ENVIRONMENTAL TAX POLICY AND INTERGENERATIONAL DISTRIBUTION: MATHEMATICAL APPENDIX

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July 1996

Rev. January 1997

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A.1. Model solution and stability

The model in Table 1 in the text can be used to derive the following propositions.

PROPOSITION A.1: *There exists a stable long-run equilibrium to* (T1)-(T7). *This equilibrium is dynamically efficient. The equilibrium interest rate satisfies* $\rho<r^{BY}<\rho+\lambda$, *and depends on the fundamental parameters according to:* $\mathrm{d}r^{BY}/\mathrm{d}\lambda>0$, $\mathrm{d}r^{BY}/\mathrm{d}\rho>0$, $\mathrm{d}r^{BY}/\mathrm{d}t_K<0$, *and* $\mathrm{d}r^{BY}/\mathrm{d}\varepsilon_L>0$.

PROPOSITION A.2: *The log-linearized model of Table A.1 has characteristic roots* $r^*>0$ *and* $-h^*<0$. *If* $t_K<r\varepsilon_L/(\lambda(1-\varepsilon_L))$, *the unstable root satisfies* $r^*>\lambda+r/(1-t_K)$ *and the stable root satisfies* $-h^*<r-(\rho+\lambda)$.

A.1.1. Construction of Figure A.1 and proof of Proposition A.1.

We first construct the phase diagram. Dropping time indices, the steady-state of the model in Table 1 can be written as:

$$C = \gamma_0 K^{1-\varepsilon_L}, \quad \dot{K}=0 \text{ line},$$

$$C = \frac{\lambda(\rho+\lambda)\left[K+B\right]}{r(K)-\rho}, \quad \dot{C}=0 \text{ line},$$

where we write $r(K)\equiv\gamma_0(1-t_K)(1-\varepsilon_L)K^{-\varepsilon_L}$, from which the following results can be derived:

$$r(K)\geq 0, \quad \lim_{K\to 0} r(K)=\infty, \quad \lim_{K\to\infty} r(K)=0,$$

$$r_K(K)\equiv -\varepsilon_L r(K)/K\leq 0, \quad \lim_{K\to 0} r_K(K)=\infty, \quad \lim_{K\to\infty} r_K(K)=0.$$

The $\dot{K}$=0 line is concave towards the origin (as drawn in Figure A.1):

$$\left(\frac{\mathrm{d}C}{\mathrm{d}K}\right)_{\dot{K}=0} = \frac{r(K)}{1-t_K}, \quad \lim_{K\to 0}\left(\frac{\mathrm{d}C}{\mathrm{d}K}\right)_{\dot{K}=0}=\infty, \quad \lim_{K\to\infty}\left(\frac{\mathrm{d}C}{\mathrm{d}K}\right)_{\dot{K}=0}=0, \quad \left(\frac{\mathrm{d}^2C}{\mathrm{d}K^2}\right)_{\dot{K}=0}=\frac{r_K(K)}{1-t_K}<0.$$

The slope of the $\dot{C}$=0 line can be written as:

$$\left(\frac{\mathrm{d}C}{\mathrm{d}K}\right)_{\dot{C}=0} = \frac{\lambda(\rho+\lambda)\left[r(K)-\rho-r_K(K)\left[K+B\right]\right]}{\left[r(K)-\rho\right]^2}.$$

The Keynes-Ramsey capital stock, K^{KR}, is implicitly defined by $r(K^{KR})=\rho$. Hence, the $\dot{C}$=0 line has a vertical asymptote at $K=K^{KR}$, is horizontal at K=0, and is upward sloping for $0<K<K^{KR}$:

$$\left(\frac{\mathrm{d}C}{\mathrm{d}K}\right)_{\dot{C}=0} \geq 0, \quad \lim_{K\to 0}\left(\frac{\mathrm{d}C}{\mathrm{d}K}\right)_{\dot{C}=0} = 0, \quad \lim_{K\to K^{KR}}\left(\frac{\mathrm{d}C}{\mathrm{d}K}\right)_{\dot{C}=0} = \infty,$$

$$\left(\frac{\mathrm{d}^2C}{\mathrm{d}K^2}\right)_{\dot{C}=0} = -\frac{\lambda(\lambda+\rho)(r(K)-\rho)\left[2r_K(K)(r(K)-\rho)+\left[(r(K)-\rho)r_{KK}(K)-2\left(r_K(K)\right)^2\right](K+B)\right]}{\left[r(K)-\rho\right]^4} > 0,$$

where $r_{KK}(K)\equiv\varepsilon_L(1+\varepsilon_L)r(K)/K^2>0$, and we have used the fact that $(r(K)-\rho)r_{KK}(K)-2r_K^2(K)<0$. Hence, the $\dot{C}$=0 line is convex towards the origin as drawn in Figure A.1. There is a unique non-trivial equilibrium at point E_0 which lies to the left of the Keynes-Ramsey point A.

A very simple expression for the equilibrium output-capital ratio can be computed by equating the $\dot{C}$=0 and $\dot{K}$=0 lines:

$$\left[r(K)-\rho\right]Y = \lambda(\rho+\lambda)\left[K+B\right] \iff (1-t_K)(1-\varepsilon_L)\left(\frac{Y}{K}\right)-\rho = \lambda(\rho+\lambda)\left[\frac{K}{Y}+\frac{B}{Y}\right] \Rightarrow$$

$$(1-t_K)(1-\varepsilon_L)\left(\frac{Y}{K}\right)^2 - \left[\rho+\lambda(\rho+\lambda)\,\omega_B\right]\left(\frac{Y}{K}\right) - \lambda(\rho+\lambda) = 0.$$

where $\omega_B\equiv B/Y$. Taking the positive root of this quadratic equation yields the expression for the output-capital ratio associated with the Yaari-Blanchard equilibrium.

$$\left(\frac{Y}{K}\right)^{BY} = \frac{\rho+\lambda(\rho+\lambda)\,\omega_B+\left[\left[\rho+\lambda(\rho+\lambda)\,\omega_B\right]^2+4\lambda(\rho+\lambda)(1-t_K)(1-\varepsilon_L)\right]^{\frac{1}{2}}}{2(1-t_K)(1-\varepsilon_L)}.$$

Obviously, since $r\equiv(1-t_K)(1-\varepsilon_L)(Y/K)$, we also have the expression for the equilibrium interest rate:

$$r^{BY} = \tfrac{1}{2}\left[\rho+\lambda(\rho+\lambda)\,\omega_B+\left[\left[\rho+\lambda(\rho+\lambda)\,\omega_B\right]^2+4\lambda(\rho+\lambda)(1-t_K)(1-\varepsilon_L)\right]^{\frac{1}{2}}\right].$$

Throughout the paper (and in the remainder of this appendix) we assume that the initial debt is zero (B=0), and r^{BY} is simplified to:

$$r^{BY} = \tfrac{1}{2}\left[\rho+\left[\rho^2+4\lambda(\rho+\lambda)(1-t_K)(1-\varepsilon_L)\right]^{\frac{1}{2}}\right], \qquad \text{(for } B=0\text{)}. \tag{A1}$$

Since $r^{BY}=\rho+\lambda(1-\omega_H)$, where $\omega_H\equiv H/(H+K)$, it follows that $\rho<r^{BY}<\rho+\lambda$ because $0<\omega_H<1$. This last inequality can be deduced from the fact that $(r^{BY}+\lambda)(H/K)^{BY} \equiv(W^N/K)^{BY}=[\varepsilon_L+t_K(1-\varepsilon_L)](Y/K)^{BY}$ implies that $(H/K)^{BY}>0$ and hence $H^{BY}>0$. Indeed, it is straightforward to show that $\rho<r^{BY}<\rho+\lambda$ implies the following inequality for 1-ω_H:

$$(1-t_K)(1-\varepsilon_L) < 1-\omega_H < (1-t_K)(1-\varepsilon_L)(1+\lambda/\rho).$$

Equation (A1) can be used to derive the following results for the steady-state interest rate.

$$0 < \frac{\mathrm{d}r^{BY}}{\mathrm{d}\rho} = \frac{r^{BY}+\lambda(1-t_K)(1-\varepsilon_L)}{2r^{BY}-\rho} = \frac{r^{BY}+\lambda(1-t_K)(1-\varepsilon_L)}{r^{BY}+\lambda(1-\omega_H)} < 1, \tag{A2}$$

$$\frac{\mathrm{d}r^{BY}}{\mathrm{d}\lambda} = \frac{(1-t_K)(1-\varepsilon_L)(\rho+2\lambda)}{2r^{BY}-\rho} > 1-\omega_H > 0, \tag{A3}$$

$$\frac{\mathrm{d}r^{BY}}{\mathrm{d}t_K} = -\frac{\lambda(\rho+\lambda)(1-\varepsilon_L)}{2r^{BY}-\rho} < 0, \tag{A4}$$

$$\frac{\mathrm{d}r^{BY}}{\mathrm{d}\varepsilon_L} = -\frac{\lambda(\rho+\lambda)(1-t_K)}{2r^{BY}-\rho} < 0, \tag{A5}$$

where we have also used the fact that $(\lambda+\rho)(K/Y)^{BY} \equiv 1-\omega_H > (1-t_K)(1-\varepsilon_L)$ because $r^{BY} < \lambda+\rho$. The proof for $\mathrm{d}r^{BY}/\mathrm{d}\lambda$ runs as follows:

$$\frac{\mathrm{d}r^{BY}}{\mathrm{d}\lambda} - (1-\omega_H) = \frac{(1-t_K)(1-\varepsilon_L)(\rho+2\lambda)}{2r^{BY}-\rho} - (1-\omega_H)$$

$$= \left(\frac{K}{Y}\right)^{BY}\left[\left(\frac{\rho+2\lambda}{r^{BY}+\lambda(1-\omega_H)}\right)(1-t_K)(1-\varepsilon_L)\left(\frac{Y}{K}\right)^{BY} - (\rho+\lambda)\right]$$

$$= \left(\frac{K}{Y}\right)^{BY}\left[\frac{(\rho+2\lambda)r^{BY} - (\rho+\lambda)\left[r^{BY}+\lambda(1-\omega_H)\right]}{r^{BY}+\lambda(1-\omega_H)}\right] = \left(\frac{K}{Y}\right)^{BY}\left(\frac{\lambda\rho\omega_H}{r^{BY}+\lambda(1-\omega_H)}\right) > 0.$$

Since $1-\omega_H \equiv (r^{BY}-\rho)/\lambda$, it is also straightforward to derive the results for the steady-state share of human wealth:

$$\frac{\mathrm{d}\omega_H}{\mathrm{d}\lambda} = \frac{1}{\lambda}\left[1-\omega_H-\frac{\mathrm{d}r^{BY}}{\mathrm{d}\lambda}\right] < 0, \quad \frac{\mathrm{d}\omega_H}{\mathrm{d}\rho} = \frac{1}{\lambda}\left[1-\frac{\mathrm{d}r^{BY}}{\mathrm{d}\rho}\right] > 0,$$
$$\frac{\mathrm{d}\omega_H}{\mathrm{d}t_K} = -\frac{1}{\lambda}\frac{\mathrm{d}r^{BY}}{\mathrm{d}t_K} > 0, \quad \frac{\mathrm{d}\omega_H}{\mathrm{d}\varepsilon_L} = -\frac{1}{\lambda}\frac{\mathrm{d}r^{BY}}{\mathrm{d}\varepsilon_L} > 0. \tag{A6}$$

The linearized model given in Table A.1 can be written in terms of a single matrix equation:

$$\begin{bmatrix} \dot{\tilde{C}}(t) \\ \dot{\tilde{K}}(t) \end{bmatrix} = \begin{bmatrix} r-\rho & -\left(r\varepsilon_L + r - \rho\right) \\ -\dfrac{r}{(1-t_K)(1-\varepsilon_L)} & \dfrac{r}{1-t_K} \end{bmatrix} \begin{bmatrix} \tilde{C}(t) \\ \tilde{K}(t) \end{bmatrix} + \begin{bmatrix} -r & -\dfrac{r-\rho}{(1-t_K)(1-\varepsilon_L)} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{t}_K(t) \\ \tilde{B}(t) \end{bmatrix}, \tag{A7}$$

where $r \equiv r^{BY}$, $\tilde{C}(t) \equiv dC(t)/C$, $\tilde{K}(t) \equiv dK(t)/K$, $\dot{\tilde{C}}(t) \equiv \dot{C}(t)/C$, $\dot{\tilde{K}}(t) \equiv \dot{K}(t)/K$, $\tilde{B}(t) \equiv r dB(t)/Y$, and $\tilde{t}_K(t) \equiv dt_K(t)/(1-t_K)$ (and we have used $d\dot{C}(t) = \dot{C}(t)$ and $d\dot{K}(t) = \dot{K}(t)$). The 2 by 2 matrix of coefficients for the endogenous variables is denoted by Δ (with typical element δ_{ij}) and the 2 by 2 matrix of coefficients for the exogenous variables is represented by Γ (with nonzero elements $\gamma_K \equiv -r$, $\gamma_B \equiv -(r-\rho)/(1-t_K)(1-\varepsilon_L)$.

We let r^* and $-h^*$ stand for the characteristic roots of Δ, and we wish to show that the equilibrium E_0 in Figure A.1 is saddle-point stable, *i.e.* that $r^*>0$ and $h^*>0$. The characteristic roots are equal to:

$$r^* = \frac{\mathrm{tr}(\Delta)}{2}\left[1 + \left(1 - \frac{4|\Delta|}{\mathrm{tr}(\Delta)^2}\right)^{1/2}\right] > \mathrm{tr}(\Delta), \qquad h^* = -\frac{\mathrm{tr}(\Delta)}{2}\left[1 - \left(1 - \frac{4|\Delta|}{\mathrm{tr}(\Delta)^2}\right)^{1/2}\right]. \tag{A8}$$

The adjustment speed of the economy is represented by h^*. The proof of saddle-point stability proceeds as follows. Recall that $|\Delta| = -r^*h^*$ and $\mathrm{tr}(\Delta) = r^* - h^*$. The determinant of Δ can be written as follows:

$$|\Delta| \equiv -r^*h^* = -\frac{r\varepsilon_L\left[r-\rho+r\right]}{(1-t_K)(1-\varepsilon_L)} < 0. \tag{A9}$$

Hence, the roots alternate in sign, and $r^*>0$ and $h^*>0$. This completes the proof of Proposition A.1.□

A.1.2. Proof of Proposition A.2

In order to prove the inequalities concerning the characteristic roots in part (ii), we define $f(s) \equiv |sI-\Delta|$.[1] Obviously, since the system is saddle-point stable, $f(s)$ is a quadratic function with roots $s_1=-h^*<0$ and $s_2=r^*>0$, and $f(0)=|\Delta|<0$. To prove the inequality for h^* ($>\rho+\lambda-r$), all we need to show is that $f(\bar{s})<0$ for $\bar{s}\equiv r-(\rho+\lambda)$. By simple substitutions we obtain:

$$\begin{aligned} f(\bar{s}) &= \lambda\left[\frac{r t_K}{1-t_K}+\rho+\lambda\right] - \frac{r}{(1-t_K)(1-\varepsilon_L)}\left[r\varepsilon_L + r - \rho\right] \\ &= \lambda\left[\frac{r t_K}{1-t_K}+\rho+\lambda\right] - \frac{\rho+\lambda}{1-\omega_H}\left[r\varepsilon_L+\lambda(1-\omega_H)\right] = \frac{r}{1-t_K}\left[\lambda t_K - \frac{r\varepsilon_L}{1-\varepsilon_L}\right], \end{aligned} \tag{A10}$$

where we have used the fact that $r-\rho=\lambda(1-\omega_H)$ and $r/[(1-t_K)(1-\varepsilon_L)]=(\rho+\lambda)/(1-\omega_H)$. The necessary and sufficient condition for $f(\bar{s})<0$ is easily derived from (A10):

$$\frac{t_K}{1-t_K} < \frac{r\varepsilon_L}{\lambda(1-t_K)(1-\varepsilon_L)} \quad \Leftrightarrow \quad t_K < \frac{r\varepsilon_L}{\lambda(1-\varepsilon_L)}. \tag{A11}$$

It is clear from (A11) that this condition is automatically satisfied if $\lambda=0$ (infinite horizons) and if $t_K=0$ initially. Since $r^*=h^*+\text{tr}(\Delta)$, the inequality for r^* follows directly from the trace condition:

$$r^* = h^* + r - \rho + \frac{r}{1-t_K} > \rho+\lambda-r+r-\rho+\frac{r}{1-t_K} = \lambda+\frac{r}{1-t_K}. \tag{A12}$$

This concludes the proof of Proposition A.2.□

Of course, since $r^*=h^*+\text{tr}(\Delta)$, the unstable root exceeds the tax-adjusted rate of interest, *i.e.*, $r^*>r/(1-t_K)$ and *a fortiori* $r^*>r-\rho$.

A.1.3. Model Solution

By taking the Laplace transform of (A7) and using

$$\mathfrak{L}\{\dot{\tilde{C}},s\} = s\mathfrak{L}\{\tilde{C},s\} - \tilde{C}(0) \quad \text{and} \quad \mathfrak{L}\{\dot{\tilde{K}},s\} = s\mathfrak{L}\{\tilde{K},s\},$$

we obtain the following expression:

[1] The form of this proof was suggested by D.P. Broer of Erasmus University.

$$(s\mathrm{I} - \Delta)\begin{bmatrix}\mathfrak{L}\{\tilde{C},s\} \\ \mathfrak{L}\{\tilde{K},s\}\end{bmatrix} = \begin{bmatrix}\tilde{C}(0) + \gamma_K \mathfrak{L}\{\tilde{t}_K,s\} + \gamma_B \mathfrak{L}\{\tilde{B},s\} \\ 0\end{bmatrix}. \tag{A13}$$

Define $\mathrm{A}(s) \equiv s\mathrm{I}-\Delta$, so that $|\mathrm{A}(s)| \equiv (s-r^*)(s+h^*)$. By pre-multiplying (A13) by $\mathrm{adj}(\mathrm{A}(r^*))$, we arrive at the initial condition for the jump in consumption:

$$\mathrm{adj}[\mathrm{A}(r^*)]\,\mathrm{A}(r^*)\begin{bmatrix}\mathfrak{L}\{\tilde{C},r^*\} \\ \mathfrak{L}\{\tilde{K},r^*\}\end{bmatrix} =$$

$$\begin{bmatrix} r^*-\delta_{22} & \delta_{12} \\ \delta_{21} & r^*-\delta_{11}\end{bmatrix}\begin{bmatrix}\tilde{C}(0) + \gamma_K \mathfrak{L}\{\tilde{t}_K,r^*\} + \gamma_B \mathfrak{L}\{\tilde{B},r^*\} \\ 0\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix}. \tag{A14}$$

Since the characteristic roots of Δ are distinct, $\mathrm{rank}(\mathrm{adj}(\mathrm{A}(r^*)))=1$ and there is exactly *one* independent equation determining the jump in consumption, $\tilde{C}(0)$. Hence, either row of (A14) may be used to find $\tilde{C}(0)$:

$$(r^*-\delta_{22})\left[\gamma_K \mathfrak{L}\{\tilde{t}_K,r^*\} + \gamma_B \mathfrak{L}\{\tilde{B},r^*\} + \tilde{C}(0)\right] = 0, \tag{A15}$$

$$\delta_{21}\left[\gamma_K \mathfrak{L}\{\tilde{t}_K,r^*\} + \gamma_B \mathfrak{L}\{\tilde{B},r^*\} + \tilde{C}(0)\right] = 0, \tag{A16}$$

Using either (A15) or (A16) to eliminate $\tilde{C}(0)$ from (A13), we arrive at the general perfect foresight solution of the model in terms of Laplace transforms. Consider the first row of (A13) in combination with (A15). After some simplification it can be written as follows:

$$\begin{aligned}(s+h^*)\mathfrak{L}\{\tilde{C},s\} &= \tilde{C}(0) + \left[\gamma_K \mathfrak{L}\{\tilde{t}_K,s\} + \gamma_B \mathfrak{L}\{\tilde{B},s\}\right] \\ &+ (r^*-\delta_{22})\left[\gamma_K\left[\frac{\mathfrak{L}\{\tilde{t}_K,s\} - \mathfrak{L}\{\tilde{t}_K,r^*\}}{s-r^*}\right] + \gamma_B\left[\frac{\mathfrak{L}\{\tilde{B},s\} - \mathfrak{L}\{\tilde{B},r^*\}}{s-r^*}\right]\right].\end{aligned} \tag{A17}$$

The second row of (A13) can be combined with (A16), after which the following expression is obtained:

$$(s+h^*)\mathfrak{L}\{\tilde{K},s\} = \delta_{21}\left[\gamma_K\left[\frac{\mathfrak{L}\{\tilde{t}_K,s\} - \mathfrak{L}\{\tilde{t}_K,r^*\}}{s-r^*}\right] + \gamma_B\left[\frac{\mathfrak{L}\{\tilde{B},s\} - \mathfrak{L}\{\tilde{B},r^*\}}{s-r^*}\right]\right]. \tag{A18}$$

The long-run effects of shocks in the capital tax ($\tilde{t}_K(\infty)$), and debt ($\tilde{B}(\infty)$) are obtained from (A17) and (A18) by applying the final-value theorem (Spiegel, 1965, p. 20).

$$\tilde{C}(\infty) \equiv \lim_{s\downarrow 0} s\mathfrak{L}\{\tilde{C},s\} = \frac{\delta_{22}[\gamma_K \tilde{t}_K + \gamma_B \tilde{B}(\infty)]}{r^* h^*}, \tag{A19}$$

$$\tilde{K}(\infty) \equiv \lim_{s\downarrow 0} s\mathfrak{L}\{\tilde{K},s\} = \frac{-\delta_{21}[\gamma_K \tilde{t}_K + \gamma_B \tilde{B}(\infty)]}{r^* h^*}. \tag{A20}$$

By making the appropriate substitutions for the γ_K, γ_B and the δ_{ij}-terms from (A7), we find the expressions (3.3) in the text by setting $\tilde{B}(\infty)=0$.

By taking the Laplace transform of equation (T.4′) in Table A.1, and imposing the fact that the quality of the environment is a predetermined variable (so that $\tilde{E}(0)=0$), we obtain the following expression:

$$\mathfrak{L}\{\tilde{E},s\} = -\frac{\alpha_E \alpha_K \mathfrak{L}\{\tilde{K},s\}}{s+\alpha_E}. \tag{A21}$$

Hence, the Laplace transform of environmental quality is linked directly to the Laplace transform for the capital stock. It follows immediately from (T.4′) that:

$$\dot{\tilde{E}}(0) = -\alpha_E\left[\tilde{E}(0) + \alpha_K \tilde{K}(0)\right] = 0, \tag{A22}$$

and:

$$\ddot{\tilde{E}}(0) = -\alpha_E\left[\dot{\tilde{E}}(0) + \alpha_K \dot{\tilde{K}}(0)\right] = -\alpha_E \alpha_K \dot{\tilde{K}}(0). \tag{A23}$$

In order to calculate the transition paths for capital and consumption, the intertemporal paths of $\tilde{t}_K(t)$ and $\tilde{B}(t)$ must be specified. Suppose that these paths are parameterized as follows:

$$\tilde{t}_K(t) = \tilde{t}_K, \qquad \tilde{B}(t) = \tilde{B}(0) + \left(1-e^{-\xi_1 t}\right)\tilde{B}_1 + \left(1-e^{-\xi_2 t}\right)\tilde{B}_2, \tag{A24}$$

with $\xi_1>0$ and $\xi_2>0$. In that case, it is possible to derive the following expressions:

$$\frac{\mathfrak{L}\{\tilde{t}_K,s\} - \mathfrak{L}\{\tilde{t}_K,r^*\}}{s-r^*} = -\frac{\tilde{t}_K}{sr^*}, \tag{A25}$$

It is also useful to recognize that:

$$\frac{\mathfrak{L}\{\tilde{B},s\} - \mathfrak{L}\{\tilde{B},r^*\}}{s-r^*} = -\frac{\tilde{B}(0)}{sr^*} - \sum_{i=1}^{2}\tilde{B}_i\left[\frac{1}{sr^*} - \frac{1}{(r^*+\xi_i)(s+\xi_i)}\right]$$

(A26)

$$\frac{1}{(s+h^*)(s+\xi_i)} = \frac{1}{\xi_i-h^*}\left[\frac{1}{s+h^*} - \frac{1}{s+\xi_i}\right] \qquad \frac{1}{(s+h^*)s} = \frac{1}{h^*}\left[\frac{1}{s} - \frac{1}{s+h^*}\right] \tag{A27}$$

for i=1,2. Using (A24)-(A26) in (A18) and recognizing (A20), we obtain the transition path for the capital stock by inverting the Laplace transforms:

$$\tilde{K}(t) = \mathrm{A}(h^*,t)\tilde{K}(\infty) + \delta_{21}\gamma_B\sum_{i=1}^{2}\left(\frac{\tilde{B}_i}{r^*+\xi_i}\right)\mathrm{T}(\xi_i,h^*,t). \tag{A28}$$

Equation (A28) contains a single adjustment term and two (temporary) single transition terms, about which the following properties can be established, respectively.

LEMMA A.1: *Let* $\mathrm{A}(\alpha_1,t)$ *be a single adjustment function of the form:*

$$\mathrm{A}(\alpha_1,t) \equiv 1-e^{-\alpha_1 t},$$

with $\alpha_1>0$. *Then* $\mathrm{A}(\alpha_1,t)$ *has the following properties: (i) (positive)* $\mathrm{A}(\alpha_1,t)>0$ $t\in(0,\infty)$, *(ii)* $\mathrm{A}(\alpha_1,t)=0$ *for* $t=0$ *and* $\mathrm{A}(\alpha_1,t)\to 1$ *in the limit as* $t\to\infty$, *(iii) (increasing)* $d\mathrm{A}(\alpha_1,t)/dt\geq 0$, *(iv) (step function as limit) As* $\alpha_1\to\infty$, $\mathrm{A}(\alpha_1,t)\to \mathrm{u}(t)$, *where* $\mathrm{u}(t)$ *is a unit step function.*

PROOF: Properties (i) and (ii) follow by simple substitution. Property (iii) follows from the fact that $d\mathrm{A}(\alpha_1,0)/dt=\alpha_1[1-\mathrm{A}(\alpha_1,t)]$ plus properties (i)-(ii). Property (iv) follows by comparing the Laplace transforms of $\mathrm{A}(\alpha_1,t)$ and $\mathrm{u}(t)$ and showing that they converge as $\alpha_1\to\infty$. Since $\mathfrak{L}\{\mathrm{u},s\}=1/s$ and $\mathfrak{L}\{\mathrm{A}(\alpha_1,t),s\}=1/s-1/(s+\alpha_1)$ this result follows. □

LEMMA A.2: *Let* $\mathrm{T}(\alpha_1,\alpha_2,t)$ *be a single transition function of the form:*

$$\mathrm{T}(\alpha_1,\alpha_2,t) \equiv \left(\frac{e^{-\alpha_1 t}-e^{-\alpha_2 t}}{\alpha_2-\alpha_1}\right),$$

with $\alpha_1>0$, $\alpha_2>0$, $\alpha_1\neq\alpha_2$. *Then* $\mathrm{T}(\alpha_1,\alpha_2,t)$ *has the following properties: (i) (positive)* $\mathrm{T}(\alpha_1,\alpha_2,t)>0$ $t\in(0,\infty)$, *(ii)* $\mathrm{T}(\alpha_1,\alpha_2,t)=0$ *for* $t=0$ *and in the limit as* $t\to\infty$, *(iii) (single-peaked)* $d\mathrm{T}(\alpha_1,\alpha_2,t)/dt>0$ *for* $t\in(0,\hat{t})$ *and*

$\mathrm{dT}(\alpha_1,\alpha_2,t)/\mathrm{d}t<0$ *for* $t\in(\hat{t},\infty)$, $\mathrm{dT}(\alpha_1,\alpha_2,t)/\mathrm{d}t=0$ *(for* $t=\hat{t}\equiv\ln(\alpha_1/\alpha_2)/(\alpha_1-\alpha_2)$ *and in the limit as* $t\to\infty$*), and* $\mathrm{dT}(\alpha_1,\alpha_2,0)/\mathrm{d}t=1$, *(iv) (point of inflexion)* $\mathrm{d}^2\mathrm{T}(\alpha_1,\alpha_2,t)/\mathrm{d}t^2=0$ *for* $t^*=2\hat{t}$.

PROOF: Property (i) follows by examining the two possible cases. If $\alpha_1<(>)\alpha_2$, then $\alpha_2-\alpha_1>(<)0$ and $\exp[-\alpha_1 t]>(<)\exp[-\alpha_2 t]$ for all $t\in(0,\infty)$, and $\mathrm{T}(\alpha_1,\alpha_2,0)>0$. Property (ii) follows by direct substitution. Property (iii) follows by examining $\mathrm{dT}(\alpha_1,\alpha_2,t)/\mathrm{d}t$:

$$\frac{\mathrm{d}\,\mathrm{T}(\alpha_1,\alpha_2,t)}{\mathrm{d}t} = \left(\frac{\alpha_1 e^{-\alpha_1 t}-\alpha_2 e^{-\alpha_2 t}}{\alpha_1-\alpha_2}\right).$$

Property (iv) is obtained by examining $\mathrm{dT}^2(\alpha_1,\alpha_2,t)/\mathrm{d}t^2$:

$$\frac{\mathrm{d}^2\mathrm{T}(\alpha_1,\alpha_2,t)}{\mathrm{d}^2 t} = \left(\frac{\alpha_1^2 e^{-\alpha_1 t}-\alpha_2^2 e^{-\alpha_2 t}}{\alpha_2-\alpha_1}\right).$$

Hence, $\mathrm{d}^2\mathrm{T}(\alpha_1,\alpha_2,0)/\mathrm{d}t^2=-(\alpha_1+\alpha_2)<0$, and $\lim_{t\to\infty}\mathrm{d}^2\mathrm{T}(\alpha_1,\alpha_2,t)/\mathrm{d}t^2=0$. The inflexion point is found by finding the value of $t=t^*$ where $\mathrm{d}^2\mathrm{T}(\alpha_1,\alpha_2,t)/\mathrm{d}t^2=0$. □

By using (A24)-(A26) in (A17) and noting (A19), we find the transition path for consumption by inverting the resulting Laplace transforms:

$$\tilde{C}(t) = \tilde{C}(0)\big(1-\mathrm{A}(h^*,t)\big) + \tilde{C}(\infty)\,\mathrm{A}(h^*,t) - \gamma_B\sum_{i=1}^{2}\left(\frac{\delta_{22}+\xi_i}{r^*+\xi_i}\right)\tilde{B}_i\,\mathrm{T}(\xi_i,h^*,t), \tag{A29}$$

where the jump in consumption that occurs at impact can be calculated by using (A16):

$$\tilde{C}(0) = -\left(\frac{\gamma_K}{r^*}\right)\tilde{t}_K - \gamma_B\left[\frac{\tilde{B}(\infty)}{r^*} - \sum_{i=1}^{2}\left(\frac{\tilde{B}_i}{r^*+\xi_i}\right)\right],\quad \tilde{B}(\infty)\equiv\tilde{B}(0)+\sum_{i=1}^{2}\tilde{B}_i. \tag{A30}$$

By substituting (A18), (A24)-(A26) into (A21) and inverting the Laplace transform, we derive the path for the environment:

$$\tilde{E}(t) \equiv -\alpha_K\mathrm{A}(\alpha_E,h^*,t)\tilde{K}(\infty) - \alpha_E\alpha_K\gamma_B\delta_{21}\sum_{i=1}^{2}\left(\frac{\tilde{B}_i}{r^*+\xi_i}\right)\mathrm{T}(\alpha_E,\xi_i,h^*,t). \tag{A31}$$

The $\mathrm{A}(\alpha_1,\alpha_2,t)$ and $\mathrm{T}(\alpha_1,\alpha_2,\alpha_3,t)$ terms in (A31) are, respectively, multiple adjustment and multiple transition terms. The forms and properties of these terms are covered in Lemma A.3 to A.5 below.

LEMMA A.3: *Let* $A(\alpha_1,\alpha_2,t)$ *be a multiple adjustment function of the form:*

$$A(\alpha_1,\alpha_2,t) \equiv 1 - \left(\frac{\alpha_2}{\alpha_2-\alpha_1}\right)e^{-\alpha_1 t} + \left(\frac{\alpha_1}{\alpha_2-\alpha_1}\right)e^{-\alpha_2 t},$$

with $\alpha_1>0$, $\alpha_2>0$, $\alpha_1\neq\alpha_2$. *Then* $A(\alpha_1,\alpha_2,t)$ *has the following properties: (i) (increasing over time)* $dA(\alpha_1,\alpha_2,t)/dt>0\ \forall t\in(0,\infty)$, $dA(\alpha_1,\alpha_2,t)/dt=0$ (*for* $t=0$ *and in the limit as* $t\to\infty$), *(ii) (between 0 and 1)* $0<A(\alpha_1,\alpha_2,t)<1\ \forall t\in(0,\infty)$ *and* $A(\alpha_1,\alpha_2,0)=1-\lim_{t\to\infty}A(\alpha_1,\alpha_2,t)=0$, *(iii) (inflexion point)* $d^2A(\alpha_1,\alpha_2,t)/dt^2=0$ *for* $t=\hat{t}\equiv\ln(\alpha_1/\alpha_2)/(\alpha_1-\alpha_2)$.

PROOF: The derivative of $A(\alpha_1,\alpha_2,t)$ with respect to time is itself proportional to a single transition term with properties covered in Lemma A.2:

$$\frac{dA(\alpha_1,\alpha_2,t)}{dt} = \alpha_1\alpha_2\left(\frac{e^{-\alpha_1 t}-e^{-\alpha_2 t}}{\alpha_2-\alpha_1}\right)\geq 0,$$

for $t\in(0,\infty)$ the inequality is strict. Hence, $A(\alpha_1,\alpha_2,t)$ itself is increasing over time. Property (ii) follows from the fact that $A(\alpha_1,\alpha_2,0)=0$ and $\lim_{t\to\infty}A(\alpha_1,\alpha_2,t)=1$ plus the fact that $dA(\alpha_1,\alpha_2,0)/dt\geq 0$. Property (iii) makes use of:

$$\frac{d^2A(\alpha_1,\alpha_2,t)}{d^2t} = \alpha_1\alpha_2\left(\frac{\alpha_1 e^{-\alpha_1 t}-\alpha_2 e^{-\alpha_2 t}}{\alpha_1-\alpha_2}\right).$$

Hence, $d^2A(\alpha_1,\alpha_2,0)/dt^2=\alpha_1\alpha_2>0$, and $\lim_{t\to\infty}d^2A(\alpha_1,\alpha_2,t)/dt^2=0$. The inflexion point is found by finding the value of t where $d^2A(\alpha_1,\alpha_2,t)/dt^2=0$. The solution is $\hat{t}\equiv\ln(\alpha_1/\alpha_2)/(\alpha_1-\alpha_2)$. □

LEMMA A.4: *Let* $f(t)$ *be a function with the following Laplace transform* $F(s)$:

$$F(s) \equiv \frac{1}{(s+\alpha_1)(s+\alpha_2)(s+\alpha_3)},$$

with $0<\alpha_1,\alpha_2,\alpha_3<\infty$ *and* $\alpha_i\neq\alpha_j\ \forall i\neq j$. *Then* $f(t)\geq 0\ \forall t\in[0,\infty)$.

PROOF: Use the convolution property of the Laplace transform (Spiegel, 1965, p. 45). Define $G(s)\equiv 1/(s+\alpha_1)$ and $H(s)\equiv 1/(s+\alpha_2)(s+\alpha_3)$, so that $F(s)=G(s)H(s)$. The inverse Laplace transforms of $G(s)$

and H(s) are:

$$g(t) \equiv e^{-\alpha_1 t}, \qquad h(t) \equiv \left(\frac{e^{-\alpha_2 t} - e^{-\alpha_3 t}}{\alpha_3 - \alpha_2}\right).$$

Then the convolution property states that f(t) is equal to:

$$f(t) \equiv \mathfrak{L}^{-1}\{G(s),H(s)\} = \int_0^t g(\tau)\,h(t-\tau)\,d\tau.$$

Since g(τ)≥0 and h(τ)≥0 $\forall\tau$ and $\alpha_1<\infty$, f(t) must be non-negative since it represents the discounted integral of a non-negative function. □

LEMMA A.5: *Let* $T(\alpha_1,\alpha_2,\alpha_3,t)$ *be a multiple transition function of the form:*

$$T(\alpha_1,\alpha_2,\alpha_3,t) \equiv \left[\frac{1}{\alpha_3-\alpha_2}\right]\left[\left(\frac{e^{-\alpha_1 t} - e^{-\alpha_2 t}}{\alpha_2-\alpha_1}\right) - \left(\frac{e^{-\alpha_1 t} - e^{-\alpha_3 t}}{\alpha_3-\alpha_1}\right)\right]$$

with $0<\alpha_1,\alpha_2,\alpha_3<\infty$ *and* $\alpha_i\neq\alpha_j$ $\forall i\neq j$. *Then* $T(\alpha_1,\alpha_2,\alpha_3,t)$ *has the following properties: (i) (positive)* $T(\alpha_1,\alpha_2,\alpha_3,t)>0$, (ii) $T(\alpha_1,\alpha_2,\alpha_3,t)=0$ *for t=0 and in the limit as* $t\rightarrow\infty$, *(iii) (single-peaked)* $dT(\alpha_1,\alpha_2,\alpha_3,t)/dt>0$ for $0<t<\bar{t}$ *and* $dT(\alpha_1,\alpha_2,\alpha_3,t)/dt<0$ *for* $t>\bar{t}$, $dT(\alpha_1,\alpha_2,\alpha_3,t)/dt=0$ *(for* $t=0$, $t=\bar{t}$, *and* $t\rightarrow\infty$*).*

PROOF: Property (i) follows applying Lemma A.4, because the Laplace transform of $T(\alpha_1,\alpha_2,\alpha_3,t)$ has the required form. Property (ii) follows trivially by substitution. Property (iii) can be proved by examining the following differential equation in $T(\alpha_1,\alpha_2,\alpha_3,t)$:

$$\frac{dT(\alpha_1,\alpha_2,\alpha_3,t)}{dt} = \left(\frac{e^{-\alpha_2 t} - e^{-\alpha_3 t}}{\alpha_3-\alpha_2}\right) - \alpha_1 T(\alpha_1,\alpha_2,\alpha_3,t).$$

The term in round brackets is a simple transition term, denoted by $T(\alpha_2,\alpha_3,t)$, the properties of which are covered in Lemma A.2. $T(\alpha_2,\alpha_3,t)$ has a single maximum at $\hat{t}\equiv\ln(\alpha_2/\alpha_3)/(\alpha_2-\alpha_3)$. $T(\alpha_1,\alpha_2,\alpha_3,t)$ itself has a maximum in $\bar{t}$ where $T(\alpha_2,\alpha_3,\bar{t})=\alpha_1 T(\alpha_1,\alpha_2,\alpha_3,\bar{t})$ and $\ddot{T}(\alpha_1,\alpha_2,\alpha_3,\bar{t})=\dot{T}(\alpha_2,\alpha_3,\bar{t})<0$. The maximum of $T(\alpha_1,\alpha_2,\alpha_3,\bar{t})$ occurs later than that of $T(\alpha_2,\alpha_3,\bar{t})$, or $\bar{t}>\hat{t}$ as $\dot{T}<0$. The maximum is unique. If there were another maximum, there would also be a minimum for which $T(\alpha_2,\alpha_3,t_{MIN})=\alpha_1 T(\alpha_1,\alpha_2,\alpha_3,t_{MIN})$ (since $\dot{T}(\alpha_1,\alpha_2,\alpha_3,t_{MIN})=0$) and $\ddot{T}(\alpha_1,\alpha_2,\alpha_3,\bar{t})=\dot{T}(\alpha_2,\alpha_3,t_{MIN})>0$. This is impossible as $t_{MIN}>\hat{t}$ and $\ddot{T}(\alpha_1,\alpha_2,\alpha_3,\bar{t})<0$. □

The transition effects can be illustrated with the phase diagram for consumption and the capital stock (see Figure A.2). On the IS curve, the goods market is in equilibrium with a constant capital stock (*i.e.* $\dot{\tilde{K}}=0$). The MKR curve represents the modified Keynes-Ramsey rule, *i.e.*, the steady-state aggregate Euler equation modified for the existence of overlapping generations (*i.e.* $\dot{\tilde{C}}=0$). In Figure A.2, the policy shock shifts MKR from MKR_0 to MKR_1 and the long-run equilibrium from E_0 to E_1. The arrows denote the dynamic forces associated with the *old* equilibrium E_0, and the saddle path associated with the *new* equilibrium E_1 is drawn as SP. As a result of the unanticipated policy shock, the economy jumps at time t=0 from E_0 to A (described by equation (3.1) in the text) onto the saddle path, after which gradual adjustment (described by (3.2)-(3.4) in the text) occurs towards E_1.

A.2. Crowding out of the capital stock

It can be shown that the long-run decline in capital (in (3.3)) is particularly large if the initial pollution tax is large, horizons are long (*i.e.*, if λ is small), or the rate of time preference is high (ρ large). The results are shown as follows. The decline in the capital stock is given by:

$$\Omega \equiv \frac{r}{\varepsilon_L\left[r-\rho+r\right]}. \tag{A32}$$

In calculating the effects on Ω of ρ, λ, ε_L, and t_K, it must be taken into account that the steady-state interest rate r depends on these parameters (see (A2)-(A5)). The comparative static effects are:

$$\frac{\mathrm{d}\Omega}{\mathrm{d}\lambda} = -\left(\frac{\rho}{\varepsilon_L\left[r-\rho+r\right]^2}\right)\frac{\mathrm{d}r}{\mathrm{d}\lambda} < 0, \tag{A33}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}\rho} = \left(\frac{r-\rho\,(\mathrm{d}r/\mathrm{d}\rho)}{\varepsilon_L\left[r-\rho+r\right]^2}\right) > 0, \tag{A34}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t_K} = -\left(\frac{\rho}{\varepsilon_L\left[r-\rho+r\right]^2}\right)\frac{\mathrm{d}r}{\mathrm{d}t_K} > 0, \tag{A35}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}\varepsilon_L} = -\left(\frac{r(2r-\rho)+\rho\,\varepsilon_L(\mathrm{d}r/\mathrm{d}\varepsilon_L)}{\varepsilon_L^2\left[r-\rho+r\right]^2}\right). \tag{A36}$$

Capital crowding out increases if the rate of time preference increases, the birth rate falls, or the initial capital tax rises. The effect of the labour share parameter is, however, ambiguous.

A.3. Comparative static effects for the Pigouvian tax rate

Equation (3.15) can be used to obtain the following effects for the Pigouvian tax rate: $\partial t_K^P/\partial\gamma_E>0$, $\partial t_K^P/\partial\alpha_K>0$, $\partial t_K^P/\partial E>0$, and $\partial t_K^P/\partial\alpha_E>0$. A strong concern for the environment generates a high Pigouvian tax as do a high sensitivity of the environment to the capital stock and a high initial quality of the environment. The higher is the speed of regeneration of the environment, the higher is the Pigouvian tax. These results can be obtained as follows. The first-order condition determining the Pigouvian tax is:

$$(1-t_K)\left(\frac{\mathrm{d}U(0)}{\mathrm{d}t_K}\right) = \frac{-\rho t_K}{(\rho+h^*)\left[\rho+h^*(1-t_K)\right]} + \frac{\gamma_E\alpha_E\alpha_K h^* E}{\rho\varepsilon_L(\rho+\alpha_E)(\rho+h^*)} \tag{A37}$$

$$\equiv \Phi(t_K;\alpha_E,\alpha_K,\gamma_E,E) = 0,$$

which yields the Pigouvian tax rate:

$$t_K^P = \frac{\alpha_E\alpha_K\gamma_E h^*(\rho+h^*)E}{\rho^2\varepsilon_L(\rho+\alpha_E)+\alpha_K\alpha_E\gamma_E\left[h^*\right]^2 E}. \tag{A38}$$

The second-order condition implies that:

$$\frac{\mathrm{d}^2U(0)}{\mathrm{d}t_K^2} \equiv \frac{\Phi_{t_K}(t_K^P;\alpha_E,\alpha_K,\gamma_E,E)}{1-t_K^P} < 0, \tag{A39}$$

where $\Phi_{t_K}\equiv\partial\Phi(t_K^P;\alpha_K,\alpha_E,\gamma_E,E)/\partial t_K<0$. The comparative static effects on t_K^P of α_K, α_E, γ_E, and E can be obtained by differentiating (A37) with respect to these variables:

$$\frac{\partial t_K^P}{\partial\alpha_E} = \frac{\Phi_{\alpha_E}}{-\Phi_{t_K}} = \frac{\gamma_E\alpha_K h^* E}{-\Phi_{t_K}\varepsilon_L(\rho+h^*)(\rho+\alpha_E)^2} > 0, \tag{A40}$$

$$\frac{\partial t_K^P}{\partial\alpha_K} = \frac{\Phi_{\alpha_K}}{-\Phi_{t_K}} = \frac{\alpha_E\gamma_E h^* E}{-\Phi_{t_K}\varepsilon_L(\rho+h^*)(\rho+\alpha_E)} > 0, \tag{A41}$$

$$\frac{\partial t_K^P}{\partial\gamma_E} = \frac{\Phi_{\gamma_E}}{-\Phi_{t_K}} = \frac{\alpha_E\alpha_K h^* E}{-\Phi_{t_K}\varepsilon_L(\rho+h^*)(\rho+\alpha_E)} > 0, \tag{A42}$$

$$\frac{\partial t_K^P}{\partial E} = \frac{\Phi_E}{-\Phi_{t_K}} = \frac{\gamma_E \alpha_E \alpha_K h^*}{-\Phi_{t_K} \varepsilon_L (\rho + h^*)(\rho + \alpha_E)} > 0. \tag{A43}$$

A.4. Intergenerational welfare analysis

The welfare implications of the different environmental policies can be derived in the manner suggested by Bovenberg (1993, 1994). The optimum utility level of generation v at time t is denoted by $U(v,t)$. It is obtained by substituting the optimum values for $C(v,\tau)$ (where τ runs from t to ∞) plus the policy-induced path for $E(\tau)$ into the utility functional (2.1): $U(v,t)=U_{NE}(v,t)+\gamma_E U_E(t)$, where $U_{NE}(v,t)$ is the private component of welfare, and $U_E(t)$ is the environmental component:

$$U_{NE}(v,t) \equiv \int_t^{\infty} \log C(v,\tau) \exp[(\rho+\lambda)(t-\tau)]\, d\tau, \tag{A44}$$

$$U_E(t) \equiv \int_t^{\infty} E(\tau) \exp[(\rho+\lambda)(t-\tau)]\, d\tau. \tag{A45}$$

Turn to the private component of welfare first. Using the Euler equation for the household, we can relate $C(v,\tau)$ to $C(v,t)$:

$$C(v,\tau) = C(v,t) \exp\left[\int_t^{\tau} (r(\mu)-\rho)\, d\mu\right], \qquad \tau \geq t. \tag{A46}$$

Substitution of this result in (A44) yields:

$$U_P(v,t) = \frac{\log C(v,t)}{\rho+\lambda} + \Delta(t), \tag{A47}$$

$$\Delta(t) \equiv \int_t^{\infty}\left[\int_t^{\tau} [r(\mu)-\rho]\, d\mu\right] \exp[(\rho+\lambda)(t-\tau)] d\tau = \int_t^{\infty}\left(\frac{r(\mu)-\rho}{\rho+\lambda}\right) \exp[(t-\mu)(\rho+\lambda)]\, d\mu. \tag{A48}$$

Linearising the expressions in (A47) and (A48), we find:

$$dU_{NE}(v,t) = \frac{\tilde{C}(v,t)}{\rho+\lambda} + d\Delta(t), \tag{A49}$$

$$d\Delta(t) = \left(\frac{r}{\rho+\lambda}\right) \int_t^{\infty} \tilde{r}(\mu) e^{-(\mu-t)(\rho+\lambda)} d\mu. \tag{A50}$$

In order to perform the welfare analysis, we must distinguish between ‘old' agents that are already alive at the time of announcement of the unanticipated shock, and ‘young' generations that are not yet alive at that time. The time of the announcement is denoted by $t_0=0$. Hence, agents with a generations index smaller than or equal to 0 ($v \le t_0=0$) are alive at the time of the shock. Those with an index larger than 0 are born later ($v=t>t_0=0$).

A.4.1. Existing Generations ($v\le 0$)

For existing generations, the change in private utility is $dU_{NE}(v,0)=\tilde{C}(v,0)/(\rho+\lambda)+d\Delta(0)$. Using (A50), and applying the initial-value theorem (Spiegel, 1965, p. 20), we find $d\Delta(0) = (r/(\rho+\lambda))\mathfrak{L}\{\tilde{r},\rho+\lambda\}$. The jump in consumption is a weighted average of the jump in human wealth and the change in financial wealth (although the capital stock is predetermined, a once-off subsidy to capital owners of s_K (per unit of capital) implies $\tilde{A}(v,0)=\tilde{A}(0)=s_K$ for $v\neq 0$). Hence, $\tilde{C}(v,0)=\alpha_{HS}\tilde{H}(0)+(1-\alpha_{HS})\tilde{A}(0)$, where $\tilde{H}(v,0)=\tilde{H}(0)$, and α_{HS} denotes the share of human wealth in total wealth of an agent belonging to generation v to be determined below. The human wealth term $\tilde{H}(0)$ can be eliminated by using the solution for the initial jump in aggregate consumption, $\tilde{C}(0)=(1-\omega_H)\tilde{A}(0)+\omega_H\tilde{H}(0)$, where $1-\omega_H \equiv K/(K+H)=(r-\rho)/\lambda$. This implies that $\tilde{C}(v,0)$ can be linked to $\tilde{C}(0)$ and $\tilde{A}(0)$ ($\equiv s_K$):

$$\tilde{C}(v,0) \equiv (1-\alpha_{HS})s_K + \alpha_{HS}\left[\frac{\tilde{C}(0)-(1-\omega_H)s_K}{\omega_H}\right], \tag{A51}$$

where the outlays on the once-off capital subsidy ($s_K K(0)$) give rise to the jump in debt at time $t=0$: $\tilde{B}(0)=s_K\omega_K$, where $\omega_K \equiv rK/Y \equiv (1-t_K)(1-\varepsilon_L)$. The human wealth share α_{HS} is determined by using the steady-state information for the optimal steady-state consumption profile of existing generations, *i.e.*, $C(v,0)=C(v,v)\exp(-(r-\rho)v)$ for $v<0$. Since all generations are born without financial wealth ($A(v,v)=0$), human wealth is the same for all agents ($H(v,t)=H$), and consumption is proportional to total wealth ($C(v,t)=(\rho+\lambda)(A(v,t)+H)$), the following expression for α_{HS} is obtained:

$$\alpha_{HS} \equiv \frac{H}{A(v,0)+H} = e^{(r-\rho)v}. \tag{A52}$$

This expression is intuitively clear. Old agents (*i.e.*, with very negative v) have had a long time to accumulate financial assets and hence feature a relatively low share of human wealth in total wealth. A very young agent ($v=0$) owns no financial wealth and hence exhibits a human wealth share of unity.

After combining all the information, we obtain the following relation:

$$(\rho+\lambda)\,dU_{NE}(v,0) = \left[1-e^{(r-\rho)v}\right]s_K + \left(\frac{e^{(r-\rho)v}}{\omega_H}\right)\left[\tilde{C}(0)-(1-\omega_H)s_K\right] + r\mathfrak{L}\{\tilde{r},\rho+\lambda\}. \tag{A53}$$

A.4.2. Future Generations (v=t>0)

Future generations are born without financial wealth. Hence their consumption at birth is given by $C(t,t)=(\rho+\lambda)H(t)$, so that $\tilde{C}(t,t)=\tilde{H}(t)$. Human wealth can be eliminated by using the aggregate relation $\tilde{C}(t)=\omega_H\tilde{H}(t)+(1-\omega_H)\tilde{K}(t)+[(1-\omega_H)/\omega_K]\tilde{B}(t)$. This enables us to write $\tilde{C}(t,t)$ in terms of $\tilde{C}(t)$, $\tilde{K}(t)$ and $\tilde{B}(t)$:

$$\tilde{C}(t,t) = \left[\frac{\tilde{C}(t) - (1-\omega_H)\tilde{K}(t) - [(1-\omega_H)/\omega_K]\tilde{B}(t)}{\omega_H}\right], \qquad t>0. \tag{A54}$$

The following Lemma can be used to calculate the Laplace transform of $d\Delta(t)$.

LEMMA A.6: *Let* X(t) *be a function defined as follows:*

$$X(t) \equiv \int_t^{\infty} z(\mu) e^{-(\rho+\lambda)(\mu-t)} d\mu.$$

Then $\mathfrak{L}\{X,s\}$ *is given by*:

$$\mathfrak{L}\{X,s\} = \left[\frac{\mathfrak{L}\{z,\rho+\lambda\} - \mathfrak{L}\{z,s\}}{s-(\rho+\lambda)}\right].$$

PROOF: X(t) satisfies the differential equation:

$$\dot{X}(t) = -z(t) + (\rho+\lambda)X(t), \qquad X(0) = \mathfrak{L}\{z,\rho+\lambda\}.$$

The Laplace transform of the differential equation amounts to:

$$s\mathfrak{L}\{X,s\} - X(0) = -\mathfrak{L}\{z,s\} + (\rho+\lambda)\mathfrak{L}\{X,s\}.$$

By substituting the initial condition $X(0)=\mathfrak{L}\{z,\rho+\lambda\}$ and gathering terms, we obtain the required result.□

The Laplace transform of $d\Delta(t)$ can be obtained by applying Lemma A.6 to (A50):

$$\mathfrak{L}\{d\Delta,s\} = \left(\frac{r}{\rho+\lambda}\right)\left[\frac{\mathfrak{L}\{\tilde{r},\rho+\lambda\} - \mathfrak{L}\{\tilde{r},s\}}{s-(\rho+\lambda)}\right]. \tag{A55}$$

By combining (A54) with (A49), taking Laplace transforms, and using (A55), we arrive at the following expression for the Laplace transform of private utility of future generations:

By substituting the Laplace transforms for $\tilde{C}$, $\tilde{K}$, $\tilde{B}$, and $\tilde{r}$ $[\equiv-\varepsilon_L\tilde{K}-\tilde{t}_K]$, and inverting, the entire path for

$$(\rho+\lambda)\mathfrak{L}\{\mathrm{d}U_{NE},s\} = \left(\frac{\mathfrak{L}\{\tilde{C},s\} - (1-\omega_H)\mathfrak{L}\{\tilde{K},s\} - [(1-\omega_H)/\omega_K]\mathfrak{L}\{\tilde{B},s\}}{\omega_H}\right) + r\left(\frac{\mathfrak{L}\{\tilde{r},\rho+\lambda\} - \mathfrak{L}\{\tilde{r},s\}}{s-(\rho+\lambda)}\right). \tag{A56}$$

$\mathrm{d}U_{NE}(t,t)$ is obtained (where t acts as the index for future generations, *i.e.*, t>0).

In order to derive the results in sections 3 and 4 of the paper, the path for $\mathrm{d}U_{NE}(t,t)$ is written in terms of $\mathrm{d}U_{NE}(0,0)$, $\mathrm{d}U_{NE}(\infty,\infty)$, and adjustment and transition terms like $\mathrm{A}(h^*,t)$ and $\mathrm{T}(\xi_i,h^*,t)$. This can be proved as follows. The crucial results that must be used are:

$$\mathfrak{L}^{-1}\left\{\frac{\mathfrak{L}\{\tilde{K},\rho+\lambda\} - \mathfrak{L}\{\tilde{K},s\}}{s-(\rho+\lambda)}\right\} = \mathfrak{L}\{\tilde{K},\rho+\lambda\} \tag{A57}$$

$$+ \frac{\mathrm{A}(h^*,t)}{\rho+\lambda+h^*}\left[\tilde{K}(\infty) - \delta_{21}\gamma_B\sum_{i=1}^{2}\left(\frac{\tilde{B}_i}{(r^*+\xi_i)(\rho+\lambda+\xi_i)}\right)\right]$$

$$+ \delta_{21}\gamma_B\sum_{i=1}^{2}\left(\frac{\tilde{B}_i}{(r^*+\xi_i)(\rho+\lambda+\xi_i)}\right)\mathrm{T}(\xi_i,h^*,t),$$

and:

$$\tilde{C}(t) - (1-\omega_H)\tilde{K}(t) - \left[(1-\omega_H)/\omega_K\right]\tilde{B}(t) \tag{A58}$$

$$= \tilde{C}(0) + \left[\tilde{C}(\infty) - \tilde{C}(0) - (1-\omega_H)\tilde{K}(\infty)\right]\mathrm{A}(h^*,t)$$

$$- \left[(1-\omega_H)/\omega_K\right]\left[\tilde{B}(0) + \sum_{i=1}^{2}\left[\mathrm{A}(h^*,t) + (\xi_i - h^*)\mathrm{T}(\xi_i,h^*,t)\right]\tilde{B}_i\right]$$

$$-\gamma_B\sum_{i=1}^{2}\left(\frac{\xi_i + \delta_{22} + \delta_{21}(1-\omega_H)}{r^*+\xi_i}\right)\tilde{B}_i\mathrm{T}(\xi_i,h^*,t).$$

By substituting these results into the inverted equation (A56), and gathering constant terms, and terms involving $\mathrm{A}(h^*,t)$ and $\mathrm{T}(\xi_i,h^*,t)$, $\mathrm{d}U_{NE}(t,t)$ can be written as follows:

$$\mathrm{d}U_{NE}(t,t) \;=\; \mathrm{d}U_{NE}(0,0) \;+\; \left[\mathrm{d}U_{NE}(\infty,\infty) - \mathrm{d}U_{NE}(0,0)\right]\mathrm{A}(h^{*},t) \;-\; \sum_{i=1}^{2} \Omega_{NE}(\xi_i)\,\tilde{B}_i\,\mathrm{T}(\xi_i,h^{*},t), \tag{A59}$$

where we have used the definitions of $\mathrm{d}U_{NE}(0,0)$, $\mathrm{d}U_{NE}(\infty,\infty)$, and $\Omega_P(\xi_i)$:

$$(\rho+\lambda)\,\mathrm{d}U_{NE}(0,0) \;=\; \frac{\tilde{C}(0) - \left[(1-\omega_H)/\omega_K\right]\tilde{B}(0)}{\omega_H} \;-\; r\varepsilon_L\,\mathfrak{L}\{\tilde{K},\rho+\lambda\} \;-\; r\,\mathfrak{L}\{\tilde{t}_K,\rho+\lambda\}, \tag{A60}$$

$$\begin{aligned}(\rho+\lambda)\,\mathrm{d}U_{NE}(\infty,\infty) \;=\; &\left[\frac{\tilde{C}(\infty) - (1-\omega_H)\,\tilde{K}(\infty) - \left[(1-\omega_H)/\omega_K\right]\tilde{B}(\infty)}{\omega_H}\right] \\ &-\left(\frac{r\varepsilon_L}{\rho+\lambda}\right)\tilde{K}(\infty) \;-\; \left(\frac{r}{\rho+\lambda}\right)\tilde{t}_K,\end{aligned} \tag{A61}$$

$$(\rho+\lambda)\,\Omega_{NE}(\xi_i) \;\equiv\; \left[\frac{(1-\omega_H)(\xi_i-h^{*})}{\omega_H\omega_K} + \frac{\gamma_B\, r\,\delta_{21}\,\varepsilon_L}{(r^{*}+\xi_i)(\rho+\lambda+\xi_i)} \;+\; \frac{\gamma_B(\xi_i+\delta_{22}+\delta_{21}(1-\omega_H))}{\omega_H(r^{*}+\xi_i)}\right]. \tag{A62}$$

In section 3 no bond policy is used ($\tilde{B}(0)=\tilde{B}_1=\tilde{B}_2=0$), so that the transition terms $\mathrm{T}(\xi_i,h^{*},t)$ disappear from the various expressions. In section 4 of the paper, $\tilde{B}_i$ and ξ_i are used as instruments to redistribute utility across generations, and the most general expressions are relevant.

A.5. Proof of Proposition 3.2.

The change in private welfare experienced by very old generations ($v \rightarrow -\infty$) can be written as:

$$(\rho+\lambda)\,\mathrm{d}U_{NE}(-\infty,0) \;=\; r\mathcal{L}\{\tilde{r},\rho+\lambda\}, \tag{A63}$$

where (3.3) and (3.5) can be used to obtain the path for $\tilde{r}(t)$:

$$\tilde{r}(t) \;=\; \left[\frac{r\,\mathrm{A}(h^*,t)}{r-\rho+r} \;-\; 1\right]\tilde{t}_K. \tag{A64}$$

By taking the Laplace transform of (A64) (evaluated for s=ρ+λ) and substituting the result into (A63) we obtain the following expression:

$$\begin{aligned}(\rho+\lambda)\,\mathrm{d}U_{NE}(-\infty,0) \;&=\; r\left[\frac{r}{\left[r-\rho+r\right]}\,\frac{h^*}{\left[\rho+\lambda+h^*\right]} \;-\; 1\right]\frac{\tilde{t}_K}{\rho+\lambda} \\ &=\; \frac{-r\tilde{t}_K}{(\rho+\lambda)\left[r-\rho+r\right]}\left[\frac{\rho+\lambda}{\rho+\lambda+h^*}r \;+\; (r-\rho)\right] \;<\; 0.\end{aligned} \tag{A65}$$

This establishes Proposition 3.2(i).

Proposition 3.2(ii) can be proved as follows. We make use of the following expression: $(\rho+\lambda)\mathrm{d}U_{NE}(0,0)=\tilde{C}(0)/\omega_H+r\mathcal{L}\{\tilde{r},\rho+\lambda\}$. We have already established that $\mathcal{L}\{\tilde{r},\rho+\lambda\}<0$ regardless of the value of t_K. From (A30) we know that:

$$\tilde{C}(0) \;\equiv\; -\left(\frac{\gamma_K}{r^*}\right)\tilde{t}_K \;=\; \left(\frac{r}{r^*}\right)\tilde{t}_K > 0,$$

so that $0<\tilde{C}(0)<\tilde{t}_K$ (as $r<r^*$). By combining the relevant information, $(\rho+\lambda)\mathrm{d}U_{NE}(0,0)$ can be written as:

$$\begin{aligned}(\rho+\lambda)\,\mathrm{d}U_{NE}(0,0) \;&\equiv\; \frac{r\tilde{t}_K}{r^*\omega_H}\left[1 \;-\; \frac{r^*\omega_H}{(\rho+\lambda)\left[r-\rho+r\right]}\left(\frac{\rho+\lambda}{\rho+\lambda+h^*}r+(r-\rho)\right)\right] \\ &=\; \frac{r\tilde{t}_K}{r^*\omega_H}\left[\frac{(\rho+\lambda)\left[r-\rho+r/\sigma_{KL}\right](\rho+\lambda+h^*-\omega_H r^*) \;-\; \omega_H(r-\rho)r^*h^*}{(\rho+\lambda)(\rho+\lambda+h^*)\left[r-\rho+r/\sigma_{KL}\right]}\right]\end{aligned} \tag{A66}$$

We designate the term in the numerator between square brackets in (A66) by $\Omega_2(t_K)$. After some manipulation $\Omega_2(t_K)$ can be simplified:

$$\Omega_2(t_K) = \left[r - \rho + r\right]\left[(\rho+\lambda)(\rho+\lambda+h^* - \omega_H r^*) - \omega_H (r-\rho) r \varepsilon_L / \omega_K\right]$$

$$= (\rho+\lambda)\left[r-\rho+r\right]\left[\rho+\lambda+h^* - \omega_H r^* - \lambda \varepsilon_L \omega_H\right]$$

$$= (\rho+\lambda)\left[r-\rho+r\right]\left[(1-\omega_H)(\rho+\lambda(1-\omega_H)+h^*) - \rho\,\omega_H\left(\frac{t_K}{1-t_K}\right) - \frac{(1-\omega_H)(\lambda\omega_H)^2}{(\rho+\lambda)(1-t_K)}\right]$$

$$= (\rho+\lambda)\left[r-\rho+r\right]\times$$

$$\left[\left(\frac{1-\omega_H}{\rho+\lambda}\right)\left[(\rho+\lambda)(\rho+\lambda(1-\omega_H)+h^*) - (\lambda\omega_H)^2\right] - \left(\frac{t_K}{1-t_K}\right)\left(\rho\,\omega_H + \frac{(1-\omega_H)(\lambda\omega_H)^2}{\rho+\lambda}\right)\right],$$

(A67)

where we have used (A9) in the first step, $r/\omega_K=(\rho+\lambda)/(1-\omega_H)$ and $r-\rho=\lambda(1-\omega_H)$ in the second step, $r^*=h^*+\mathrm{tr}(\Delta)$ in the third step, and $(1-\varepsilon_L)=(1-\omega_H)[\rho+\lambda(1-\omega_H)]/[(1-t_K)(\rho+\lambda)]$ in the fourth step. If the initial capital tax is zero, equation (A67) shows that $\Omega_2(0)>0$ (since Proposition A.2 implies that $h^*> \rho+\lambda-r=\lambda\omega_H$). This establishes Proposition 3.2(ii).

Proposition 3.2(iii) is proved by computing $t_K=\bar{t}_K$ such that the term in square brackets in (A67) is zero. After some manipulation we find:

$$\bar{t}_K = \frac{(1-\omega_H)\left[(\rho+\lambda)(\rho+\lambda(1-\omega_H)+h^*) - (\lambda\omega_H)^2\right]}{(\rho+\lambda)\left[\rho+(1-\omega_H)(h^*+\lambda(1-\omega_H))\right]}. \qquad \text{(A68)}$$

Note that (A68) is not a reduced form expression for $\bar{t}_K$, since h^* and ω_H implictly depend on $\bar{t}_K$. If $t_K<r\varepsilon_L/(\lambda(1-\varepsilon_L))$, $h^*>\lambda\omega_H$ (by Proposition A.2) and $\bar{t}_K>0$ is guaranteed. This establishes Proposition 3.2(iii).

Proposition 3.2(iv) is proved by noting that we can write steady-state utility as:

$$(\rho+\lambda)\,dU_{NE}(\infty,\infty) = \left[\frac{\tilde{C}(\infty) - (1-\omega_H)\tilde{K}(\infty)}{\omega_H}\right] + \frac{r}{\rho+\lambda}\tilde{r}(\infty), \qquad \text{(A69)}$$

where expressions for $\tilde{K}(\infty)$ and $\tilde{C}(\infty)$ are given in (3.3) and an expression for $\tilde{r}(\infty)$ is given in (3.6). The term in square brackets in (A69) represents the long-run effect on human capital, $\tilde{H}(\infty)$. By using (3.3) it can be written as:

After some manipulation, $\varepsilon_L-\omega_H$ can be written as:

$$\tilde{H}(\infty) = \frac{r(\varepsilon_L - \omega_H)\tilde{t}_K}{\omega_H \varepsilon_L \left[r - \rho + r \right]}. \tag{A70}$$

$$\varepsilon_L - \omega_H = \frac{(1-\omega_H)\left[\lambda\omega_H - (\rho+\lambda)t_K\right]}{(\rho+\lambda)(1-t_K)}. \tag{A71}$$

By substituting (A71) into (A70), the following expression for $\tilde{H}(\infty)$ is obtained:

$$\tilde{H}(\infty) = \frac{r(1-\omega_H)\left[\lambda\omega_H - (\rho+\lambda)t_K\right]\tilde{t}_K}{\omega_H \varepsilon_L (1-t_K)(\rho+\lambda)\left[r-\rho+r\right]}. \tag{A72}$$

If λ=0 and t_K=0 then $\tilde{H}(\infty)$=0. There are neither redistribution effects nor (first-order) efficiency losses. If λ>0 and t_K=0 then $\tilde{H}(\infty)$>0. There is redistribution towards future generations, but no efficiency loss. Finally, if λ=0 and t_K>0 then $\tilde{H}(\infty)$<0. There is no redistribution effect but there exists a first-order efficiency loss.

The long-run welfare effects are obtained by substituting (A72) and (3.6) into (A69). After some simplification we obtain:

$$(\rho+\lambda)\,\mathrm{d}U_{NE}(\infty,\infty) = \frac{r(1-\omega_H)\left[\lambda\omega_H\left[1-\varepsilon_L(1-t_K)\right] - (\rho+\lambda)t_K\right]\tilde{t}_K}{(\rho+\lambda)\varepsilon_L\omega_H(1-t_K)\left[r-\rho+r\right]}. \tag{A73}$$

The statements in Proposition 3.2(iv) can be deduced from (A73). If λ=0 and t_K=0 then d$U_{NE}(\infty,\infty)$=0, if λ>0 and t_K=0 then d$U_{NE}(\infty,\infty)$>0, and if $\lambda{\downarrow}0$ and t_K>0 then d$U_{NE}(\infty,\infty)$<0. This establishes Proposition 3.2(iv).

In order to prove Proposition 3.2(v), we use (A67), (A66), and (A73) and write dU_{NE}(0,0)-d$U_{NE}(\infty,\infty)$ as follows:

$$(\rho+\lambda)\left[\mathrm{d}U_{NE}(0,0) - \mathrm{d}U_{NE}(\infty,\infty)\right] = \left(\frac{r(1-\omega_H)\tilde{t}_K}{(\rho+\lambda)\omega_H r^*}\right)\times$$

$$\left[\frac{(\rho+\lambda)^2 + (\rho+\lambda)(h^* - \lambda\omega_H) - (\lambda\omega_H)^2 - \dfrac{t_K}{1-t_K}\left[\rho r\omega_H/\omega_K + (\lambda\omega_H)^2\right]}{\rho+\lambda+h^*} + \frac{r\left[(\rho+\lambda)t_K - \lambda\omega_H(1-\omega_H)\right]}{(1-\omega_H)(1-t_K)h^*}\right]$$

$$= \left(\frac{r(1-\omega_H)\tilde{t}_K}{(\rho+\lambda)\omega_H r^*}\right)\left(\left[\frac{(\rho+\lambda)^2+(\rho+\lambda)(h^*-\lambda\omega_H)-(\lambda\omega_H)^2}{\rho+\lambda+h^*} - \frac{r\lambda\omega_H}{h^*}\right]\right.$$

$$\left. + \frac{t_K}{1-t_K}\left[\frac{r\left[\rho+\lambda-\lambda\omega_H(1-\omega_H)\right]}{(1-\omega_H)h^*} - \frac{\omega_H\rho r/\omega_K+(\lambda\omega_H)^2}{\rho+\lambda+h^*}\right]\right) \tag{A74}$$

The two terms in square brackets are denoted by Ω_3 and Ω_4, respectively. After some manipulation we can simplify Ω_3 as follows:

$$\Omega_3 = \frac{h^*\left[(\rho+\lambda)^2+(\rho+\lambda)(h^*-\lambda\omega_H)-(\lambda\omega_H)^2\right] - (\rho+\lambda+h^*)\lambda\omega_H\left[\rho+\lambda(1-\omega_H)\right]}{h^*(\rho+\lambda+h^*)}$$

$$= \frac{(\rho+\lambda)\left[h^*+\rho+\lambda(1-\omega_H)\right](h^*-\lambda\omega_H)}{h^*(\rho+\lambda+h^*)} > 0, \tag{A75}$$

where we have used Proposition A.2 ($h^*>\lambda\omega_H$) and $r=\rho+\lambda(1-\omega_H)$. In a similar fashion, Ω_4 can be simplified to:

$$(1-\omega_H)\Omega_4 = \frac{(\rho+\lambda+h^*)r\left[r+\lambda\omega_H^2\right] - h^*\left[\omega_H\rho(\rho+\lambda)+(1-\omega_H)(\lambda\omega_H)^2\right]}{h^*(\rho+\lambda+h^*)}$$

$$= \frac{(\rho+\lambda)r\left[\rho+\lambda\omega_H^2\right]+h^*\left[r(r+\lambda\omega_H^2)-\omega_H\rho(\rho+\lambda)-(1-\omega_H)(\lambda\omega_H)^2\right]}{h^*(\rho+\lambda+h^*)} > 0, \tag{A76}$$

where we have used that the term involving h^* is positive:

$$r(r+\lambda\omega_H^2)-\omega_H\rho(\rho+\lambda)-(1-\omega_H)(\lambda\omega_H)^2$$

$$= (1-\omega_H)(\rho+\lambda)r > 0, \tag{A77}$$

Hence, since both $\Omega_3>0$ and $\Omega_4>0$, $dU_{NE}(0,0)>dU_{NE}(\infty,\infty)$ for all t_K, as is stated in Proposition 3.2(v).

In order to prove Proposition 3.2(vi), we write the utility of the representative agent at the time of the shock (equation (3.12)) as:

$$\mathrm{d}U_{NE}(0) \equiv \frac{-\rho\, t_K \tilde{t}_K}{\left[\rho + h^*(1-t_K)\right](\rho + h^*)},$$

from which the result follows. This completes the proof of Proposition 3.2(vi). □

In Figure A.3 non-environmental welfare profiles are illustrated for different values of λ (panel a) and the initial capital tax t_K (panel b).

A.6. Environmental utility and the proof of Proposition 3.3.

A.6.1. Environmental utility

The environmental component of total utility is given in equation (A45). By linearising (A45), and using Lemma A.6, the following expression for the change in environmental utility can be obtained:

$$\frac{1}{E}\mathfrak{L}\{\mathrm{d}U_E, s\} = \frac{\mathfrak{L}\{\tilde{E}, \rho+\lambda\} - \mathfrak{L}\{\tilde{E}, s\}}{s-(\rho+\lambda)}. \tag{A78}$$

For existing generations the effect on environmental utility is obtained by applying the initial value theorem, *i.e.*, $\mathrm{d}U_E(0)\equiv\lim_{s\to\infty}s\mathfrak{L}\{\mathrm{d}U_E,s\}=E\mathfrak{L}\{\tilde{E},\rho+\lambda\}$.

In order to derive the results in sections 3 and 4 of the paper, the path of $\mathrm{d}U_E(t)$ is written in terms of $\mathrm{d}U_E(0)$, $\mathrm{d}U_E(\infty)$, and generalized multiple adjustment and transition terms like $\mathrm{A}(\alpha_E,h^*,\rho+\lambda,t)$, and $\mathrm{T}(\alpha_E,h^*,\xi_i,\rho+\lambda,t)$. Equation (A78) implies:

$$\mathrm{d}U_E(0) = E\mathfrak{L}\{\tilde{E},\rho+\lambda\}, \qquad \mathrm{d}U_E(\infty) = E\left(\frac{\tilde{E}(\infty)}{\rho+\lambda}\right). \tag{A79}$$

Equation (A78) can be expanded by using (A31) in the following way:

$$\begin{aligned}\frac{\mathfrak{L}\{\tilde{E},\rho+\lambda\} - \mathfrak{L}\{\tilde{E},s\}}{s-(\rho+\lambda)} &= \tilde{E}(\infty)\left[\frac{\mathfrak{L}\{\mathrm{A}(\alpha_E,h^*,t),\rho+\lambda\} - \mathfrak{L}\{\mathrm{A}(\alpha_E,h^*,t),s\}}{s-(\rho+\lambda)}\right] \\ &\quad -\alpha_E\alpha_K\gamma_B\delta_{21}\sum_{i=1}^{2}\left(\frac{\tilde{B}_i}{r^*+\xi_i}\right)\left[\frac{\mathfrak{L}\{\mathrm{T}(\alpha_E,h^*,\xi_i,t),\rho+\lambda\} - \mathfrak{L}\{\mathrm{T}(\alpha_E,h^*,\xi_i,t),s\}}{s-(\rho+\lambda)}\right].\end{aligned} \tag{A80}$$

It is straightforward to show that the first term in square brackets on the RHS of (A80) can be expanded as follows:

$$\begin{aligned}\frac{\mathfrak{L}\{\mathrm{A}(\alpha_E,h^*,t),\rho+\lambda\} - \mathfrak{L}\{\mathrm{A}(\alpha_E,h^*,t),s\}}{s-(\rho+\lambda)} &= \frac{1}{s}\mathfrak{L}\{\mathrm{A}(\alpha_E,h^*,t),\rho+\lambda\} \\ &\quad - \mathrm{G}(s)\left[\mathfrak{L}\{\mathrm{A}(\alpha_E,h^*,t),\rho+\lambda\} - \frac{1}{\rho+\lambda}\right],\end{aligned} \tag{A81}$$

where $\mathrm{G}(s)$ is the Laplace transform of a generalized multiple adjustment term, the properties of which are stated in Lemma A.7 below:

Furthermore, the second term in square brackets on the RHS of (A80) can be expanded as:

Denoting the inverse Laplace transform of $\mathrm{H}_i(s)$ by $\mathrm{h}_i(t)$, it is clear that $\mathrm{h}_i(0)\equiv\mathfrak{L}\{\mathrm{T}(\alpha_E,h^*,\xi_i,t),\rho+\lambda\}$ and $\mathrm{h}_i(\infty)=\mathrm{T}(\alpha_E,h^*,\xi_i,\infty)/(\rho+\lambda)=0$. It is possible to write $\mathrm{H}_i(s)$ as:

$$G(s) \equiv \mathfrak{L}\{A(\alpha_E,h^*,\rho+\lambda,t),s\} = \frac{1}{s} + \frac{h^*(\rho+\lambda+h^*)}{(\alpha_E-h^*)(\rho+\lambda+\alpha_E+h^*)}\frac{1}{s+\alpha_E} - \frac{\alpha_E(\rho+\lambda+\alpha_E)}{(\alpha_E-h^*)(\rho+\lambda+\alpha_E+h^*)}\frac{1}{s+h^*}. \tag{A82}$$

$$\frac{\mathfrak{L}\{T(\alpha_E,h^*,\xi_i,t),\rho+\lambda\} - \mathfrak{L}\{T(\alpha_E,h^*,\xi_i,t),s\}}{s-(\rho+\lambda)} \equiv H_i(s) = \frac{1}{(\alpha_E-\xi_i)(\alpha_E-h^*)(\xi_i-h^*)}\times \tag{A83}$$

$$\left[\frac{(\xi_i-h^*)}{(\rho+\lambda+\alpha_E)}\frac{1}{s+\alpha_E} + \frac{(\alpha_E-\xi_i)}{(\rho+\lambda+h^*)}\frac{1}{s+h^*} + \frac{(h^*-\alpha_E)}{(\rho+\lambda+\xi_i)}\frac{1}{s+\xi_i}\right].$$

$$H_i(s) \equiv \mathfrak{L}\{T(\alpha_E,h^*,\xi_i,t),\rho+\lambda)\}\left[\frac{1}{s} - G(s)\right] + J_i(s), \tag{A84}$$

where $J_i(s)$ is the Laplace transform of a generalized multiple transition term, the properties of which are covered in Lemma A.8 below.

$$J_i(s) \equiv \mathfrak{L}\{T(\alpha_E,h^*,\xi_i,\rho+\lambda,t),s\} = \mathfrak{L}\{T(\alpha_E,h^*,\xi_i,t),\rho+\lambda\}\left[\frac{\omega_{1,i}}{s+\alpha_E} + \frac{\omega_{2,i}}{s+h^*} + \frac{\omega_{3,i}}{s+\xi_i}\right] \tag{A85}$$

where $\omega_{j,i}$ are defined as follows:

$$\omega_{1,i} = \frac{(\rho+\lambda+h^*)(\rho+\lambda+\xi_i)}{(\alpha_E-\xi_i)(\alpha_E-h^*)} + \frac{h^*(\rho+\lambda+h^*)}{(\alpha_E-h^*)(\rho+\lambda+\alpha_E+h^*)}, \quad \omega_{2,i} = \frac{(\rho+\lambda+\alpha_E)(\rho+\lambda+\xi_i)}{(\alpha_E-h^*)(\xi_i-h^*)} - \frac{\alpha_E(\rho+\lambda+\alpha_E)}{(\alpha_E-h^*)(\rho+\lambda+\alpha_E+h^*)},$$

$$\omega_{3,i} = -\frac{(\rho+\lambda+\alpha_E)(\rho+\lambda+h^*)}{(\alpha_E-\xi_i)(\xi_i-h^*)}.$$

Hence, $\sum_j\omega_{j,i}=0$. Denoting the inverse Laplace transform of $J_i(s)$ by $j_i(t)$, it is clear that $j_i(0)=j_i(\infty)=0$.

By using (A81)-(A85) in (A80), we can write (A78) as:

$$\frac{1}{E}\mathfrak{L}\{dU_E,s\} = \tilde{E}(\infty)\left[\frac{1}{s}\mathfrak{L}\{A(\alpha_E,h^*,t),\rho+\lambda\} - G(s)\left[\mathfrak{L}\{A(\alpha_E,h^*,t),\rho+\lambda\} - \frac{1}{\rho+\lambda}\right]\right]$$

$$- \alpha_E\alpha_K\gamma_B\delta_{21}\sum_{i=1}^{2}\left(\frac{\tilde{B}_i}{r^*+\xi_i}\right)\left[J_i(s) + \left(\frac{1}{s}-G(s)\right)\mathfrak{L}\{T(\alpha_E,h^*,\xi_i,t),\rho+\lambda\}\right]$$

$$= \frac{1}{s}\left[\tilde{E}(\infty)\mathfrak{L}\{\mathrm{A}(\alpha_E,h^*,t),\rho+\lambda\} - \alpha_E\alpha_K\gamma_B\delta_{21}\sum_{i=1}^{2}\left(\frac{\tilde{B}_i}{r^*+\xi_i}\right)\mathfrak{L}\{\mathrm{T}(\alpha_E,h^*,\xi_i,t),\rho+\lambda\}\right]$$

$$- \mathrm{G}(s)\left[\tilde{E}(\infty)\mathfrak{L}\{\mathrm{A}(\alpha_E,h^*,t),\rho+\lambda\} - \alpha_E\alpha_K\gamma_B\delta_{21}\sum_{i=1}^{2}\left(\frac{\tilde{B}_i}{r^*+\xi_i}\right)\mathfrak{L}\{\mathrm{T}(\alpha_E,h^*,\xi_i,t),\rho+\lambda\} - \frac{\tilde{E}(\infty)}{\rho+\lambda}\right]$$

$$- \alpha_E\alpha_K\gamma_B\delta_{21}\sum_{i=1}^{2}\left(\frac{\tilde{B}_i}{r^*+\xi_i}\right)\mathrm{J}_i(s). \qquad \text{(A86)}$$

By taking the inverse Laplace transform of (A86) we obtain:

$$\mathrm{d}U_E(t) = \mathrm{d}U_E(0) + \mathrm{A}(\alpha_E,h^*,\rho+\lambda,t)\left[\mathrm{d}U_E(\infty) - \mathrm{d}U_E(0)\right] - \alpha_E\alpha_K\gamma_B\delta_{21}E\sum_{i=1}^{2}\left(\frac{\tilde{B}_i}{r^*+\xi_i}\right)\mathrm{T}(\alpha_E,h^*,\xi_i,\rho+\lambda,t), \qquad \text{(A87)}$$

where we have used the definitions for $\mathrm{d}U_E(0)$ and $\mathrm{d}U_E(\infty)$.

$$\frac{\mathrm{d}U_E(0)}{E} \equiv \mathfrak{L}\{\tilde{E},\rho+\lambda\} = \tilde{E}(\infty)\mathfrak{L}\{\mathrm{A}(\alpha_E,h^*,t),\rho+\lambda\} - \alpha_E\alpha_K\gamma_B\delta_{21}\sum_{i=1}^{2}\left(\frac{\tilde{B}_i}{r^*+\xi_i}\right)\mathfrak{L}\{\mathrm{T}(\alpha_E,h^*,\xi_i,t),\rho+\lambda\},$$

$$\frac{\mathrm{d}U_E(\infty)}{E} \equiv \frac{\tilde{E}(\infty)}{\rho+\lambda}.$$

The results in section 3 of the text are obtained by setting $\tilde{B}(0)=\tilde{B}_1=\tilde{B}_2=0$.

LEMMA A.7: *Let* $\mathrm{A}(\alpha_1,\alpha_2,\beta,t)$ *be a generalized multiple adjustment function of the form:*

$$\mathrm{A}(\alpha_1,\alpha_2,\beta,t) \equiv 1 - \left(\frac{\alpha_2(\beta+\alpha_2)}{(\alpha_2-\alpha_1)(\beta+\alpha_1+\alpha_2)}\right)e^{-\alpha_1 t} + \left(\frac{\alpha_1(\beta+\alpha_1)}{(\alpha_2-\alpha_1)(\beta+\alpha_1+\alpha_2)}\right)e^{-\alpha_2 t},$$

with α_1, α_2, $\beta>0$, $\alpha_1\neq\alpha_2$. *Then* $\mathrm{A}(\alpha_1,\alpha_2,\beta,t)$ *has the following properties: (i) (increasing over time)* $\mathrm{dA}(\alpha_1,\alpha_2,\beta,t)/\mathrm{d}t>0$ $\forall t\in(0,\infty)$, $\mathrm{dA}(\alpha_1,\alpha_2,\beta,t)/\mathrm{d}t=\alpha_1\alpha_2/(\beta+\alpha_1+\alpha_2)>0$ *for* $t=0$ *and* $\mathrm{dA}(\alpha_1,\alpha_2,\beta,t)/\mathrm{d}t\downarrow 0$ *in the limit as* $t\to\infty$), *(ii) (between 0 and 1)* $0<\mathrm{A}(\alpha_1,\alpha_2,\beta,t)<1$ $\forall t\in(0,\infty)$ *and* $\mathrm{A}(\alpha_1,\alpha_2,\beta,0) = 1-\lim_{t\to\infty}\mathrm{A}(\alpha_1,\alpha_2,\beta,t)=0$, *(iii) (inflexion point)* $\mathrm{d}^2\mathrm{A}(\alpha_1,\alpha_2,\beta,t)/\mathrm{d}t^2=0$ *for* $t=\hat{t}$, *where* $\hat{t}\equiv\ln[\alpha_1(\beta+\alpha_2)/(\alpha_2(\beta+\alpha_1))]/(\alpha_1-\alpha_2)>0$.

PROOF: The derivative of $A(\alpha_1,\alpha_2,\beta,t)$ with respect to time is equal to:

$$\frac{dA(\alpha_1,\alpha_2,\beta,t)}{dt} = \frac{\alpha_1\alpha_2}{(\alpha_2-\alpha_1)(\beta+\alpha_1+\alpha_2)}\left[(\beta+\alpha_2)e^{-\alpha_1 t}-(\beta+\alpha_1)e^{-\alpha_2 t}\right]>0 \quad \text{for } t\in(0,\infty).$$

where the inequality follows by examining the two possible cases. If $\alpha_1>(<)\alpha_2$, then $\alpha_2-\alpha_1<(>)0$ and $(\beta+\alpha_2)\exp[-\alpha_1 t]<(>)(\beta+\alpha_1)\exp[-\alpha_2 t]$ for all $t\in(0,\infty)$. Property (ii) follows from the fact that $A(\alpha_1,\alpha_2,\beta,0)=0$ and $\lim_{t\to\infty}A(\alpha_1,\alpha_2,\beta,t)=1$ plus the fact that $dA(\alpha_1,\alpha_2,0)/dt\geq 0$. Property (iii) makes use of:

$$\frac{d^2A(\alpha_1,\alpha_2,\beta,t)}{d^2t} = -\frac{\alpha_1\alpha_2}{(\alpha_2-\alpha_1)(\beta+\alpha_1+\alpha_2)}\left[\alpha_1(\beta+\alpha_2)e^{-\alpha_1 t}-\alpha_2(\beta+\alpha_1)e^{-\alpha_2 t}\right].$$

Hence, $d^2A(\alpha_1,\alpha_2,t)/dt^2=\alpha_1\alpha_2\beta/(\beta+\alpha_1+\alpha_2)>0$ for $t=0$, and $\lim_{t\to\infty}d^2A(\alpha_1,\alpha_2,\beta,t)/dt^2=0$. The inflexion point is found by finding the value of t where $d^2A(\alpha_1,\alpha_2,\beta,t)/dt^2=0$. The solution is $\hat{t}\equiv\ln[\alpha_1(\beta+\alpha_2)/(\alpha_2(\beta+\alpha_1))]/(\alpha_1-\alpha_2)$.□

REMARK: As $\beta\to\infty$ the generalized multiple adjustment term converges to the multiple adjustment terms, *i.e.*, $\lim_{\beta\to\infty}A(\alpha_1,\alpha_2,\beta,t)=A(\alpha_1,\alpha_2,t)$.

LEMMA A.8: *Let* $T(\alpha_1,\alpha_2,\alpha_3,\beta,t)$ *be a generalized multiple transition function of the form:*

$$T(\alpha_1,\alpha_2,\alpha_3,\beta,t) \equiv \mathfrak{L}\{T(\alpha_1,\alpha_2,\alpha_3,t),\beta\}\left[\omega_1 e^{-\alpha_1 t}+\omega_2 e^{-\alpha_2 t}+\omega_3 e^{-\alpha_3 t}\right],$$

with $0<\alpha_1,\alpha_2,\alpha_3<\infty$, $\beta>0$, $\sum\omega_i=0$, *and* $\alpha_i\neq\alpha_j$ $\forall i\neq j$. *Then* $T(\alpha_1,\alpha_2,\alpha_3,\beta,t)$ *has the following properties: (i) (positive)* $T(\alpha_1,\alpha_2,\alpha_3,\beta,t)>0$, (ii) $T(\alpha_1,\alpha_2,\alpha_3,\beta,t)=0$ *for* $t=0$ *and in the limit as* $t\to\infty$, *(iii) (single-peaked)* $dT(\alpha_1,\alpha_2,\alpha_3,\beta,t)/dt>0$ for $0<t<\bar{t}$ *and* $dT(\alpha_1,\alpha_2,\alpha_3,\beta,t)/dt<0$ *for* $t>\bar{t}$, $dT(\alpha_1,\alpha_2,\alpha_3,\beta,t)/dt=0$ *(for* $t=0$, $t=\bar{t}$, *and* $t\to\infty$*).*

PROOF: To be added. The properties that are stated in Lemma A.8 were verified with numerical simulations. The generalised multiple transition function only plays a role in the numerical results in section 4.□

A.6.1. Proposition 3.3

In order to prove Proposition 3.3(i) we write, by using (A21), $dU_E(0)/E=\mathfrak{L}\{\tilde{E}(t),\rho+\lambda\}=-\alpha_E\alpha_K\mathfrak{L}\{\tilde{K},\rho+\lambda\}/(\rho+\lambda+\alpha_E)$. The Laplace transform of (A28) can be rewritten as follows:

$$\mathcal{L}\{\tilde{K},\rho+\lambda\} = -\frac{\gamma_K\delta_{21}}{r^*(\rho+\lambda)(\rho+\lambda+h^*)}\tilde{t}_K < 0. \tag{A88}$$

Hence, $dU_E(0)/E>0$. This completes the proof of Propositions 3.3(i). The proof of Proposition 3.3(i) is immediate:

$$\frac{dU_E(\infty)}{E} = -\alpha_K\frac{\tilde{K}(\infty)}{\rho+\lambda} = \frac{\alpha_K\gamma_K\delta_{21}}{(\rho+\lambda)r^*h^*}\tilde{t}_K > 0. \tag{A89}$$

In order to prove Proposition 3.3(iii), we write:

$$\frac{dU_E(\infty)}{E} - \frac{dU_E(0)}{E} = \frac{\alpha_K\gamma_K\delta_{21}}{r^*(\rho+\lambda)}\left[\frac{1}{h^*} - \frac{\alpha_E}{(\rho+\lambda+\alpha_E)(\rho+\lambda+h^*)}\right]\tilde{t}_K > 0. \tag{A90}$$

This completes the proof of Proposition 3.3.□

Figure A.3.a shows how the birth rate impacts the intergenerational distributional effects due to the introduction of a pollution tax (*i.e.* the initial pollution tax is zero). In the absence of overlapping generations (*i.e.* a zero birth rate), the welfare profile is flat. With a zero initial pollution tax, there are no first-order effects on private welfare. If the birth rate is positive, the oldest generations lose out in terms of private welfare while future and younger, existing generations gain. Environmental welfare rises for all generations with future generations experiencing a larger increase in environmental welfare because they are born in a cleaner world.

A larger birth rate, which increases the disconnectedness of generations, widens intergenerational inequities. Indeed, it raises disparities in welfare between the oldest generations, which suffer a private welfare loss, and the youngest, future generations, which enjoy a gain in both private and environmental welfare. In particular, a larger death rate implies that older generations feature a shorter planning horizon and thus internalise less of the recovery of capital incomes in the future. Therefore, they suffer larger losses in private welfare. Moreover, the shorter horizon associated with a higher death rate implies that existing generations internalise less of the future improvement in environmental quality and thus enjoy smaller increases in environmental welfare (see the second panel of Figure A.3.a).

Generations that are born far in the future, in contrast, enjoy larger private welfare gains with a larger birth rate. The reason is that a larger birth rate contains the decline in the long-run capital stock by partly offsetting the negative saving effect on account of intertemporal substitution by positive saving effects associated with intergenerational redistribution. The smaller decline in the long-run capital stock limits the fall in gross wages, thereby allowing human capital, which determines the welfare of future generations, to rise in the long run.

Figure A.3.b shows how the initial pollution tax affects intergenerational welfare (for a birth rate of λ=0.02). If the initial tax rate is zero, young existing and all future generations benefit in terms of private welfare at the expense of older, existing generations. With a higher initial tax rate of 0.25, the larger increase in the overall tax burden drags down the private welfare levels of the generations that benefited from the introduction of such a tax. Only those generations born close to the time of the unanticipated policy shock are still better off in terms of private welfare as a result of a further marginal increase in the pollution tax. As explained in sub-section 3.5, a higher tax burden alleviates the private welfare losses suffered by the oldest generations by making capital scarcer. By thus reducing both the losses of the old and the gains of young, a higher initial tax rate flattens out the private welfare profile across generations. Furthermore, a higher initial tax rate boosts the gains in environmental welfare because the larger aggregate tax burden reduces saving, capital accumulation, and economic activity, thereby reducing pollution.

Figure A.4 contains, for various initial tax rates, the generation-specific environmental weights, $\gamma_E(v,v')$, for which each generation is indifferent to a marginal increase in the pollution tax rate. For existing generations, this weight is rising with age. This is consistent with the analysis in sub-section 3.5, which showed that private welfare falls with age while environmental welfare is the same for all generations.

For generations that are yet to be born, the relationship between the environmental weight and the time of birth is less straightforward because younger generations enjoy both less private welfare (associated with a smaller physical capital stock) and more environmental welfare (associated with a larger natural capital stock). If the initial tax rate is low, the required environmental weight is increasing monotonically with the date of birth because the negative effect of the birth date on private welfare dominates. Intuitively, the rising required welfare weight reflects the decline in the stock of physical capital. With a higher initial tax rate, however, the critical environmental utility weight may decline for generations born far into the future because younger generations benefit from larger increases in environmental welfare. In this case, the declining utility weight reflects the increase in the stock of natural capital. Even in that case, however, the weight increases for generations born immediately after the unanticipated policy shock so that the environmental weight varies non-monotonically with the date of birth. The reason is that the environment (and hence natural capital and environmental welfare) adjusts more slowly than the economy (and physical capital and private welfare) does. As a result of the relative slow adjustment of natural versus physical capital, the decline in private welfare dominates for generations born immediately after the increase in the tax.

A.7. Redistribution issues

The simulations underlying Table 3 in the text are performed as follows. The policy maker uses the path of debt as parameterized in (A24). This implies that the paths for the capital stock, consumption, and the environment are given by (A28)-(A31). The paths for private utility are given in (A53) and (A59) and the path for environmental utility is given in (A87). Total utility is the sum of private and environmental utility - see (3.24)-(3.25) in the text. The requirements for the egalitarian policy are that all generations gain to the same extent. Denoting this common gain by π, the requirements are summarized by:

$$\mathrm{d}U(v,0) = \mathrm{d}U(t,t) = \pi, \qquad v \leq 0,\ t \geq 0.$$

The instruments at the disposal of the policy maker are a once-off subsidy to capital owners at the time of the shock (s_K which implies a value for $\tilde{B}(0)$), plus the parameters influencing the shape of the bond path (ξ_i and $\tilde{B}_i$, i=1,2).

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Table A.1: The log-linearised model

$$\dot{\tilde{C}}(t) = (r-\rho)\tilde{C}(t) + r\tilde{r}(t) - (r-\rho)\left[\tilde{K}(t) + \omega_K^{-1}\tilde{B}(t)\right] \tag{T.1′}$$

$$\dot{\tilde{K}}(t) = \left(\frac{r}{(1-t_K)(1-\varepsilon_L)}\right)\left[\tilde{Y}(t) - \tilde{C}(t)\right] \tag{T.2′}$$

$$\tilde{B}(t) = (1-t_K)(1-\varepsilon_L)\left[\tilde{t}_K + \left(\frac{t_K}{1-t_K}\right)\tilde{Y}(t)\right] + \tilde{Z}(t) + r^{-1}\dot{\tilde{B}}(t) \tag{T.3′}$$

$$\dot{\tilde{E}}(t) = -\alpha_E\left[\tilde{E}(t) + \alpha_K\tilde{K}(t)\right] \tag{T.4′}$$

$$\tilde{Y}(t) = (1-\varepsilon_L)\tilde{K}(t) \tag{T.5′}$$

$$\tilde{W}(t) = \tilde{Y}(t) \tag{T.6′}$$

$$\tilde{r}(t) + \tilde{t}_K = -\varepsilon_L\tilde{K}(t) \tag{T.7′}$$

Shares and semi-elasticities:

ε_L	WL/Y	Share of labour income in net output.
α_E	$-f_E$	Speed of regeneration of the environment (α_E>0).
α_K	$f_K K/f_E E$	Substitution elasticity between environmental and physical capital in the steady state (α_K>0).

Note:

In the derivation of this table from Table 1, we assume that *initial* debt is zero (B=0).

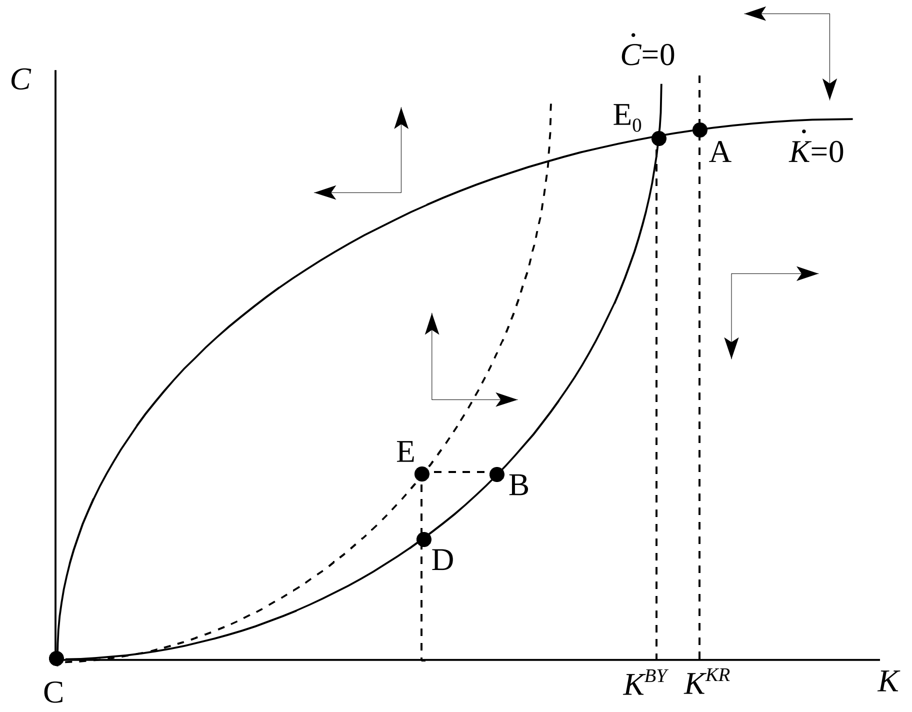


Figure A.1. Equilibrium and stability.

Key: K^{KR} is the Keynes-Ramsey capital stock, for which $r=(1-t_K)\partial F/\partial K=\rho$, and K^{BY} is the Blanchard-Yaari equilibrium capital stock.

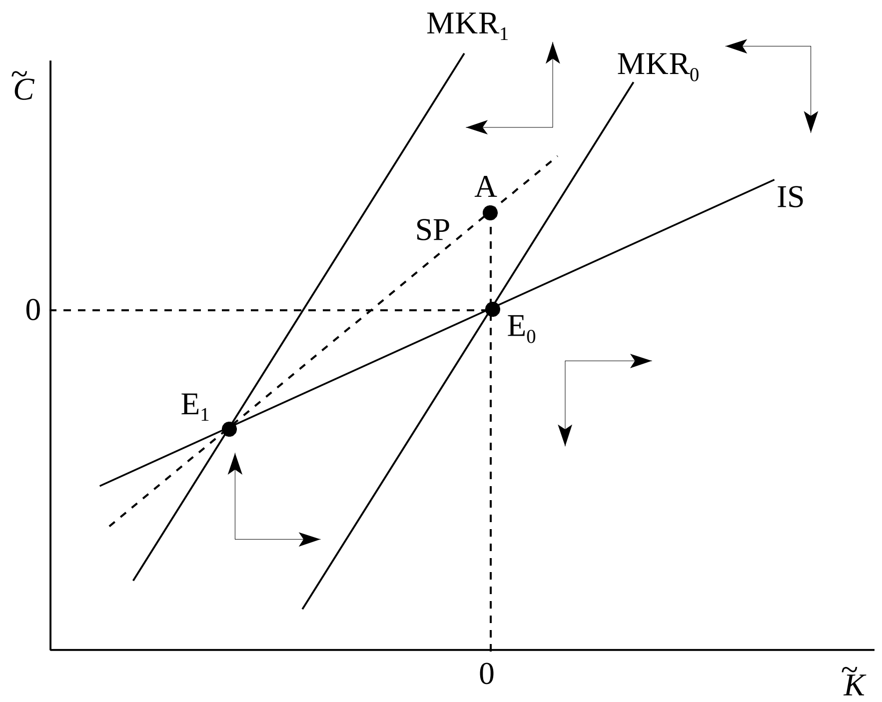
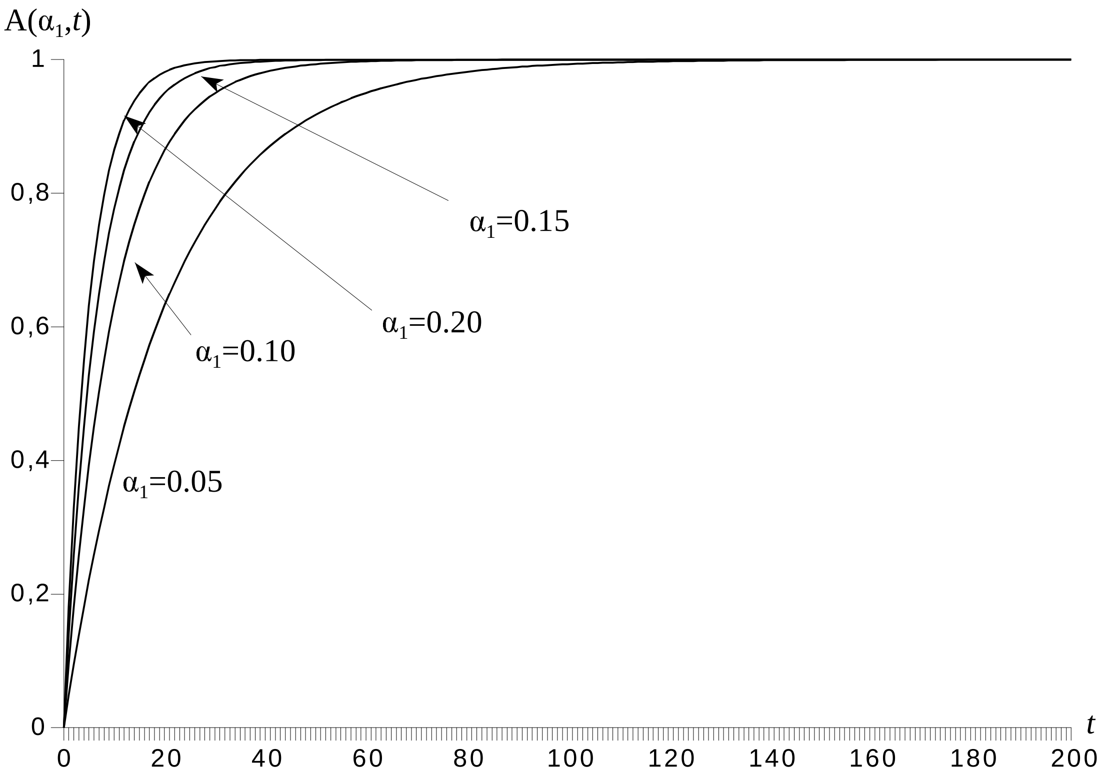


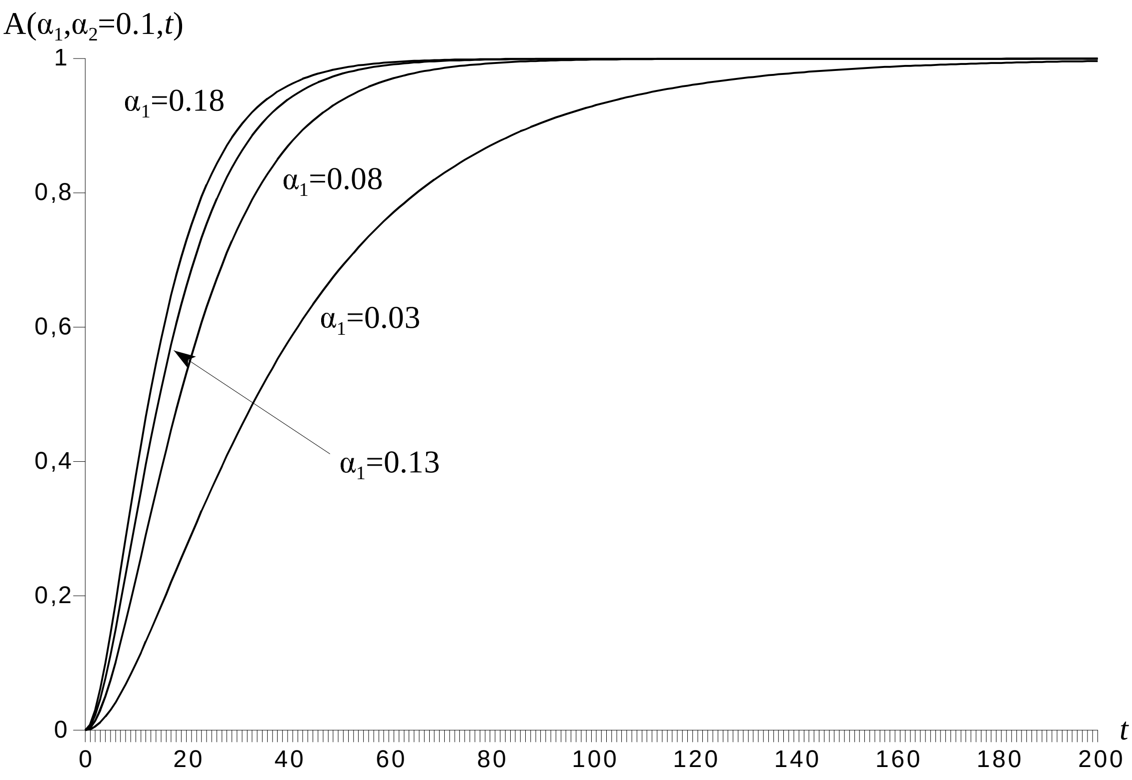
Figure A.2. Macroeconomic effects of capital taxation.

Key: The capital tax shifts the MKR curve from MKR_0 to MKR_1 and the steady-state moves from E_0 to E_1. The transition path consists of a discrete adjustment from E_0 to A at the time of the shock, followed by gradual adjustment along the saddle path SP from A to E_1.

Figure A.3. Adjustment and transition terms



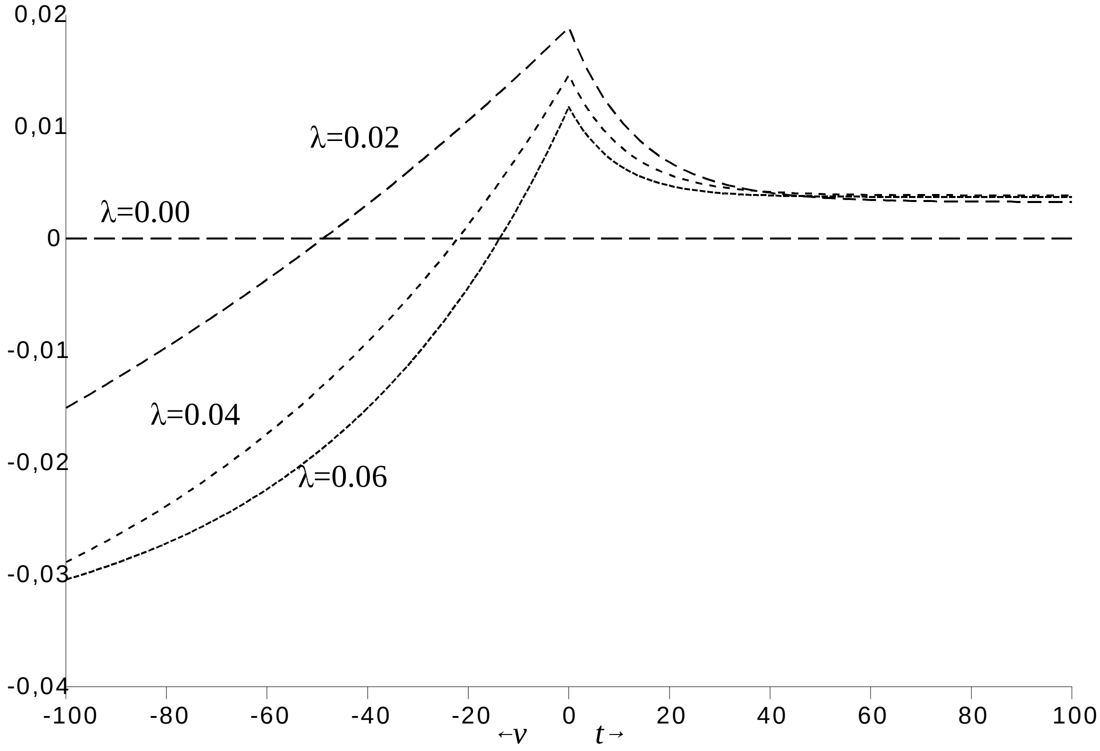
Key: The single adjustment term $A(\alpha_1,t)$ is increasing over time. As α_1 rises, the closer $A(\alpha_1,t)$ approximates a step function.



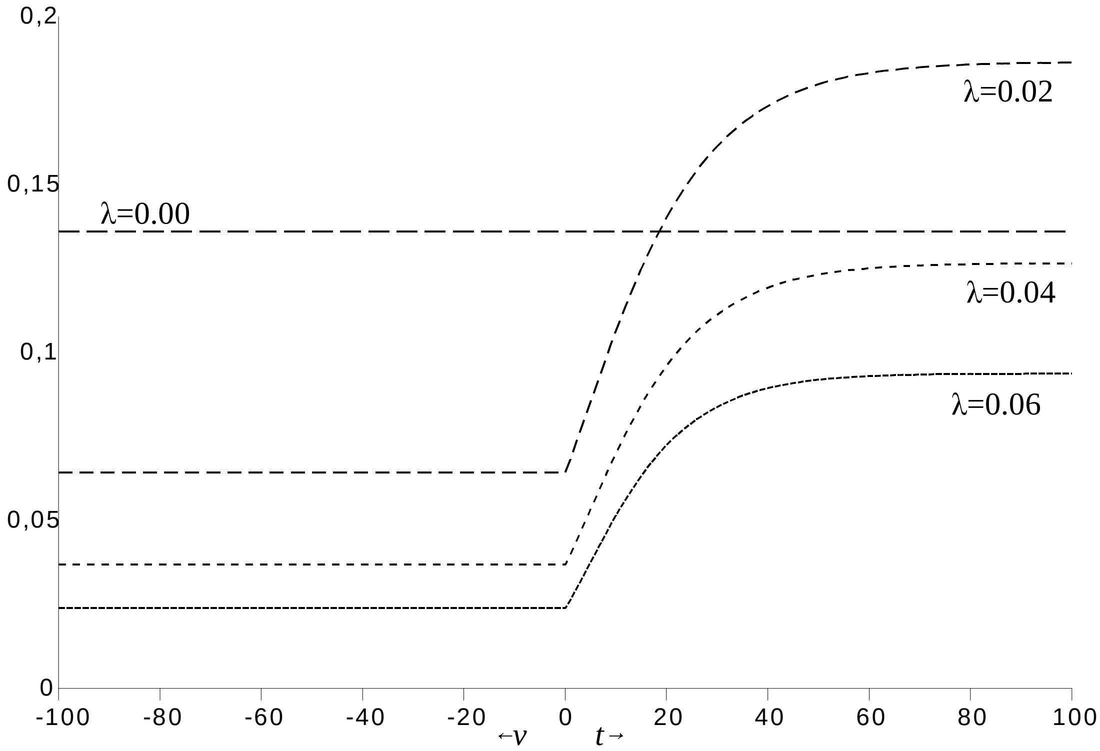
Key: The multiple adjustment term $A(\alpha_1,\alpha_2,t)$ is S-shaped and increasing over time. As α_1 rises (for a given value of α_2), $A(\alpha_1,\alpha_2,t)$ more and more approximates a step function.

Figure A.3.a. The effect of the birth rate on the welfare paths

Private welfare: dU_P



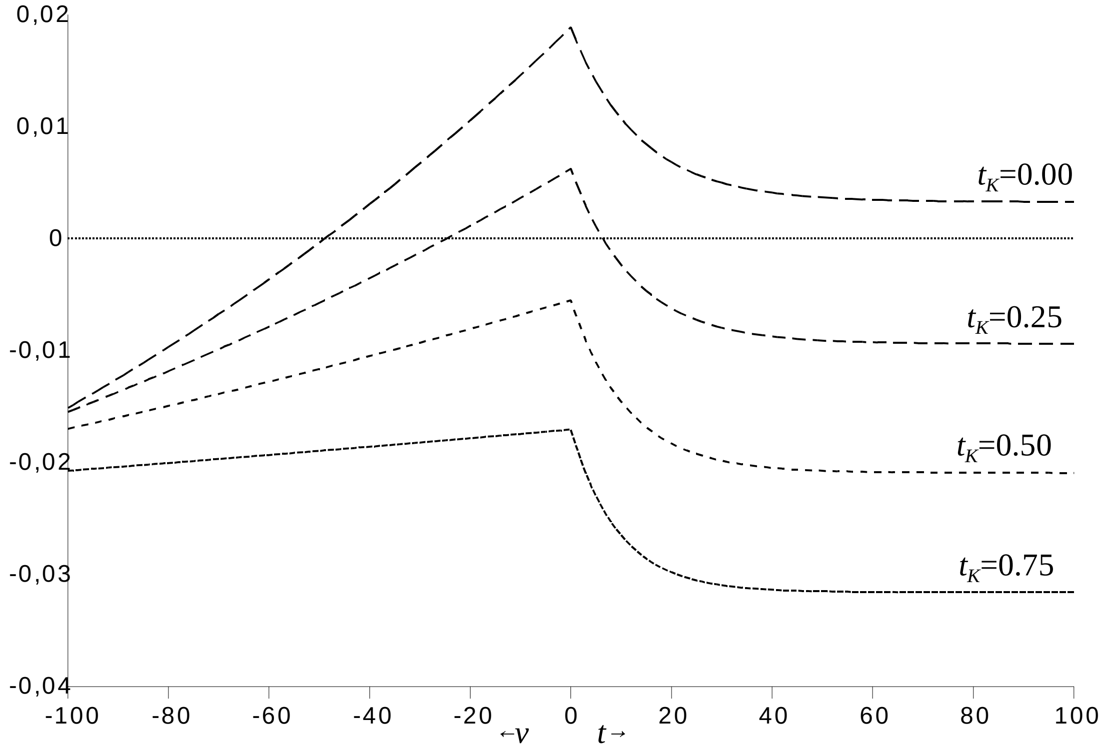
Environmental welfare: dU_E



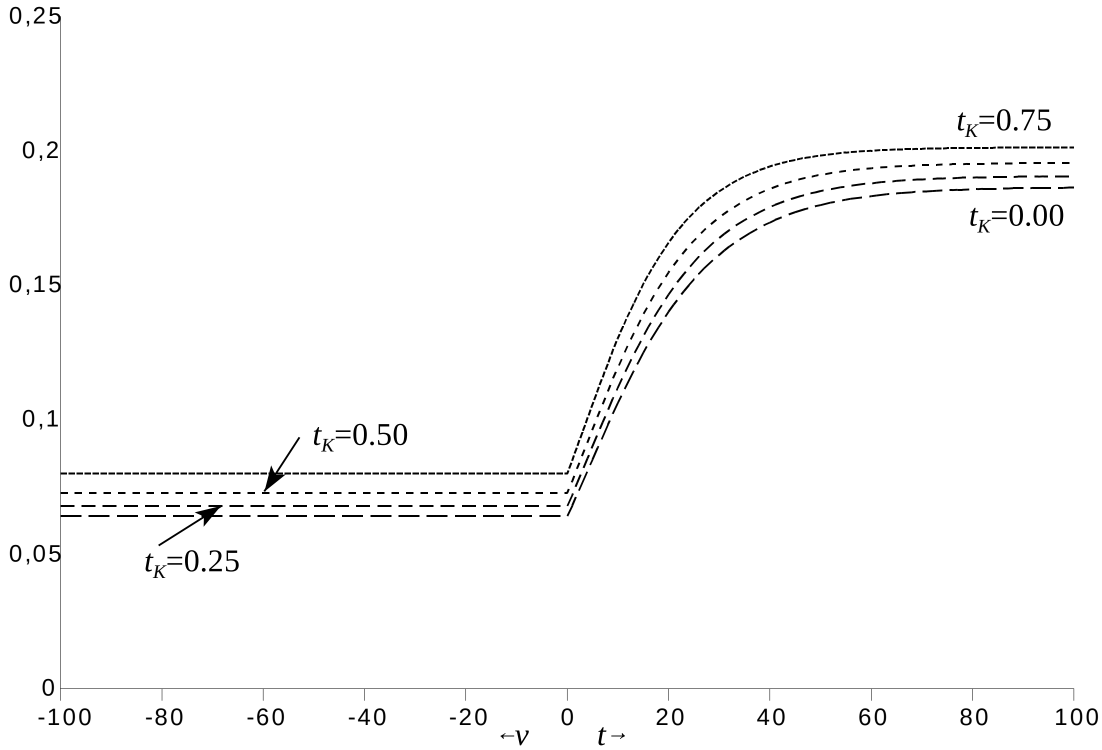
Key: The tax rate on capital is t_K=0, v is the generations index for existing generations, and t is the index for historical time, which also represents the generations index for future generations whose welfare is evaluated at birth.

Figure A.3.b. The effect of the initial tax rate on the welfare paths

Private welfare: dU_P

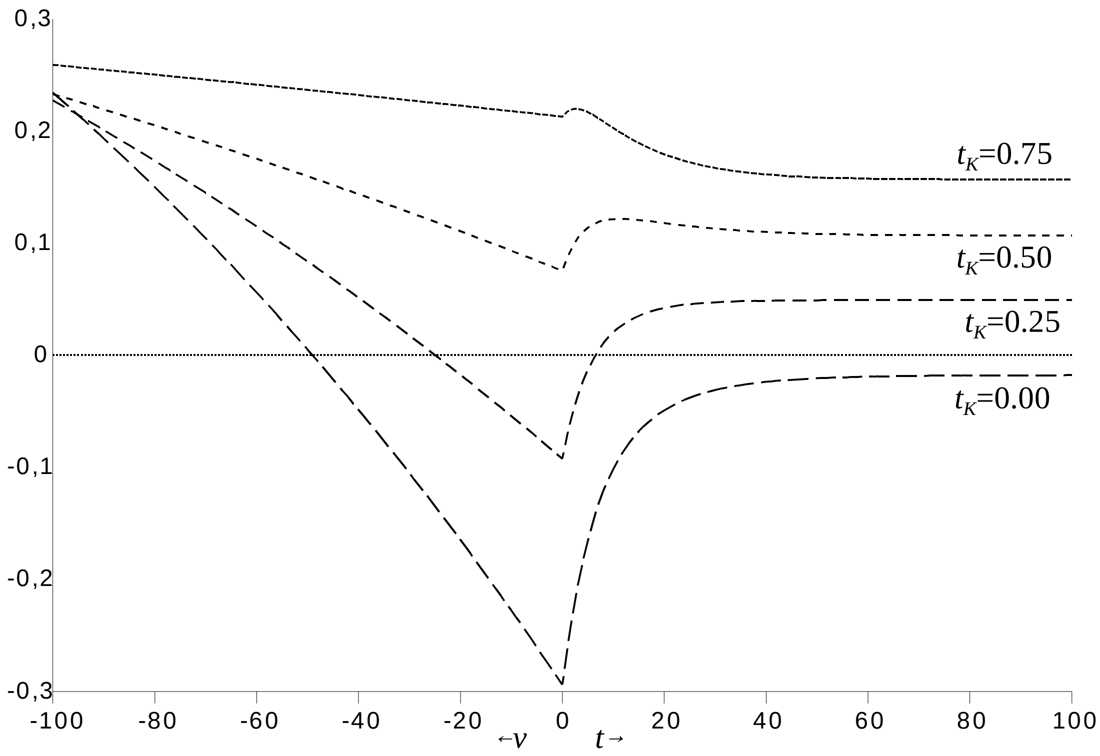


Environmental welfare: dU_E



Key: The birth rate is λ=0.02, *v* is the generations index for existing generations, and *t* is the index for historical time, which also represents the generations index for future generations whose welfare is evaluated at birth.

Figure A.4. Required environmental weights for different initial tax rates by generation



Key: The required environmental weights are given for each generation. These weights, $\gamma_E(v,v')$, are such that the particular generation to which it refers is indifferent to a marginal increase in the pollution tax rate.