
\begin{aligned}
dEU(\tilde{Z}) &= U'(\bar{Z})d\bar{Z} - 2\eta\sigma_Z d\sigma_Z = 0 \qquad \Leftrightarrow \\
\frac{d\bar{Z}}{d\sigma_Z} &= \frac{2\eta\sigma_Z}{U'(\bar{Z})} > 0 \\
\frac{d^2\bar{Z}}{d\sigma_Z^2} &= 2\eta \left[\frac{U'(\bar{Z}) - \sigma_Z U''(\bar{Z})\left(d\bar{Z}/d\sigma_Z\right)}{\left[U'(\bar{Z})\right]^2}\right] > 0
\end{aligned}
$$

  - mean return on the risky exceeds that on money ($\bar{r} > r^M$), otherwise a risk-averse agent would never hold any risky assets
  - in top panel AB is the budget line (A) [described paramaterically, i.e. by varying ω in the interval $[0, 1]$]
  - in bottom panel AB represents the relation (B): $\sigma_Z = S(1-\omega)\sigma_R$

– optimum at E_0: a risk-averse investor will typically choose a *diversified portfolio.* Money held even if its return is zero ($r^M = 0$) because it represents a riskless means of investing (at least, under the present set of assumptions)

– at point E_0 the indifference curve is tangential to the budget line:

$$\left(\frac{d\bar{Z}}{d\sigma_Z}\right)_{IC} \equiv \frac{2\eta\sigma_Z}{U'(\bar{Z})} = \frac{\bar{r} - r^M}{\sigma_R} \equiv \left(\frac{d\bar{Z}}{d\sigma_Z}\right)_{BL} \quad \text{(F1)}$$

– write optimal portfolio share of money as:

$$\omega^* = \omega^*(r^M, \bar{r}, \sigma_R^2, \eta) \quad \text{(OMD)}$$

- effect of an increase in the yield on money r^M:
  - in Figure 12.5 the budget line shifts up and becomes flatter; see A′B in the top panel
  - ultimate effect on the portfolio share of money (and thus money demand) can be decomposed into income and pure substitution effects
  - SE: an increase in r^M narrows the yield gap between money and the risky asset which induces the investor to substitute towards the safe asset and to hold a higher portfolio share of money (move E_0 to E'.)
  - IE: an increase in r^M also increases expected wealth and the resulting income (or wealth) effect also leads to an upward shift in ω (move from E' to E_1)
  - both IE and SE work in the same direction and the new optimum lies at point E_1.

– formally the Slutsky equation is:

$$\frac{\partial\omega}{\partial r^M} = \left(\frac{\partial\omega}{\partial r^M}\right)_{dEU=0} + \omega S\left(\frac{\partial\omega}{\partial\bar{Z}}\right) > 0$$

where the first term on the right-hand side represents the pure substitution or "compensated" effect and the second term is the income effect:

$$\left(\frac{\partial\omega}{\partial r^M}\right)_{dEU=0} \equiv \frac{(1-\omega)\sigma_R^2}{(\bar{r}-r^M)\left[\sigma_R^2+(r^M-\bar{r})^2\right]} > 0$$

$$\left(\frac{\partial\omega}{\partial\bar{Z}}\right) \equiv \frac{\bar{r}-r^M}{S\left[\sigma_R^2+(r^M-\bar{r})^2\right]} > 0$$

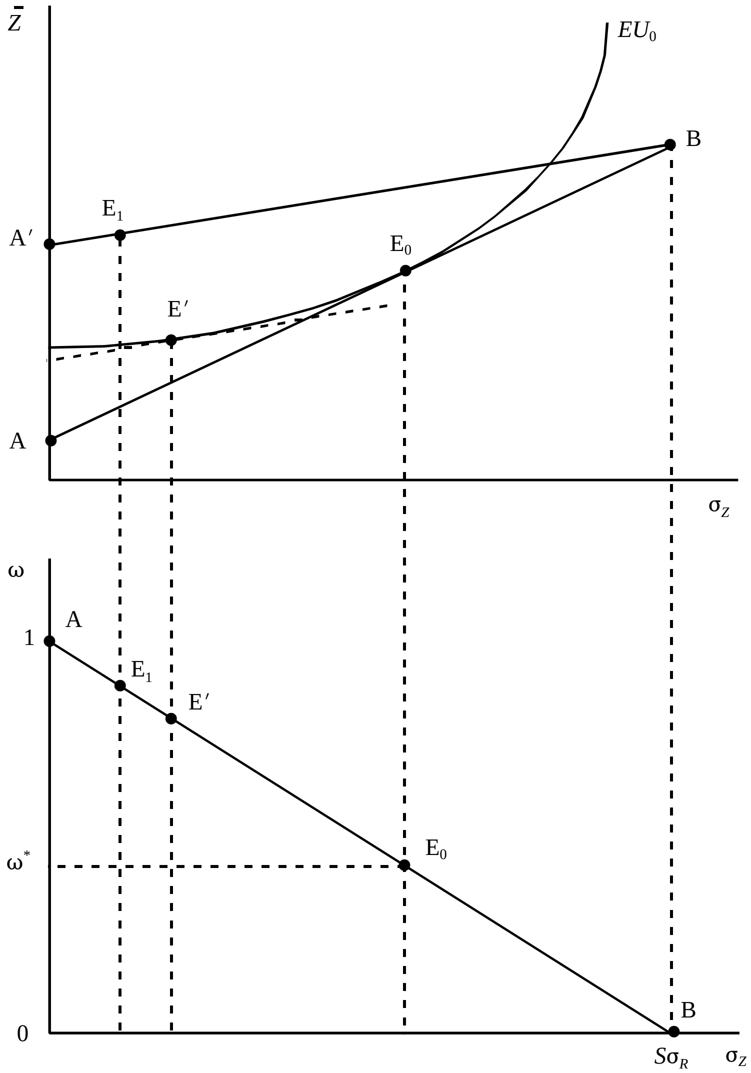

Figure 12.5: Portfolio Choice

- Effect of incease in expected yields on risky asset: slope of money demand
  - In **Figure 12.6** an increase in $\bar{r}$ causes the budget line to rotate in a counter-clockwise fashion around point A
  - income and substitution effects now operate in opposite directions and the Slutsky equation becomes:

$$\frac{\partial \omega}{\partial \bar{r}} = \left(\frac{\partial \omega}{\partial \bar{r}}\right)_{dEU=0} + (1-\omega) S \left(\frac{\partial \omega}{\partial \bar{Z}}\right) > 0$$

where:

$$\left(\frac{\partial \omega}{\partial \bar{r}}\right)_{dEU=0} = -\left(\frac{\partial \omega}{\partial r^M}\right)_{dEU=0} = \frac{-(1-\omega)\sigma_R^2}{(\bar{r}-r^M)\left[\sigma_R^2 + (r^M - \bar{r})^2\right]} < 0$$

$$\frac{\partial \omega}{\partial \bar{Z}} \equiv \frac{\bar{r} - r^M}{S\left[\sigma_R^2 + (r^M - \bar{r})^2\right]} > 0$$

- SE: move from E_0 to E'
- IE: move from E' to E_1^1 (if SE dominates) or to E_1^2 (if IE dominates)
- It is thus quite possible that money demand depends positively on the expected yield on the risky asset in the portfolio model of Tobin (1958).

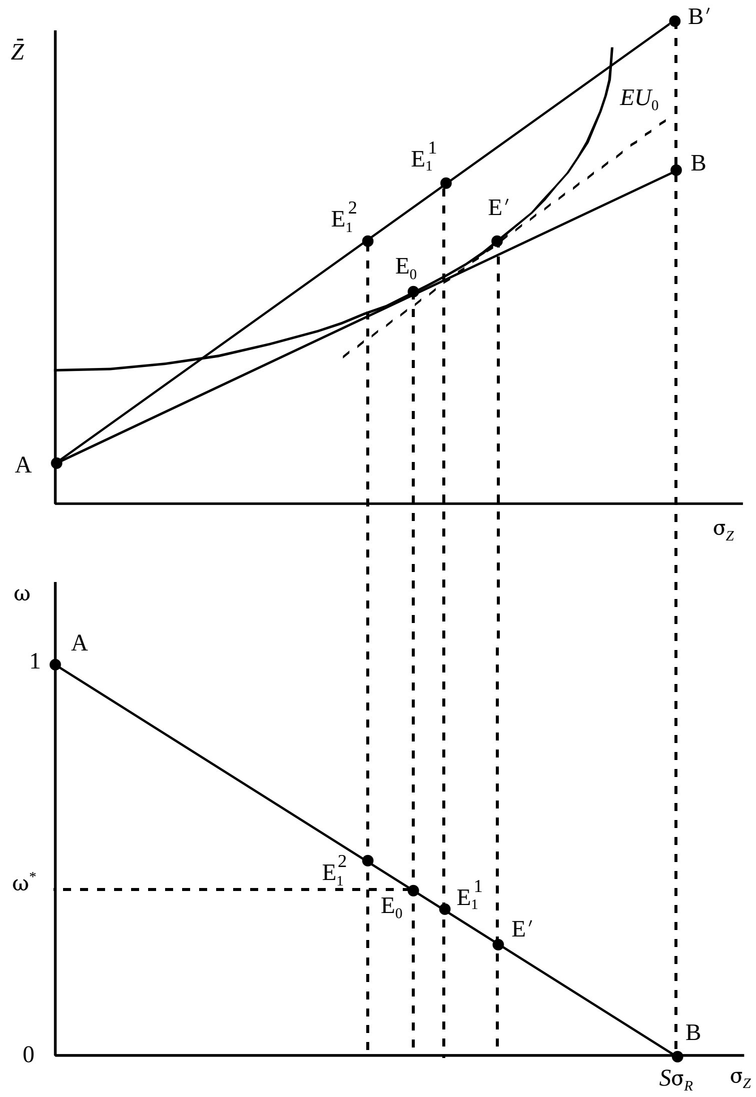

Figure 12.6: Portfolio Choice: Change in $\bar{r}$

- Effect of the degree of risk associated with the risky asset as measured by the standard deviation of the yield, σ_R.
  - **Figure 12.7** studies effect of increase in σ_R
  - in top panel the budget line becomes flatter and rotates in a clockwise fashion around point A
  - in bottom panel, the line relating the standard deviation of the portfolio to the portfolio share of money becomes flatter and rotates in a counter-clockwise fashion around point A.

– Slutsky equation:

$$\frac{\partial\omega}{\partial\sigma_R} = \left(\frac{\partial\omega}{\partial\sigma_R}\right)_{dEU=0} - (1-\omega)S\left[(\bar{r}-r^M)/\sigma_R\right]\left(\frac{\partial\omega}{\partial\bar{Z}}\right) > 0$$

where:

$$\left(\frac{\partial\omega}{\partial\sigma_R}\right)_{dEU=0} \equiv \frac{(1-\omega)\left[2\sigma_R^2+(r^M-\bar{r})^2\right]}{\sigma_R\left[\sigma_R^2+(r^M-\bar{r})^2\right]} > 0$$

$$\frac{\partial\omega}{\partial\bar{Z}} \equiv \frac{\bar{r}-r^M}{S\left[\sigma_R^2+(r^M-\bar{r})^2\right]} > 0$$

– SE dominates IE and money demand rises if the return on the risky asset becomes more volatile: safe haven of money holdings.

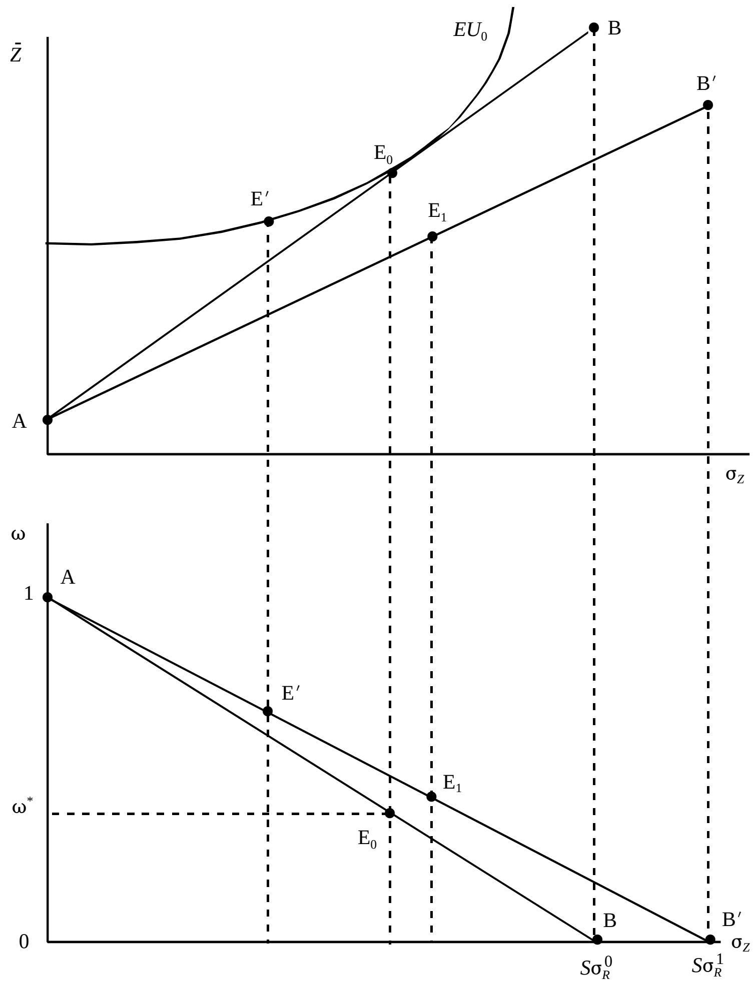

Figure 12.7: Portfolio Choice: Change in σ_R

Optimal Quantity of Money

- What is the socially optimal quantity of money?
- Friedman (1969): *satiation* or *full liquidity* result
  - MC of fiat money close to zero
  - MC=MB is optimal
  - ergo: drive MB to zero by satiating households with money
- How to implement the rule?
  - Opportunity cost of money is nominal interest rate on bonds (R_t)
  - Since $R_t = r_t + \pi_t^e$ and r_t is determined by real factors (classical dichotomy) we should set $\pi_t^e = -r_t$
  - In steady state we have $\pi_t = \mu_t$, where μ_t is the money growth rate
  - With fullfilled expectations we have $\pi_t^e = \pi_t$

- To sum up, the strong version of the Friedman rule says:

$$\mu_t = \pi_t = -r < 0, \quad \text{(FR)}$$

  where r is the steady-state real interest rate. Reduce the money supply and price level at the same rate as the steady-state real interest rate.

- Outline of the argument:
  - simple general equilibrium model
  - derivation of the satiation result
  - critiques of the full liquidity rule

Basic Model

- Utility function of representative household:

$$V = U(C_1, m_1) + \left(\frac{1}{1+\rho}\right) U(C_2, m_2) \quad \text{(UF)}$$

$\longrightarrow$ m_t is real money balances held at the *end* of period t.

- Abstracting from bonds and with exogenous and constant production, the budget identities are:

$$P_1 Y + M_0 + P_1 T_1 = P_1 C_1 + M_1 \quad \text{(BI1)}$$

$$P_2 Y + M_1 + P_2 T_2 = P_2 C_2 + M_2 \quad \text{(BI2)}$$

  – M_0 is predetermined

  – $P_t T_t$ are lump-sum cash transfers received from the government.

- Money supply process (constant money growth, μ):

$$\frac{\Delta M_t}{M_{t-1}} = \mu \qquad \text{(MSR)}$$

- Relation transfers money growth:

$$P_t T_t = \Delta M_t \qquad \text{(TR)}$$

- The household chooses C_t and M_t (for $t = 1, 2$) in order to maximize (UF) subject to (BI1)-(BI2). Assuming an interior solution, the FONCs are:

$$\frac{U_C(C_1, m_1)}{P_1} = \frac{U_m(C_1, m_1)}{P_1} + \left(\frac{1}{1+\rho}\right) \frac{U_C(C_2, m_2)}{P_2} \quad \text{(F1)}$$

$$U_C(C_2, m_2) = U_m(C_2, m_2) \quad \text{(F2)}$$

  - $U_C(\cdot) \equiv \partial U(\cdot)/\partial C_t$ and $U_m(\cdot) \equiv \partial U(\cdot)/\partial m_t$
  - (F1): marginal utility of spending one dollar on consumption (the left-hand side) must be equated to the marginal utility obtained by holding one dollar in the form of money balances (the right-hand side). The latter is itself equal to the marginal utility due to reduced transaction costs (first term) plus that due to the store-of-value function of money (second term).
  - (F2): In the final (second) period, money is not used as a store of value so only the transactions demand for money motive is operative.

- Goods market clearing condition:

$$Y = C_1 = C_2 \tag{GME}$$

- The perfect foresight equilibrium for the economy can be written as:

$$[U_C(Y,m_1) - U_m(Y,m_1)]\, m_1 = \frac{m_2 U_C(Y,m_2)}{(1+\rho)(1+\mu)} \tag{F1$'$}$$

$$U_C(Y,m_2) = U_m(Y,m_2) \tag{F2$'$}$$

  - equations recursively determine the equilibrium values for the real money supply
  - work backwards in time
  - (F2$'$) is solved for m_2 (yielding m_2^*)
  - (F1$'$) with m_2^* substituted can then determine m_1^*
  - since M_t is determined by the policy maker, the nominal price level is $P_t^* \equiv M_t/m_t^*$

- Example: the felicity function is *additively separable*:

$$U(C_t, m_t) \equiv u(C_t) + v(m_t) \qquad \text{(ASFF)}$$

  – $u'(C_t) > 0$ and $u''(C_t) < 0$

  – $v'(m_t) > 0$ for $0 < m_t < m^*$, $v'(m_t) = 0$ for $m_t = m^*$, $v'(m_t) < 0$ for $m_t > m^*$ and $v'(m_t) < 0$.

  – Marginal utility of consumption is positive throughout but satiation with money balances is possible provided the real money supply is sufficiently high.

- Model simplifies to:

$$\left[u'(Y) - v'(m_1)\right] m_1 = \frac{m_2 u'(Y)}{(1+\rho)(1+\mu)} \tag{EE}$$

$$u'(Y) = v'(m_2) \tag{TC}$$

  - See **Figure 12.8** for illustration
  - TC is the “terminal condition”
  - EE is an Euler-like equation (upward-sloping)
  - equilibrium is at point E_0.

- Comparative static result: increase in μ.
  - upward shift in the EE line, say to EE$_1$
  - equilibrium shifts to E$_1$
  - real money balances in the first period fall, i.e. $dm_1^*/d\mu < 0$
  - even though only the level of *future* nominal money balances is affected (M_1 stays the same and M_2 rises), the rational representative agent endowed with perfect foresight foresees the consequences of higher money growth and as a result ends up bidding up the nominal price level not only in the future but also in the present. A similar effect is obtained if the rate of pure time preference is increased.

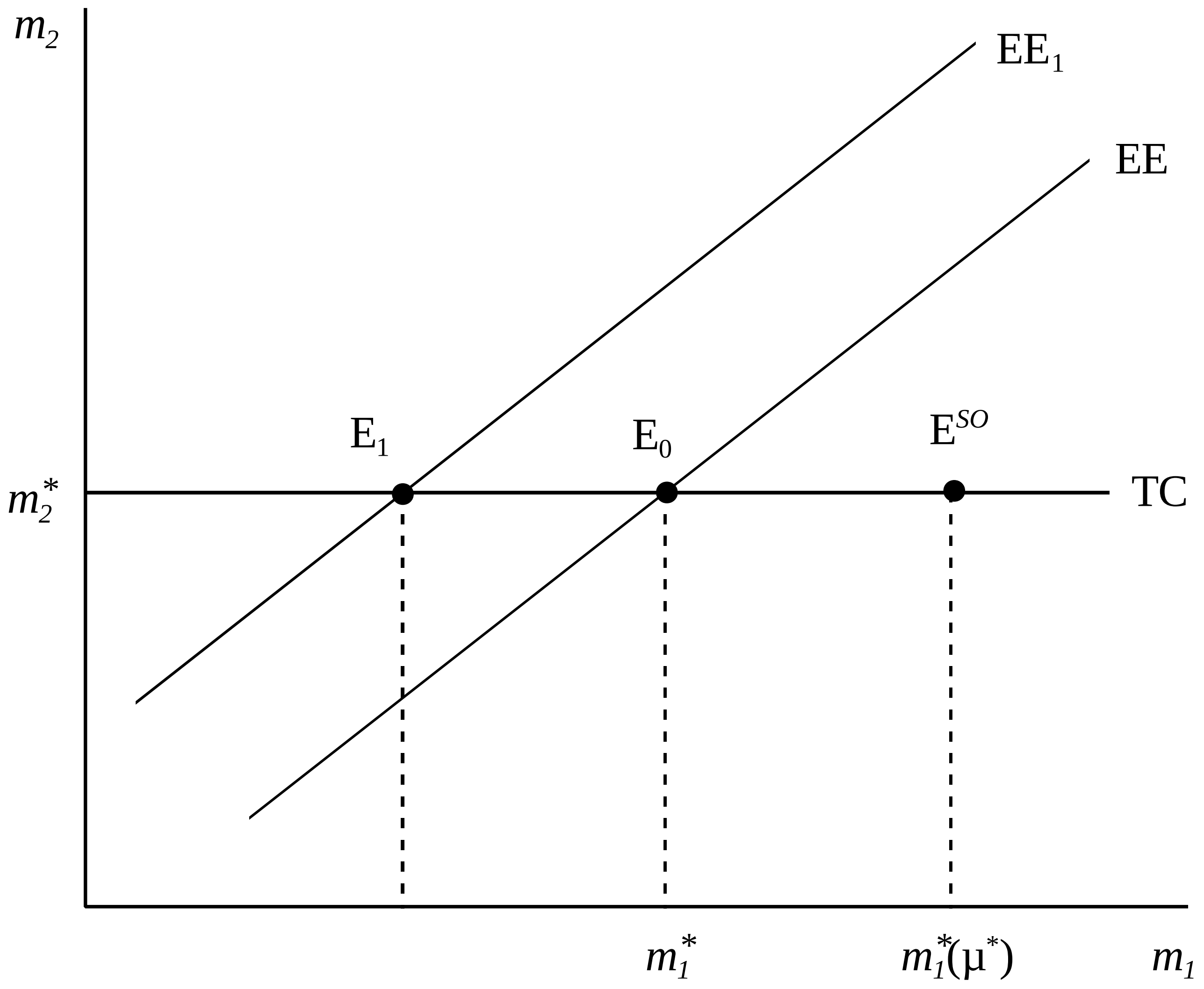

Figure 12.8: Monetary Equilibrium in a Perfect Foresight Model

Satiation Result in Basic Model

- Optimal household decisions summarized by: $m_1^* = m_1^*(\rho, Y, \mu)$ and $m_2^* = m_2^*(\rho, Y, \mu)$, with $\partial m_1^*/\partial\mu < 0$ and $\partial m_2^*/\partial\mu = 0$ (for the felicity function (ASFF))

- μ is policy maker's instrument to influence the equilibrium of money balances (in the first period).

- substitutie $m_t^*(\cdot)$ and (GME) into the utility function of the representative agent:

$$V = u(Y) + v\left(m_1^*(\rho, Y, \mu)\right) + \left(\frac{1}{1+\rho}\right)\left[u(Y) + v\left(m_2^*(\rho, Y)\right)\right] \qquad \text{(IUF)}$$

- Utilitarian policy maker pursues optimal monetary policy by choosing the money growth rate for which the welfare of the representative agent is maximized. By maximizing (IUF) w.r.t. μ we obtain the Friedman satiation result:

$$\frac{dV}{d\mu} = v'\left(m_1^*(\rho, Y, \mu^*)\right)\left(\frac{dm_1^*}{d\mu}\right) = 0 \quad \Rightarrow$$

$$v'\left(m_1^*(\rho, Y, \mu^*)\right) = 0, \qquad \text{(FSR)}$$

– μ^* is the optimal money growth rate

– μ^* is such that the corresponding demand for current real money balances chosen by the representative household is such that the marginal utility of these balances is zero

– in Figure 12.8 the social optimum is at point E^{SO} (higher level of real money balances and a lower money growth rate than at point E_0)

– the satiation result does not hold in the final period, of course, as the terminal condition pins down a positive marginal utility of money balances needed for transaction purposes

- Generalization of the Friedman result to a setting with an infinitely lived representative agent
  - Utility function:

$$V = U(C_t, m_t) + \left(\frac{1}{1+\rho}\right) U(C_{t+1}, m_{t+1}) + \left(\frac{1}{1+\rho}\right)^2 U(C_{t+1}, m_{t+1}) + \cdots$$

  - Budget identities:

$$P_t Y + M_{t-1} + P_t T_t = P_t C_t + M_t$$

  - FONCs of private optimum:

$$\left[u'(Y) - v'(m_t)\right] m_t = \frac{m_{t+1} u'(Y)}{(1+\rho)(1+\mu)}, \quad (t = 1, 2, 3, \cdots, \infty). \qquad \text{(EE)}$$

  - the terminal condition is no longer relevant

– equilibrium solution to (EE) will in fact be the steady-state solution for which $m_t = m_{t+1} = m^*$:

$$v'(m^*) = \left[1 - \frac{1}{(1+\rho)(1+\mu)}\right] u'(Y) \;\Rightarrow \qquad \text{(SEE)}$$

$$\frac{dm^*}{d\mu} = \frac{u'(Y)}{(1+\rho)(1+\mu)^2 v''(m^*)} < 0$$

– Since both the endowment and real money balances are constant over time, lifetime utility of the infinite lived representative agent is equal to:

$$\frac{\rho V}{1+\rho} = u(Y) + v\left(m^*(\mu)\right) \qquad \text{(IUF)}$$

– Maximizing (IUF) by choice of μ yields the result that the optimal money supply is such as to ensure that $v'(m^*) = 0$ for all periods. In view of (SEE), this is achieved if the money supply is shrunk at the rate at which the representative household discounts future utility:

$$
\begin{aligned}
1 &= \frac{1}{(1+\rho)(1+\mu^*)} \qquad \Leftrightarrow \\
\mu^* &= -\frac{\rho}{1+\rho}. \qquad \text{(FSR)}
\end{aligned}
$$

– Although there are no interest-bearing assets in our model, (FSR) can nevertheless be interpreted as a zero-interest rate result.

  * the pure rate of time preference represents the psychological costs associated with waiting and $\rho/(1+\rho)$ ($\approx \rho$) can be interpreted as the real rate of interest
  * since real money balances are constant, the money growth rate μ^* also represents the rate of price inflation
  * the nominal rate of interest in the optimum is thus:

$$R \equiv \frac{\rho}{1+\rho} + \pi = \frac{\rho}{1+\rho} + \mu^* = 0$$

Critiques of Full Liquidity Rule

- In model with endogenous labour supply (and thus endogenous production) the satiation result no longer holds unless:
  - preferences are non-separable in $(C_t, 1 - L_t)$ and m_t but the initial tax rate on labour income is zero (see (12.110) in text). Intuition: for $\tau_t = 0$, μ has no first-order welfare cost due to distortion of consumption-leisure choice.
  - initial tax on labour income is positive but preferences are separable in $(C_t, 1 - L_t)$ and m_t. Intuition: μ does not affect MRS$_{C,1-L}$ and thus does not affect consumption
- In absence of lump-sum taxes/transfers, μ may act as inflation tax needed to raise revenue. Optimal μ is set as optimal Ramsey tax. No full satiation result in general (see Section 12.4.4 in the text)

Punchlines

- Identified major functions of money:
  - medium of exchange
  - store of value
  - medium of account
- Key models of money in its different roles
  - medium of account: shopping costs, cash in advance, money in utility
  - store of value: consumption smoothing in presence of frictions, portfolio choice under uncertainty
- Socially optimal quantity of money
  - full satiation result
  - critiques of Friedman Rule
  - money and public finance issues