Foundations of Modern Macroeconomics

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Problem set for Chapter 14

The questions with a star (★) are difficult.

Question 1

[Harrod (1939)-Domar (1946)] One of the key notions underlying the Solow-Swan model is the substitutability between capital and labour incorporated in the aggregate production function [viz. equation (14.1) or (14.6)]. Even before Solow-Swan made their contributions, Roy Harrod and Evsey Domar proposed a growth model which negates the possibility of substitution between capital and labour. They postulated the following aggregate production function:

$$Y(t) = \min\left[\frac{K(t)}{\upsilon}, \frac{L(t)}{\alpha}\right], \quad \upsilon > 0,\ \alpha > 0, \tag{1}$$

where $Y(t)$, $K(t)$, and $L(t)$ are, respectively, aggregate output, the capital stock, and employment, and the coefficients υ and α are fixed. The rest of the model is the same as in the text:

$$\begin{aligned} S(t) &= sY(t), \qquad 0 < s < 1, & (2)\\ I(t) &= \delta K(t) + \dot{K}(t), & (3)\\ I(t) &= S(t), & (4)\\ \frac{\dot{L}(t)}{L(t)} &= n_L, & (5) \end{aligned}$$

where $S(t)$, $I(t)$, s, and n_L are, respectively, aggregate saving, aggregate investment, the (constant) propensity to save, and the (constant) growth rate of the population.

(a) Draw the isoquants of the production function given in (1). Derive expressions for Y/K, Y/L, and K/L under the assumption that both production factors are fully employed. What happens if the actual K/L is less than υ/α? What if it is larger than υ/α?

(b) Show that, in order to maintain full employment of capital in the model, output and investment must grow at the so-called "warranted rate of growth" which is equal to $[s - \delta v]/v$.

(c) Show that, in order to maintain full employment of labour in the model, output and investment must grow at the so-called "natural rate of growth" which is equal to n_L.

(d) Derive the condition under which the economy grows with full employment of both factors of production. This is called the Harrod-Domar condition. What happens if $[s - \delta v]/v$ falls short of (exceeds) n_L?

(e) Show that a Harrod-Domar like condition also appears in the Solow-Swan model but that the latter model does not suffer from the instability (or knife-edge stability) of the Harrod-Domar model. Make sure that you explain the critical role of capital-labour substitution.

Question 2

Consider the Solow-Swan model discussed in the book in Section 14.2. Assume that the economy features perfectly competitive firms.

(a) Derive the phase diagram for the model without technological progress with output per worker, y, on the vertical axis and capital per worker, k, on the horizontal axis. Show that in the balanced growth path the relative output shares of capital and labour are constant. Show those shares in the diagram.

(b) Next assume that there is Harrod-neutral technological progress. Derive the expressions for the rental rate on capital and the wage rate. Show that in the balanced growth path the wage rate grows at the rate of technological progress (n_A). Demonstrate the constancy of income shares and illustrate with a diagram.

(c) Abstract from technological progress but assume that the population growth rate is not constant (as in the standard Solow-Swan model) but instead depends on economic conditions. In particular, assume that n_L is low for low levels of output per worker (y), rises quite rapidly as y exceeds some "subsistence level," and starts to slow down again as y becomes very large. Show that is possible for the model to exhibit multiple steady-state equilibria. Investigate the stability of these equilibria and explain how the model can provide a description of the theory of the "big push" (known from development economics).

Question 3

[Kaldor (1955), Pasinetti (1962), Samuelson and Modigliani (1966)] Consider the Solow-Swan model discussed in the text in Section 14.2. Assume that the economy features perfectly competitive firms and abstract from technological progress. Assume that the savings function takes the form suggested by Nicholas Kaldor:

$$S(t) = s_w W(t)L(t) + s_p \left[Y(t) - W(t)L(t)\right], \tag{1}$$

where $S(t)$, $W(t)$, $L(t)$, and $Y(t)$ are, respectively, aggregate saving, the wage rate, employment, and output. The propensity to save out of labour income is s_w, and the propensity to save out of profit income is s_p. Both these savings propensities are constant and it is assumed that $0 < s_w < s_p < 1$. The rest of the model is standard and is given by equations (14.3)-(14.6) in the book.

(a) Show that the savings function can be written as $S(t) = s(t)Y(t)$, where $s(t)$ is defined as:

$$s(t) \equiv s_w + (s_p - s_w)\omega_K(k(t)), \tag{3}$$

and where $\omega_K(\cdot) \equiv (r(t) + \delta)K(t)/Y(t)$ is the income share of capital.

(b) Derive the fundamental differential equation for output per worker. Assume that the production function is CES with substitution elasticity $0 < \sigma_{KL} \leq 1$. Demonstrate stability and uniqueness of the steady-state equilibrium. Illustrate your answer using a diagram with output per worker, y, on the vertical axis, and capital per worker, k, on the horizontal axis.

(c) Show that income shares are constant along the balanced growth path and illustrate this result in the diagram.

(d) ★ What do we mean by the "Pasinetti paradox" and how do Samuelson and Modigliani (1966) deal with it?

Question 4

Consider the standard Solow-Swan model in which the population, $L(t)$, grows at a constant exponential rate ($\dot{L}(t)/L(t) = n_L$). Abstract from technological progress and assume that the labour force *participation rate* is a function of the real wage rate, $W(t)$, according to:

$$p(W(t)) = \frac{N(t)}{L(t)}, \tag{1}$$

where $N(t)$ is employment. Assume that the production function is Cobb-Douglas:

$$Y(t) = K(t)^{\alpha} N(t)^{1-\alpha}, \tag{2}$$

with $0 < \alpha < 1$.

(a) Develop the fundamental differential equation for the per capita capital stock ($k \equiv K/L$) and show that it depends on the elasticity of the participation rate with respect to the wage (η_{pW}) and on the elasticity of wages with respect to per capita capital (η_{Wk}).

(b) What are the likely signs of η_{pW} and η_{Wk}? Explain intuitively.

(c) Explain both formally and intuitively what the effect of an endogenous participation rate is on the adjustment speed of the economy.

Question 5

Assume that the production function, $Y = F(K, L)$, satisfies assumptions (P1)-(P3) stated in Section 14.2.1 in the book. Define the per-worker production function, $f(k)$, as in equation (14.8).

(a) Show that the marginal products of capital and labour (F_K and F_L, respectively) can be written in terms of the per-worker production function.

(b) Prove that the per-worker production function has the following properties:

$$\begin{aligned} f'(k) &\geq 0 \\ \lim_{k\to 0} f'(k) &= +\infty \\ \lim_{k\to\infty} f'(k) &= 0. \end{aligned}$$

(c) Assume that the production factors receive their respective marginal products. Derive the expressions for the wage rate, W, and the rental rate on capital, $r + \delta$, when technology is Cobb-Douglas. What happens to (W, r) as $k \to 0$ and as $k \to \infty$? Derive the expression for the *factor price frontier* (FPF), i.e. the expression linking W and r, and illustrate it graphically. Show what happens to the FPF if general productivity increases.

Question 6

[Burmeister and Dobell (1970)] Assume that the factors of production are paid according to their respective marginal products, i.e. $W = F_L(K, L)$ and $r + \delta = F_K(K, L)$ as in equation (14.73) in the text. Abstract from technological progress.

(a) The wage-rental ratio, ω, is defined as follows: $\omega \equiv W/[r+\delta]$. Show that ω can be written as a function of k only, i.e. $\omega = \omega(k)$. Show that this function can be inverted to yield k as a function of ω. Denote this function by $k = \xi(\omega)$.

(b) Identify the wage-rental ratio in a diagram with k on the horizontal axis and y and $W(k)$ on the vertical axis. (See also Question 2(a) above for a similar diagram.)

(c) The elasticity of substitution in production is defined as:

$$\sigma_{KL} \equiv \frac{F_L F_K}{F(K,L) F_{LK}}. \tag{1}$$

Show that the *elasticity* of the $\xi(\omega)$ function, defined as $\frac{dk}{d\omega}\frac{\omega}{k}$, is equal to σ_{KL}.

Question 7

[Intriligator (1971)] In the neoclassical model, under conditions of competition, in equilibrium the wage equals the marginal product of labour, i.e. $W^*(t) \equiv f(k(t)) - k(t)f(k(t))$. In disequilibrium, with sticky real wages, we assume that $W(t)$ is adjusted toward the equilibrium according to:

$$\dot{W}(t) = \psi\left[W(t) - W^*(t)\right], \tag{1}$$

with $\psi(0) = 0$ and $\psi' < 0$. We abstract from technological change and assume that the population grows at a constant exponential rate, n_L. Assume that the Inada conditions are satisfied.

(a) Assume that the workers consume their entire wage income. The capitalists have a constant propensity to save s_C (with $0 < s_C < 1$). Derive the fundamental differential equation for capital per worker.

(b) Show in the $(k(t), W(t))$ plane the equilibrium levels of $k(t)$ and $W(t)$. Assume for convenience that technology is Cobb-Douglas, i.e. $f(k) = k^\alpha$ with $0 < \alpha < 1$..

(c) Show several possible paths toward this equilibrium following an outbreak of the plague in which much of the labour force is destroyed but the capital stock remains intact. Under what conditions will the paths eventually move toward equilibrium rather than spiralling around it?

Question 8

[Cass (1965), Koopmans (1967)] In Section A.8 of the Mathematical Appendix we introduce some key results from the theory of optimal control. This theory is very useful to solve dynamic optimization problems, such as the ones faced by the representative household (in Section 14.5.1) or the social planner (in Section 14.5.4).

(a) Derive the consumption Euler equation (14.63) by means of the *Hamiltonian method* explained in the Mathematical Appendix. Denote the co-state variable by $\mu(t)$. Which variable is the control variable? Which one is the state variable?

(b) Develop the phase diagram for the Ramsey model with $\mu(t)$ on the vertical and $k(t)$ on the horizontal axis (rather than in the (c, k) space, as in Figure 14.8).

In the text we assume that the representative household has a lifetime utility function as in (14.53). Assume now that the household has a slightly different lifetime utility function:

$$\Lambda(0) \equiv \int_0^\infty L(t)U[c(t)]e^{-\rho t}dt, \quad \rho > 0, \tag{1}$$

where $L(t)$ is the size of the population [i.e. the size of the dynastic family]. Assume a constant population growth rate, i.e. $\dot{L}(t)/L(t) = n$.

(c) Compare and contrast (14.53) and (1). Derive the consumption Euler equation for the modified model.

Question 9

[Intriligator (1971)] Consider the standard neoclassical growth model without technological change. Assume that consumption per worker is fixed (at $c(t) = \bar{c}$), that capital does not depreciate ($\delta = 0$), and that the population grows at a constant exponential rate n_L.

(a) Show that the growth rate of the capital stock, $\gamma_K(t) \equiv \dot{K}(t)/K(t)$, is maximized when this growth rate equals the marginal product of capital, i.e. when $\gamma_K(t)$ equals the interest rate.

(b) Illustrate your result in part (a) graphically. Assume for convenience that technology is Cobb-Douglas. *Hint*: express $\gamma_K(t)$ as a function of the capital-labour ratio $k(t)$.

(c) Generalize your result to the case with a positive depreciation rate ($\delta > 0$).

Question 10

[Ramsey (1928), Intriligator (1971)] In the original treatment of the problem of optimal economic growth, Frank Ramsey argued on the basis of ethical beliefs that there should be no discounting of future felicity ($\rho = 0$). Since the welfare integral will then not generally converge, Ramsey suggested a different approach. He assumed that there is a finite upper limit for either the production function or the felicity function, in either case leading to a finite upper limit to utility called *bliss*, B:

$$B \equiv \max_{\{c\}} U(c) = U(c_B), \tag{1}$$

where c_B is the bliss consumption per worker which is assumed to be finite. He then postulated the following (undiscounted) objective function that is to be *minimized*:

$$R \equiv \int_0^\infty [B - U(c(t))] \, dt, \tag{2}$$

where R is a measure of "regret" (i.e. the social cost associated with deviating from the bliss point). Solve the Ramsey problem of minimizing regret subject to the neoclassical growth model. Assume that there is no technological change and that the population is constant. Illustrate your answer with the aid of a diagram and show that the model is saddle-point stable.

Question 11 ★

[Kurz (1968)] One of the objections that has been raised against the Solow-Swan model concerns the *ad hoc* nature of the savings function. In the so-called "inverse optimum" problem we try to determine the class of household objective functions which will in fact yield the Solow-Swan savings function as an optimal policy rule. In this question we study this inverse optimum problem in detail. We consider the following model:

$$\dot{k}(t) = f(k(t)) - c(t) - (\delta + n_L) k(t), \tag{1}$$

$$c(t) = (1 - s) y(t), \qquad 0 < s < \alpha, \tag{2}$$

$$y(t) = Ak(t)^\alpha, \qquad 0 < \alpha < 1, \tag{3}$$

where $k(t)$, $y(t)$, and $c(t)$ are, respectively, capital, output, and consumption per worker. Capital depreciates at a constant rate, δ, and the population grows at an exponential rate, n_L. The savings rate, s, is constant. The economy is perfectly competitive so the usual marginal productivity conditions for the production factors are valid. Household behaviour is described by the Ramsey model of Section 14.5.

(a) Show that equation (1) is a rewritten version of equation (14.55) in the book.

(b) Show that in the steady state, the various parameters are related according to:

$$\frac{\delta + n_L}{s} = \frac{\rho + \delta + n_L}{\alpha}, \tag{4}$$

where ρ is the rate of time preference.

(c) Show that the only solution for the inverse optimum problem is the family of utility functions featuring a constant elasticity of substitution. Show also that $\sigma = s$ and explain why we need the condition $s < \alpha$ for this result to make sense.

Question 12 ★

In the book we develop and solve the Ramsey model in terms of its implications for consumption and capital per worker. In this question we study the Ramsey model in terms of its implications for capital per worker and the savings ratio. Recall that the savings ratio, $s(t)$, is the proportion of income that is saved and invested:

$$s(t) \equiv \frac{S(t)}{Y(t)} = \frac{I(t)}{Y(t)} = \frac{\dot{K}(t) + \delta K(t)}{Y(t)}. \tag{1}$$

It follows from (1) that consumption per worker is:

$$c(t) = [1 - s(t)]\, y(t), \tag{2}$$

where $y(t) = f(k(t))$ is the intensive-form production function. The fundamental differential equation for the capital stock is:

$$\dot{k}(t) = s(t) f(k(t)) - (\delta + n_L)\, k(t). \tag{3}$$

Assume that the technology is Cobb-Douglas, i.e. $y(t) = Ak(t)^{\alpha}$ with $0 < \alpha < 1$, and that the felicity function is iso-elastic with intertemporal substitution elasticity σ. Abstract from technological change.

(a) Solve the optimization problem in terms of the savings rate and the capital stock per worker.

(b) Derive the fundamental differential equations for k and s.

(c) Denote the steady-state value for the savings rate by s^*. Illustrate the phase diagram for the three possible cases, namely case (1) $s^* = \sigma$, case (2) $s^* > \sigma$, and case (3) $s^* < \sigma$. Explain the economic intuition behind the dynamic adjustment for all three cases.

(d) Now solve for the optimal time path for the saving rate if the production function is $f(k(t)) = Ak(t)$.

Question 13

[Intriligator (1971)] Consider the neoclassical growth model with technological progress and assume that the aggregate production function is given by:

$$Y(t) = A_P(t) F\left[A_K(t)K(t), A_L(t)L(t)\right], \tag{1}$$

where $A_P(t) \equiv e^{n_H t}$ summarizes "product augmenting" technical changes, $A_K(t) \equiv e^{n_S t}$ summarizes "capital augmenting" technical change, and $A_L(t) \equiv e^{n_A t}$ summarizes "labour augmenting" technical change. Assume that the representative household has the lifetime utility function (14.53) and faces the budget identity (14.54) and an appropriately defined NPG condition.

(a) Show that the only technical progress consistent with a balanced growth equilibrium is purely labour augmenting ("Harrod neutral") technical change.

(b) Develop the solution to the household optimization problem in the case of Harrod neutral technological change, where $n_A > 0$ and $n_S = n_H = 0$. Assume that the utility function features a constant intertemporal substitution elasticity, σ. *Hint*: recall that the wage rate grows exponentially in the balanced growth path.

(c) Develop the solution to the neoclassical problem of optimal economic growth for purely product augmenting ("Hicks neutral") technical change, where $A_P(t) = e^{n_A t}$ and $A_L(t) = A_K(t) = 1$.

(d) Develop the solution to the neoclassical problem of optimal economic growth for purely capital augmenting ("Solow neutral") technical change, where $A_K(t) = e^{n_A t}$ and $A_P(t) = A_L(t) = 1$.

Question 14

[Constant marginal utility] In the text we focus attention on the case in which household utility features a finite intertemporal substitution elasticity. As an extension we now study the case for which marginal utility is constant. The lifetime utility function (14.53) is replaced by:

$$\Lambda(0) \equiv \int_0^\infty c(t)e^{-\rho t}dt, \tag{1}$$

where $c(t)$ is consumption per worker. We study the social planning solution to the household optimization problem. The fundamental differential equation for the capital stock per worker, $k(t)$, is:

$$\dot{k}(t) = f(k(t)) - c(t) - (\delta + n_L)\, k(t), \tag{2}$$

where $y(t) = f(k(t))$ is output per worker. The production function satisfies the Inada conditions and there is no technological progress. The solution for consumption must satisfy the following constraints:

$$\bar{c} \leq c(t) \leq f(k(t)), \tag{3}$$

where $\bar{c}$ is some minimum consumption level (it is assumed that $0 < \bar{c} < c^{GR}$, where c^{GR} is the maximum attainable "golden rule" consumption level).

(a) Set up the appropriate current-value Hamiltonian and derive the first-order conditions. Show that the solution for consumption is a so-called "bang-bang" solution:

$$c(t) = \begin{cases} \bar{c} & \text{for } \mu(t) > 1 \\ \text{free} & \text{for } \mu(t) = 1 \\ f(k(t)) & \text{for } \mu(t) < 1 \end{cases}, \tag{4}$$

where $\mu(t)$ is the co-state variable.

(b) Derive the phase diagram for the model and show that there exists a unique saddle point-stable equilibrium. Show that this equilibrium is in fact reached provided the initial capital stock per worker lies in the interval (k_L, k_U).

Question 15

[Rebelo (1992)] Consider a simple model of endogenous growth. The representative household has the following life-time utility function:

$$\Lambda(t) \equiv \int_t^{\infty} \left[\frac{\left[C(\tau) - \bar{C}\right]^{1-1/\sigma} - 1}{1 - 1/\sigma} \right] e^{\rho(t-\tau)} d\tau, \tag{1}$$

where $\bar{C}$ denotes the subsistence (or minimum) level of private consumption. The production function displays constant returns to scale with respect to a very broad measure of capital, i.e. $Y(t) = AK(t)$. Ignore technological change and assume a constant population. Assume furthermore that $r > \rho$ and $(1-\sigma)r + \sigma\rho > 0$ where $r \equiv A - \delta$.

(a) Derive an expression for the intertemporal substitution elasticity and show that it depends on $\bar{C}$. Explain the intuition.

(b) Derive an expression for the growth rate of the economy, both in the short run and in the long run. Show that poor countries grow at a slower rate than rich countries do.

(c) Consider a Ramsey model of classical growth with the same preferences as before, but with a Cobb-Douglas production function and thus decreasing returns to capital, i.e. $Y(t) = F(K(t)) \equiv AK(t)^{\alpha}$. Derive an expression for the growth rate of the economy, both in the short run and in the long run. Is it now possible for poor countries to grow faster than rich countries and catch up?

Notes:

See Solow (2000) book for two further questions on pp.114-116 and 127-133.

References

Burmeister, E. and Dobell, A. R. (1970). *Mathematical Theories of Economic Growth*. Macmillan, London.

Cass, D. (1965). Optimum growth in an aggregative model of capital accumulation. *Review of Economic Studies*, 32:233–240.

Domar, E. D. (1949). Capital expansion, rate of growth, and employment. *Econometrica*, 14:137–147.

Harrod, R. F. (1939). An essay in dynamic theory. *Economic Journal*, 49:14–33.

Intriligator, M. D. (1971). *Mathematical Optimization and Economic Theory*. Prentice Hall, Englewood Cliffs, NJ.

Kaldor, N. (1955). Alternative theories of distribution. *Review of Economic Studies*, 23:83–100.

Koopmans, T. C. (1967). Objectives, constraints, and outcomes in optimal growth models. *Econometrica*, 35:1–15.

Pasinetti, L. L. (1962). Rate of profit and income distribution in relation to the rate of economic growth. *Review of Economic Studies*, 29:267–279.

Samuelson, P. A. and Modigliani, F. (1966). The Pasinetti paradox in neoclassical and more general models. *Review of Economic Studies*, 33:269–301.

Solow, R. M. (2000). *Growth Theory: An Exposition*. Oxford University Press, Oxford, second edition.