Foundations of Modern Macroeconomics

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Problem set for Chapter 17

The questions with a star (★) are difficult.

Question 1

[Jones and Manuelli (1992)] In this question we consider a capital-fundamentalist growth model in which agents have finite lives. We show that intergenerational redistribution of resources from the old to the young is needed to jump-start the growth process. We use a simplified version of the Diamond-Samuelson model considered in Section 17.1 of the book. The representative young household has the following lifetime utility function:

$$\Lambda_t^Y \equiv \log C_t^Y + \left(\frac{1}{1+\rho}\right) \log C_{t+1}^O, \tag{1}$$

where Λ_t^Y is lifetime utility, C_t^Y is consumption during youth, C_{t+1}^O is consumption during old age, and ρ is the pure rate of time preference. The household faces the usual budget identities:

$$C_t^Y + S_t = W_t, \tag{2}$$

$$C_{t+1}^O = (1 + r_{t+1}) S_t, \tag{3}$$

where S_t is saving, W_t is the wage rate, and r_{t+1} is the interest rate. There is no population growth and the size of each generation is normalized to unity ($L_t = L_{t-1} = 1$). Households only work during youth.

The representative firm hires labour L_t (from the young) and capital K_t (from the old) in order to maximize profit:

$$\Pi_t \equiv F(K_t, L_t) - W_t L_t - R_t^K K_t, \tag{4}$$

where $F(\cdot)$ is a linear homogeneous production function, and R_t^K is the rental rate on capital. It is assumed that the production function features a constant elasticity of substitution:

$$F(K_t, L_t) \equiv A \left[\alpha K_t^{(\sigma-1)/\sigma} + (1-\alpha) L_t^{(\sigma-1)/\sigma}\right]^{\sigma/(\sigma-1)}, \quad 0 < \alpha < 1, \tag{5}$$

where A is some index of general productivity. It is assumed that capital and labour can be substituted quite easily, i.e. $\sigma > 1$.

(a) Use a simple arbitrage argument to explain why the rental rate on capital is equal to $R_t^K = r_t + \delta$.

(b) Solve the optimization problems for the representative (old and young) households and the representative firm. Show that profits are zero and that saving is proportional to the wage rate. Derive the expression linking S_t and K_{t+1}.

(c) Derive an expression for W_t/K_t and show that $\lim_{K_t \to 0} W_t/K_t = \infty$ and $\lim_{K_t \to \infty} W_t/K_t = 0$. Define the growth rate of capital (per worker) by $\gamma_t^K \equiv (K_{t+1} - K_t)/K_t$ and show that it goes to zero in the long run. Illustrate your answer with the aid of a diagram with K_t on the vertical axis and K_{t+1}/K_t on the horizontal axis.

(d) Why is there no endogenous growth in this model despite the fact that capital and labour can be easily substituted?

Assume that the government introduces an output tax, τ, so that the profit function is now $\Pi_t \equiv (1-\tau)F(K_t, L_t) - W_t L_t - R_t^K K_t$. Assume that the tax revenue, $\tau F(K_t, L_t)$, is rebated to young households in the form of lump-sum transfers. Hence, these transfers are given by $T_t^Y = \tau F(K_t, L_t)$.

(e) ★ Solve the extended model and prove that the long-run growth rate in the capital stock is given by:

$$\gamma_t^K = \max\{0, \gamma^*\}, \qquad \gamma^* \equiv \left(\frac{\tau A}{2+\rho}\right) \alpha^{\sigma/(\sigma-1)} - 1. \tag{6}$$

Explain intuitively why endogenous growth becomes feasible if a sufficient amount of income is distributed to the young.

Question 2

[Welfare effects of debt] Consider the basic Diamond-Samuelson model studied in Section 17.1 of the book and assume that the felicity function is logarithmic:

$$\Lambda_t^Y \equiv \log C_t^Y + \left(\frac{1}{1+\rho}\right) \log C_{t+1}^O, \tag{1}$$

where Λ_t^Y is lifetime utility, C_t^Y is consumption during youth, C_{t+1}^O is consumption during old age, and ρ is the pure rate of time preference. The technology is Cobb-Douglas and $y_t = k_t^{1-\epsilon_L}$ where $y_t \equiv Y_t/L_t$ and $k_t \equiv K_t/L_t$.

(a) Introduce government consumption, government debt and lump-sum taxes levied on young and old generations into the model. Denote these variables by, G_t, B_t, T_t^Y, and T_t^O respectively. Define per capita debt and government consumption as, respectively, $b_t \equiv B_t/L_t$ and $g_t \equiv G_t/L_t$. Derive and interpret the government budget identity.

(b) Solve the household optimization problem. Establish the link between household saving and the future capital stock. Show that one of b_t, T_t^Y, and T_t^O is redundant to finance a given path for government consumption.

(c) Determine the macroeconomic effects of a once-off increase in government consumption, g, which is financed by means of lump-sum taxes on the young. Abstract from government debt (i.e. $b_t = b_{t+1} = 0$). Derive the stability condition and explain the intuition behind your results.

(d) Redo part (c) but now assume that financing is by means of lump-sum taxes on the old. Comment on the key differences with the earlier case.

(e) Assume that the government is somehow unable to levy taxes on the old (so that $T_t^O = 0$ for all t) and that it maintains a constant amount of debt per member of the young generation (i.e. $b_t = b$ for all t). Show that a once-off increase in b leads to crowing out of capital in the long run in a dynamically efficient economy.