Foundations of Modern Macroeconomics

Ben J. Heijdra
University of Groningen

April 2002

# Problem set for Chapter 15

The questions with a star (★) are difficult.

## Question 1

We extend the continuous-time RBC model of Chapter 15 by assuming more general preferences and technology. In particular, the felicity function ($\Phi(\tau)$, appearing in (15.1)) now takes the following form:

$$\Phi(\tau) \equiv \frac{U(\tau)^{1-1/\sigma_X} - 1}{1 - 1/\sigma_X}, \tag{1}$$

where $\sigma_X$ is the intertemporal substitution elasticity and $U(\tau)$ is the sub-felicity function which depends on consumption ($C(\tau)$) and leisure ($1 - L(\tau)$):

$$U(\tau) \equiv \left[\epsilon_C C(\tau)^{(\sigma_{CL}-1)/\sigma_{CL}} + (1-\epsilon_C)[1 - L(\tau)]^{(\sigma_{CL}-1)/\sigma_{CL}}\right]^{\sigma_{CL}/(\sigma_{CL}-1)}. \tag{2}$$

We assume that the production function (15.12) is replaced by a more general CES function featuring a constant elasticity of substitution, $\sigma_{KL}$:

$$F[K(\tau), L(\tau)] \equiv \left[\epsilon_L L(\tau)^{(\sigma_{KL}-1)/\sigma_{KL}} + (1-\epsilon_L) K(\tau)^{(\sigma_{KL}-1)/\sigma_{KL}}\right]^{\sigma_{KL}/(\sigma_{KL}-1)}. \tag{3}$$

The rest of the model is unchanged, i.e. the household budget identity is (15.3), the NPG condition is (15.5), and (15.13)-(15.14) and (15.16)-(15.17) are all still appropriate.

(a) Prove that the extended model incorporates the unit-elastic model studied in the text as a special case.

(b) Use the method of two-stage budgeting (which is discussed *inter alia* in Section 16.4 of the book) to solve the optimal plans of the representative households. Show that the Euler equation for *full* consumption is given by:

$$\frac{\dot{X}(\tau)}{X(\tau)} = \sigma_X \left[r(\tau) - \rho\right] + (1 - \sigma_X)\left(\frac{\dot{P}_U(\tau)}{P_U(\tau)}\right), \tag{4}$$

where $P_U(\tau)$ is the true cost-of-living index:

$$P_U(\tau) \equiv \begin{cases} \left[\epsilon_C{}^{\epsilon_C}(1-\epsilon_C)^{1-\epsilon_C}\right]^{-1} W(\tau)^{1-\epsilon_C} & \text{if } \sigma_{CL}=1 \\ \\ \left[\epsilon_C^{\sigma_{CL}} + (1-\epsilon_C)^{\sigma_{CL}} W(\tau)^{1-\sigma_{CL}}\right]^{1/(1-\sigma_{CL})} & \text{if } \sigma_{CL}\neq 1 \end{cases} \tag{5}$$

*Hint:* start by postulating that full consumption and subfelicity are related according to:

$$X(\tau) \equiv C(\tau) + W(\tau)[1-L(\tau)] = P_U(\tau)U(\tau). \tag{6}$$

(c) Derive the marginal productivity conditions for labour and capital. Relate the wage rate and the interest rate to, respectively, output per worker and output per unit of capital. Explain the role of $\sigma_{KL}$ in the factor demand equations.

(d) Show that the "great ratios" result still holds for the extended model studied here.

(e) Derive the long-run output multiplier with respect to government consumption ($dY(\infty)/dG$) and show that it does not depend on the intertemporal substitution elasticity, $\sigma_X$. Give an economic interpretation for this result.

## Question 2

[Home production in the RBC model] It is a well-known fact of life that the non-market sector is quite sizeable in advanced economies. For example, in the United States, an average married couple spends 33 percent of its discretionary time working for a wage in the market sector and 28 percent of its time working in the home. Home production activities can include things like cooking, cleaning, child care, gardening, shopping, etcetera. Similarly, the figures indicate that investment in household capital (such as consumer durables and residential structures) exceeds investment in market capital (producer durables, non-residential structures). In this question we study the effects of introducing home production into the RBC model.

The representative household has the following lifetime utility function:

$$E_t\Lambda_t \equiv E_t \sum_{\tau=t}^{\infty} \left(\frac{1}{1+\rho}\right)^{\tau-t} \Big[\epsilon_C \log C_\tau + (1-\epsilon_C)\log[1-L_\tau]\Big], \quad 0<\epsilon_C<1, \tag{1}$$

where $C_\tau$ is *composite* consumption and $1-L_\tau$ is leisure. Composite consumption itself depends on the consumption of a market-produced good ($C_\tau^M$) and a home-produced good ($C_\tau^H$):

$$C_\tau \equiv \left[\epsilon_H \left(C_\tau^H\right)^{(\sigma-1)/\sigma} + (1-\epsilon_H)\left(C_\tau^M\right)^{(\sigma-1)/\sigma}\right]^{\sigma/(\sigma-1)}, \quad 0<\epsilon_H<1, \tag{2}$$

where $\sigma \geq 0$ is the substitution elasticity between $C_\tau^H$ and $C_\tau^M$ in composite consumption. Labour time is spent either in the market sector ($L_\tau^M$) or on home production activities ($L_\tau^H$):

$$L_\tau \equiv L_\tau^M + L_\tau^H. \tag{3}$$

Home-produced goods are only used for home consumption and the technology is given by:

$$C_\tau^H = Z_\tau^H \left(L_\tau^H\right)^{\eta_L} \left(K_\tau^H\right)^{1-\eta_L}, \quad 0 < \eta_L < 1, \tag{4}$$

where $K_\tau^H$ is the stock of household capital and $Z_\tau^H$ is a stochastic productivity term affecting home production. In addition to household capital, the household also accumulates business (or market) capital, $K_\tau^M$, that it rents out to firms in the market sector. The household budget constraint can be written as:

$$C_\tau^M + K_{\tau+1}^M + K_{\tau+1}^H = W_\tau L_\tau^M + R_\tau^K K_\tau^M + (1-\delta_M)K_\tau^M + (1-\delta_H)K_\tau^H, \tag{5}$$

where $\delta_M$ and $\delta_H$ denote the depreciation rates of, respectively, business capital and household capital ($0 < \delta_M < 1$ and $0 < \delta_H < 1$). Furthermore $W_\tau$ is the real wage rate and $R_\tau^K$ is the rental rate on business capital.

(a) Interpret the household budget constraint (5). What do we assume about the substitutability of the two types of capital?

(b) Solve the household optimization problem using the methods explained in the text (viz. pages 505-506 of Chapter 15). Interpret the various expressions you obtain.

The representative firm is perfectly competitive and produces homogeneous output, $Y_\tau$, by renting business capital, $K_\tau^M$, and labour, $L_\tau^M$, from the household sector. The production function is:

$$Y_\tau \equiv Z_\tau^M \left(L_\tau^M\right)^{\epsilon_L} \left(K_\tau^M\right)^{1-\epsilon_L}, \quad 0 < \epsilon_L < 1, \tag{6}$$

where $Z_\tau^M$ is a stochastic productivity term affecting market productivity. The firm maximizes profit, $\Pi_\tau \equiv Y_\tau - W_\tau L_\tau - R_\tau^K K_\tau$.

(c) Derive the first-order conditions for the firm's optimal plans.

(d) What do we assume about the substitutability of working in the market sector and in home production? Show how you can modify the household utility function to capture the notion that working in the market is actually preferred to working around the home. [*Hint*: you may want to look at Benhabib, Rogerson, and Wright (1991)]

## References

Benhabib, J., Rogerson, R., and Wright, R. (1991). Homework in macroeconomics: Household production and aggregate fluctuations. *Journal of Political Economy*, 99:1166–1187.