Foundations of Modern Macroeconomics

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Problem set for Chapter 16

The questions with a star (★) are difficult.

Question 1

[Saint-Paul (1992)] In this question we extend the Blanchard-Yaari model (studied in Chapter 16) to allow for endogenous growth. Assume that the private production function, $F(K, L)$, is given by:

$$Y(t) = F(K(t), L(t)) \equiv Z(t)L(t)^{\epsilon_L}K(t)^{1-\epsilon_L}, \tag{1}$$

where $Z(t)$ is the level of general technology (taken as given by individual firms). There is an external effect which ensures that general technology is positively affected by the aggregate capital stock:

$$Z(t) = AK(t)^{\epsilon_L}. \tag{2}$$

The rest of the model is unchanged, i.e. the expressions in Table 16.1 are all still appropriate. Assume for simplicity that government consumption is zero ($G(t) = 0$).

(a) Derive the marginal productivity conditions for labour and capital. Show that the real interest rate is constant.

(b) Assume that the government maintains a constant tax rate on labour income, so that $T(t) = t_L W(t)$, where $W(t)$ is the real wage rate. Assume furthermore that the government also maintains a constant ratio between debt ($B(t)$) and aggregate output ($Y(t)$) which we denote by ζ (i.e. $B(t) = \zeta Y(t)$). Derive the growth rate of the economy.

(c) Is the rate of economic growth affected by the birth rate, β? If so, explain the economic intuition behind this dependence.

(d) Show that a decrease in ζ increases the rate of growth in the economy. Explain the economic intuition behind the result.

Question 2

[Current account dynamics] Consider the following Blanchard-Yaari model of a small open economy:

$$\dot{C}(t) = (r-\rho)C(t) - \beta(\rho+\beta)A_F(t), \tag{1}$$
$$\dot{A}_F(t) = rA_F(t) + W(t) - C(t), \tag{2}$$

where C is consumption, r is the world interest rate (exogenous), ρ is the pure rate of time preference, β is the birth rate (equals to the death rate), A_F is the stock of net foreign assets, and W is the real wage rate. As usual, a variable with a dot above it represents that variable's time rate of change, i.e. $\dot{C}(t) \equiv dC(t)/dt$ and $\dot{A}_F(t) \equiv dA_F(t)/dt$. For simplicity we assume that there is no capital in the model so that the production function depends on labour only:

$$Y(t) = Z(t)L(t), \tag{3}$$

where Y, Z, and L denote, respectively, aggregate output, an index for general technology (exogenous), and aggregate employment. Aggregate labour supply is exogenous and equal to unity. Firms are perfectly competitive and hire labour from households.

(a) Interpret equations (1)-(2). Explain the role of the birth rate in the aggregate Euler equation.

(b) Show that the real wage is uniquely related to the index of general technology. Explain this result intuitively.

(c) Derive the phase diagram for the model under the assumption that the country's citizens are relatively patient (so that $\rho < r$). Show that the model is saddle-path stable and that the saddle path is upward sloping in (C, A_F) space. Which variable is the jumping variable? Which one is the pre-determined variable?

(d) Derive the impact-, transitional-, and long-run effects of a permanent and unanticipated improvement of general technology. Use the phase diagram derived in part (c) and show the transition paths for consumption, net foreign assets, output, employment, and the real wage.

(e) Assume that the world interest rate is equal to the rate of pure time preference (i.e. $r = \rho$). Show that the model is still saddle-point stable and demonstrate the effects on consumption and net foreign assets of a temporary productivity shock. Explain the economic intuition.

(f) ★ Now assume that the birth/death rate is zero ($\beta = 0$). Show that the world interest rate must equal the rate of pure time preference for the model to make any sense at all (i.e. $r = \rho$). Show that the model exhibits the hysteresis property in that a temporary productivity shock has permanent effects on consumption and the stock of net foreign assets.

Question 3

[Buiter (1990, ch. 7)] In this question we modify the standard Blanchard model (studied in Section 16.2.2 of the book) by assuming that there are no markets for insuring against the risks associated with an unexpected death. In this case the household will generally make "accidental bequests" which may be positive or negative. It is assumed that the estate of a household who has died accrues to the government. The government reimburses the revenue of this scheme to surviving agents in an age-independent lump-sum fashion. Surviving agents take these lump-sum transfers as given.

(a) Solve the optimization problem for individual households. Explain carefully what the household's budget identity looks like and state the NPG condition that you use.

(b) Derive the aggregate consumption rule and the aggregate consumption Euler equation for this model.

(c) Show that the validity of the Ricardian equivalence theorem hinges on the birth rate and not on the existence of life insurance possibilities *per se*. Explain the intuition behind this result.

Question 4 ★

[Buiter (1988)] In this question we modify the standard Blanchard model (studied in Section 16.2.2 of the book) by assuming that: (i) the probability of death, β, is not necessarily equal to the birth rate, which we denote by $\eta \geq 0$; (ii) that there is Harrod-neutral technological change at an exogenously given rate $n_A \geq 0$. The expected lifetime utility function of a representative household of vintage v in period t is:

$$E\Lambda(v,t) = \int_t^{\infty} \log \bar{c}(v,\tau) e^{(\rho+\beta)(t-\tau)} d\tau, \tag{1}$$

where $\bar{c}(v,\tau)$ is consumption and ρ is the pure rate of time preference. We assume that $\rho > n_L$, where n_L is the growth rate of the population. The budget identity of the household is:

$$\dot{\bar{a}}(v,\tau) = [r(\tau) + \beta]\, \bar{a}(v,\tau) + \bar{w}(\tau) - \bar{z}(\tau) - \bar{c}(v,\tau), \tag{2}$$

where $r(\tau)$ is the interest rate, $\bar{w}(\tau)$ is the wage rate, $\bar{z}(\tau)$ is the lump-sum tax, $\bar{a}(v,\tau)$ are real financial assets, and $\dot{\bar{a}}(v,\tau) \equiv d\bar{a}(v,\tau)/d\tau$. The solvency condition is:

$$\lim_{\tau\to\infty} e^{-R^A(t,\tau)}\bar{a}(v,\tau) = 0, \quad R^A(t,\tau) \equiv \int_t^{\tau} [r(s) + \beta]\, ds. \tag{3}$$

The population at time t is denoted by $L(t)$ and the population at time $t = 0$ is normalized to unity ($L(0) = 1$). With a positive birth rate ($\eta > 0$), the cohort born at time v is related to the total population in existence at that time according to $L(v, v) = \eta L(v)$.

(a) Solve the optimization problem for individual households.

(b) The growth rate of the total population is given by n_L. Prove that the size of cohort v at time $t \geq v$ is equal to:

$$L(v,t) = \begin{cases} \eta e^{\eta v} e^{-\beta t} & \text{if } \eta > 0 \\ \begin{cases} e^{-\beta t} & \text{for } v = 0 \\ 0 & \text{for } v > 0 \end{cases} & \text{if } \eta = 0 \end{cases}$$

Prove furthermore that $n_L = \eta - \beta$. Explain the intuition behind the expressions.

(c) Derive the aggregate consumption rule, the aggregate consumption Euler equation, and the differential equation for aggregate human wealth for this model. Denote the aggregate variables by $C(t)$, $A(t)$, $W(t)$, $Z(t)$, and $H(t)$, etcetera.

(d) The aggregate production function is given by $Y(t) = K(t)^{\alpha} N(t)^{1-\alpha}$, where $Y(t)$ is aggregate output, $K(t)$ is the capital stock, and $N(t) \equiv A_L(t)L(t)$ is the labour input measured in efficiency units. The index of Harrod-neutral technical change grows exponentially at rate n_A, i.e. $\dot{A}_L(t)/A_L(t) = n_A > 0$. The capital stock depreciates at a constant rate δ and firms are perfectly competitive. Derive the marginal productivity conditions for labour and capital and express them in terms of the intensive-form production function, $y(t) = k(t)^{\alpha}$, where $y(t) \equiv Y(t)/N(t)$ and $k(t) \equiv K(t)/N(t)$.

(e) Assume that the government budget identity is given by $\dot{B}(\tau) = r(\tau)B(\tau) + G(\tau) - Z(\tau)$, where $B(\tau)$ is government debt and $G(\tau)$ is government consumption. Write the model derived thus far in terms of efficiency unit of labour and state the system of differential equations characterizing the economy. Use the notation $c(t) \equiv C(t)/N(t)$, $a(t) \equiv A(t)/N(t)$, $b(t) \equiv B(t)/N(t)$ etcetera.

(f) Show that the Ricardian equivalence theorem is valid when the birth rate is zero ($\eta = 0$) even if the probability of death is positive ($\beta > 0$ so that lifetimes are uncertain) or there is positive technological change ($n_A > 0$). Explain the intuition behind this result.

(g) Assume that government consumption, $G(t)$, is financed by means of lump-sum taxes, $Z(t)$, and that there is no government debt. Compute the effects on growth in consumption, output, wages, and the capital stock, of a balanced-budget increase in government consumption expressed in terms of efficiency units of labour. Illustrate your answer with the aid of a phase diagram and draw the impulse-response functions.

Question 5

[Marini and van der Ploeg (1988)] In this question we study monetary superneutrality in an overlapping generations model of the Blanchard-Yaari type. We use the model of Section 16.2.2 in the book but modify it as follows. The lifetime utility function of a household of vintage $v \leq t$ is given by:

$$E\Lambda(v,t) \equiv \int_t^{\infty} \log U(v,\tau) e^{(\rho+\beta)(t-\tau)} d\tau, \tag{1}$$

where $U(\cdot)$ is a CES sub-felicity function:

$$U(v,\tau) \equiv \left[\epsilon C(v,\tau)^{(\sigma-1)/\sigma} + (1-\epsilon) M(v,\tau)^{(\sigma-1)/\sigma}\right]^{\sigma/(\sigma-1)}, \quad 0 < \epsilon < 1, \ \sigma > 0, \tag{2}$$

where $C(v,\tau)$ and $M(v,\tau)$ denote, respectively, consumption and *real* money balances of the household at time τ. The household's budget identity is:

$$\dot{A}(v,\tau) = [r(\tau) + \beta] A(v,\tau) + W(\tau) - T(\tau) - C(v,\tau) - [r(\tau) + \pi(\tau)] M(v,\tau), \tag{3}$$

where $\pi(\tau)$ is the inflation rate. The monetary authority sets a constant rate of growth, θ, of the *nominal* money supply so that the real money supply changes according to:

$$\dot{M}(\tau) = [\theta - \pi(\tau)] M(\tau). \tag{4}$$

The production side of the model is as in the book. For simplicity we assume that the production function is Cobb-Douglas, i.e. $Y(t) = K(t)^{\alpha} L(t)^{1-\alpha}$. Labour supply is exogenous so $L(t) = 1$. We consider a closed economy, so the goods market clearing condition is given by $Y(t) = C(t) + I(t) + G(t)$.

(a) Interpret equations (3) and (4) of the model. Make sure you explain why the inflation rate appears in these equations.

(b) Use the method of two-stage budgeting to solve the optimization problem for individual households.

(c) Derive the aggregate consumption rule, the aggregate consumption Euler equation, and the differential equations for aggregate human and financial wealth for the model.

(d) State the government budget identity and the solvency condition for the government.

(e) Show that the model features *superneutrality* of money if and only if the birth/death rate is zero and the sub-felicity function is Cobb-Douglas ($\sigma = 1$). Show that, for the case with $\beta = 0$ and $\sigma = 1$, an unanticipated and permanent increase in the money growth rate leads to a discrete jump in the price level at impact but causes no further transitional dynamics. Show the phase diagram for real money balances and the impulse response functions for inflation, the price level, and the nominal money supply.

(f) Assume that $\beta > 0$, $\sigma = 1$, $B(\tau) = 0$ (for all τ), and that the government balances its budget by means of lump-sum taxes. Show that an increase in the money growth rate leads to a steady-state increase in consumption and the capital stock but causes an ambiguous effect on real money balances. Explain the intuition behind your results.

(g) Make the same assumption as in part (f) and derive the long-run effects on capital, consumption, and real money balances of a tax-financed increase in government consumption. Explain the intuition behind your results.

Question 6

[Nielsen (1994)] In this question we the effects of social security in a Blanchard-Yaari model of a small open economy with mandatory retirement. The economy under consideration is small in world financial markets and faces a constant interest rate r. There is no capital and the production function is:

$$Y(\tau) = ZL(\tau), \tag{1}$$

where Y is output, Z is an index of productivity (exogenous), and L is employment. Households are as in Section 16.2.2 of the book but there is a system of mandatory retirement which prohibits agents older than π to work. Instead such retired agents receive an untaxed pension, P, from the government. The pension system is financed by means of age-independent lump-sum taxes, T, levied on working generations. The income at time τ of a generation born at time $v \leq \tau$ is thus:

$$I(v, \tau) = \begin{cases} W(\tau) - T(\tau) & \text{for } \tau - v < \pi \\ \\ P & \text{for } \tau - v \geq \pi \end{cases} \tag{2}$$

We abstract from government consumption and government debt. The country is populated by patient agents so that $\rho < r$.

(a) Solve the household optimization problem. Explain why human wealth is age-dependent in this case.

(b) Derive the government budget constraint. Show what happens to the lump-sum tax if the retirement age is increased but the pension payment is held constant.

(c) Derive the aggregate consumption rule, the aggregate consumption Euler equation, and the differential equation for aggregate financial wealth for the model. Explain the intuition behind the terms involving the pension system.

(d) Derive the impact, transitional, and long-run effects on consumption and net foreign assets of an increase in the pension payment. Illustrate with the aid of a phase diagram and explain the intuition.

(e) Derive the impact, transitional, and long-run effects on consumption and net foreign assets of an increase in the mandatory retirement age. Illustrate with the aid of a phase diagram and explain the intuition.

Question 7

[Mourmouras & Lee (1999)] Consider the following Blanchard-Yaari model with productive public infrastructure. Individual households have the utility function (16.16) and face the lifetime budget constraint (16.19). The production function (16.30) is replaced by:

$$Y(t) = Z_0 K(t)^{1-\epsilon_L} \left[G(t)L(t)\right]^{\epsilon_L}, \tag{1}$$

where Z_0 is a time-invariant index of general technology and $G(t)$ is the quantity of productive government services (measured as a flow). There is a general output tax, t_Y, so the objective function of the firm is given by:

$$V(t) = \int_t^\infty \left[(1-t_Y)Y(\tau) - W(\tau)L(\tau) - I(\tau)\right] e^{-R(t,\tau)} d\tau. \tag{2}$$

The tax receipts are used to finance government services, i.e. $t_Y Y(t) = G(t)$. There is no government debt and lump-sum taxes are zero. Labour supply is exogenous and normalized to unity ($L(\tau) = 1$).

(a) Derive the first-order conditions for the (representative and competitive) firm's optimization problem. Show that factor payments exhaust after-tax revenues.

(b) Show that the marginal product of capital is uniquely related to Z_0 and t_Y only. Derive an expression for the marginal product of labour.

(c) Prove that there is no transitional dynamics in the model.

(d) Show that the model exhibits endogenous growth. Compute the growth rate and show that it depends negatively on the birth-death rate β. Give the intuition for this result.

(e) Characterize the maximum growth rate in this economy. Show that the growth maximizing tax rate does not depend on the birth-death rate. Provide the intuition behind this result. Is the maximum growth rate independent of the birth/death rate?

[Heijdra & Meijdam (2002)] Now consider the more realistic scenario in which $G(t)$ is the *stock* of public infrastructure (rather than the flow of services) which must be built up gradually with the aid of public investment, $I_G(t)$. The accumulation identity for public capital is:

$$\dot{G}(t) = I_G(t) - \delta_G G(t), \tag{3}$$

where δ_G is the constant rate of depreciation of public capital. The government budget constraint is now $t_Y Y(t) = I_G(t)$.

(f) ★ Show that the modified model displays nontrivial transitional dynamics. Compute the asymptotic growth rate as a function of t_Y.

(g) ★ Show that the maximum asymptotic growth rate depends on δ_G and explain the intuition behind this result.

References

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