Intergenerational and international welfare leakages of a product subsidy in a small open economy: Mathematical appendix

Leon J.H. Bettendorf* Ben J. Heijdra♯

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1 Introduction

In this appendix all computations for the main paper are presented. A more general case than the one discussed in the main paper is dealt with here. Specifically, the model discussed here is more general in three aspects. First, here (as in an earlier version of the paper), preference for diversity (η) and the markup ($\mu \equiv \sigma_C/(\sigma_C - 1)$) are distinguished separately. So instead of (2) in the text we use:

$$\begin{aligned} C_D(v,\tau) &\equiv N(\tau)^{\eta}\left[N(\tau)^{-1}\sum_{i=1}^{N(\tau)} C_{D,i}(v,\tau)^{1/\mu}\right]^{\mu}, \;\; \sigma_C > 1, \;\; \eta \geq 1, \\ C_F(v,\tau) &\equiv (N^*)^{\eta}\left[(N^*)^{-1}\sum_{j=1}^{N^*} C_{F,j}(v,\tau)^{1/\mu}\right]^{\mu}. \end{aligned}$$

The consequences of using these more general specifications are found in equations (A.13)-(A.16) below. Since setting $\eta = \mu$ does not excluded anything of interest *in the present model* this simplification has been adopted.

Second, more general versions of the sub-utility function (1) and the export demand equation (8) have been used in this appendix:

$$\begin{aligned} U(v,t) &\equiv \left[\gamma_D C_D(v,t)^{\frac{\sigma_A-1}{\sigma_A}} + (1-\gamma_D) C_F(v,t)^{\frac{\sigma_A-1}{\sigma_A}}\right]^{\frac{\gamma\sigma_A}{\sigma_A-1}} \left[1-L(v,t)\right]^{1-\gamma}, \\ C^*_{F,i}(t) &= C^*_F N(t)^{-(\sigma_C+\eta)+\eta\sigma_C} E(t)^{\sigma_T}\left(\frac{P_{D,i}(t)}{P_D(t)}\right)^{-\sigma_C}, \end{aligned}$$

*OCFEB, Erasmus University, P.O. Box 1738, 3000 DR Rotterdam, The Netherlands. Phone: +31-10-408-1808. Fax: +31-10-408-9173. E-mail: `bettendorf@few.eur.nl`.

♯NAKE, Department of Economics, University of Groningen, P.O. Box 800, 9700 AV Groningen, The Netherlands. Phone: +31-50-363-7303, Fax: +31-50-363-7207, E-mail: `b.j.heijdra@eco.rug.nl`.

where σ_A is the Armington elasticity and σ_T is the real exchange rate effect in export demand. The consequences of these modifications are as follows. If $\sigma_A > 1$ (< 1) an increase in the real exchange rate leads to an increase (decrease) in the share of consumption that is spent on domestic goods. Similarly, if $\sigma_T > 1$ (< 1) an increase in the real exchange rate causes an increase in the demand for the domestic goods from abroad. The consequences of these generalisations are found throughout the appendix starting in (A.10).

Third, in the appendix we allow for a downward sloping marginal cost curve at firm level by letting the production function of individual firms be homogeneous of degree $\lambda \geqslant 1$. So instead of (9) in the text we use:

$$Y_i(\tau) + f \equiv G(K_i(\tau), L_i(\tau)) = L_i(\tau)^{\lambda\epsilon} K_i(\tau)^{\lambda(1-\epsilon)},$$

The consequences of this more general case are given in equations (A.22)-(A.24). Since the product subsidy does not affect the equilibrium size of the firm, setting $\lambda = 1$ does not affect anything substantial in the analysis.

2 Optimal choices of the household

The optimisation problem faced by the representative consumer can be solved in two stages. In stage 1 the dynamic problem is solved. This yields a path of full consumption, $X(v,\tau)$. In stage 2 the static allocation problem is solved. First, full consumption is allocated between its components $C(v,\tau)$ and $L(v,\tau)$. Then $C(v,\tau)$ is allocated over $C_D(v,\tau)$ and $C_F(v,\tau)$. Finally, $C_D(v,\tau)$ and $C_F(v,\tau)$ are allocated, respectively, over the different varieties of the differentiated product, $C_{D,i}(v,\tau)$ and $C_{F,j}(v,\tau)$.

Stage 1. Define the ideal cost-of-living index in period τ as $\mathrm{P}_U(\tau)$:

$$P_U(\tau)U(v,\tau) = X(v,\tau), \tag{A.1}$$

where $U(v,\tau) \equiv C(v,\tau)^{\gamma}[1 - L(v,\tau)]^{1-\gamma}$. In the first stage the following optimisation problem is solved.

$$\begin{aligned} & \max_{\{U(v,\tau)\}} \int_t^{\infty} \log\left[U(v,\tau)\right] \exp\left[(\alpha+\beta)(t-v)\right] d\tau \\ \text{s.t. } \frac{dA(v,\tau)}{d\tau} &= \left[r(\tau)+\beta\right] A(v,\tau) + W(\tau) - T(\tau) - P_U(\tau)U(v,\tau), \end{aligned} \tag{A.2}$$

where $A(v,t)$ is taken as given in period t. This leads to the following first-order conditions:

$$\frac{1}{U(v,\tau)} = \lambda(v,\tau) P_U(\tau), \tag{A.3}$$

$$\frac{d\lambda(v,\tau)}{d\tau} = \left[\alpha - r(\tau)\right]\lambda(v,\tau), \tag{A.4}$$

where $\lambda(v,\tau)$ is the co-state variable of the flow budget restriction. The integrated (life-time) budget restriction (with a NPG condition imposed) is:

$$\begin{aligned} A(v,t) + H(t) &= \int_t^{\infty} P_U(\tau)U(v,\tau)\exp\left[-\int_t^{\tau}[r(\mu)+\beta]d\mu\right] d\tau \\ &= \int_t^{\infty} \lambda(v,\tau)^{-1}\exp\left[-\int_t^{\tau} r(\mu)+\beta]d\mu\right] d\tau, \end{aligned} \tag{A.5}$$

where $H(t)$ is human wealth, i.e. the net present value of the household's time endowment:

$$H(t) = \int_t^{\infty} [W(\tau) - T(\tau)] \exp\left[- \int_t^{\tau} [r(\mu) + \beta] d\mu \right] d\tau$$

The path of $\lambda(v, \tau)$ is described by (A.4). Using this in (A.5) yields:

$$(\alpha + \beta) [A(v,t) + H(t)] = \frac{1}{\lambda(v,t)} = X(v,t). \tag{A.6}$$

Full consumption is a constant proportion of total wealth.

Stage 2-a. Full consumption $X(v,t)$ is now allocated over goods, $C(v,t)$, and leisure, $1 - L(v,t)$.

$$\begin{aligned} \max_{\{C(v,t), 1-L(v,t)\}} U(v,t) &= C(v,t)^{\gamma} [1 - L(v,t)]^{1-\gamma} \\ \text{s.t. } X(v,t) &= P_C C(v,t) + W(t) [1 - L(v,t)] . \end{aligned} \tag{A.7}$$

This implies that:

$$W(t) [1 - L(v,t)] = (1-\gamma) X(v,t), \;\; P_C(t) C(v,t) = \gamma X(v,t). \tag{A.8}$$

By substituting (A.8) into the utility function $U(v,t)$ and noting (A.1), we recover the expression for $P_U(t)$:

$$P_U(t) \equiv \left[\gamma^{\gamma} (1-\gamma)^{1-\gamma} \right]^{-1} P_C(t)^{\gamma} W(t)^{1-\gamma}. \tag{A.9}$$

Stage 2-b. Total goods consumption $C(v,t)$ is now allocated over consumption of the composite differentiated goods, $C_D(v,t)$ and $C_F(v,t)$.

$$\begin{aligned} \max_{\{C_D(v,t), C_F(v,t)\}} C(v,t) &= \left[\gamma_D C_D(v,t)^{1-1/\sigma_A} + (1-\gamma_D) C_F(v,t)^{1-1/\sigma_A} \right]^{\sigma_A/(\sigma_A - 1)} \\ \text{s.t. } P_C(t) C(v,t) &= C_D(v,t) + E(t) C_F(v,t). \end{aligned} \tag{A.10}$$

This implies that:

$$\frac{C_D(v,t)}{C(v,t)} = \gamma_D^{\sigma_A} P_C(t)^{\sigma_A}, \;\; \frac{C_F(v,t)}{C(v,t)} = (1-\gamma_D)^{\sigma_A} \left(\frac{P_C(t)}{E(t)} \right)^{\sigma_A}. \tag{A.11}$$

By substituting (A.11) into the subutility function $C(v,t)$, we recover the expression for $P_C(t)$:

$$P_C(t) \equiv \begin{array}{ll} \left[\gamma_D^{\gamma_D} (1-\gamma_D)^{1-\gamma_D} \right]^{-1} E(t)^{1-\gamma_D} & (\text{if } \sigma_A = 1), \\ \left[\gamma_D^{\sigma_A} + (1-\gamma_D)^{\sigma_A} E(t)^{1-\sigma_A} \right]^{1/(1-\sigma_A)} & (\text{if } \sigma_A \neq 1), \end{array} \tag{A.12}$$

In view of (A.11)-(A.12) and (A.8), we can recover the expressions in the text (6) by setting $\sigma_A = 1$.

Stage 2-c. The agent now chooses $C_{D,i}(v,t)$ such that the following static maximisation program is solved.

$$\begin{aligned} \max_{\{C_{D,i}(v,t)\}} C_D(v,t) &\equiv N(t)^{\eta} \left[N(t)^{-1} \sum_{i=1}^{N(t)} C_{D,i}(v,t)^{\frac{\sigma_C - 1}{\sigma_C}} \right]^{\frac{\sigma_C}{\sigma_C - 1}} \\ \text{s.t. } \sum_{i=1}^{N(t)} P_{D,i}(t) C_{D,i}(v,t) &= P_D(t) C_D(v,t). \end{aligned} \tag{A.13}$$

Straightforward manipulation yields the demand functions for the domestically produced varieties of the differentiated commodity by the agent of vintage v:

$$\frac{C_{D,i}(v,t)}{C_D(v,t)} = N(t)^{-(\sigma_C+\eta)+\eta\sigma_C}\left(\frac{P_{D,i}(t)}{P_D(t)}\right)^{-\sigma_C}. \tag{A.14}$$

By substituting (A.14) into the definition for $C_D(v,t)$ the expression for $P_D(t)$ is obtained:

$$P_D(t) \equiv N(t)^{-\eta}\left[N(t)^{-\sigma_C}\sum_{i=1}^{N(t)} P_{D,i}(t)^{1-\sigma_C}\right]^{1/(1-\sigma_C)}. \tag{A.15}$$

The demand for foreign varieties ($C_{F,j}(v,t)$) and the foreign price index ($P_F(t)$) are derived in a similar fashion:

$$\begin{aligned}\frac{C_{F,j}(v,t)}{C_F(v,t)} &= (N^*)^{-(\sigma_C+\eta)+\eta\sigma_C}\left(\frac{P_{F,j}(t)}{P_F(t)}\right)^{-\sigma_C}, \\ P_F(t) &\equiv (N^*)^{-\eta}\left[(N^*)^{-\sigma_C}\sum_{j=1}^{N^*} P_{F,j}(t)^{1-\sigma_C}\right]^{1/(1-\sigma_C)}.\end{aligned} \tag{A.16}$$

In the text we have analyzed the standard Dixit-Stiglitz case for which $\eta = \mu = \sigma_C/(\sigma_C - 1)$.

3 Optimal choices of a representative firm

The representative firm i aims to maximise (10) subject to the demand restriction and the production function. The Lagrangian is defined as follows.

$$\begin{aligned}\pounds(t) &= P_{D,i}(t)\left[1+s_P(t)\right]Y_i(t) - W(t)P_D(t)L_i(t) - R_L(t)P_D(t)K_i(t) \\ &\quad +\lambda_Y(t)\left[F(L_i(t),K_i(t)) - f - Y_i(t)\right],\end{aligned} \tag{A.17}$$

where $Y_i(t) \equiv C_{D,i}(t) + C^*_{F,i}(t)$ is the price-elastic demand facing firm i. The main first-order necessary conditions are:

$$\frac{\partial \pounds(t)}{\partial K_i(t)} = 0 : -R_L(t)P_D(t) + \lambda_Y(t)\left(\frac{\partial Y_i(t)}{\partial K_i(t)}\right) = 0, \tag{A.18}$$

$$\frac{\partial \pounds(t)}{\partial L_i(t)} = 0 : -W(t)P_D(t) + \lambda_Y(t)\left(\frac{\partial Y_i(t)}{\partial L_i(t)}\right) = 0, \tag{A.19}$$

$$\frac{\partial \pounds(t)}{\partial P_{D,i}(t)} = 0 : \left[1+s_P(t)\right]Y_i(t) + \left[P_{D,i}(t)\left[1+s_P(t)\right] - \lambda_Y(t)\right]\left(\frac{\partial Y_i(t)}{\partial P_{D,i}(t)}\right) = 0. \tag{A.20}$$

Equation (A.20) can be used to solve for $\lambda_Y(t)$ in terms of the mark-up ($\mu_i(t)$) and the price chosen by the firm: $\lambda_Y(t) = P_{D,i}(t)[1+s_P(t)]/\mu_i(t)$, where $\lambda_Y(t)$ has the interpretation of marginal cost, $\mu_i(t) \equiv \epsilon_i(t)/[\epsilon_i(t)-1]$ and $\epsilon_i(t)$ is the (absolute value of the) price elasticity of demand:

$$\epsilon_i(t) \equiv -\frac{P_{D,i}(t)}{Y_i(t)}\frac{\partial Y_i(t)}{\partial P_{D,i}(t)} = \sigma_C, \tag{A.21}$$

where the last equality follows from the fact that both $C_{D,i}(t)$ and $C^*_{F,i}(t)$ feature a price elasticity of σ_C. In view of (A.21) the markup is constant and equal across firms: $\mu = \mu_i = \sigma_C/(\sigma_C - 1)$. By substituting the expression for $\lambda_Y(t)$ into (A.18) and (A.19) and noting that $\mu = \eta$ the expressions in (11) in the text are obtained. Under free exit and entry of firms, profits of all active firms go to zero, $\Pi_i(t) = 0$. The gross production function is homogeneous of degree λ:

$$\frac{\partial G(.)}{\partial L_i(t)} L_i(t) + \frac{\partial G(.)}{\partial K_i(t)} K_i(t) = \lambda G(K_i(t), L_i(t)) = \lambda \left[Y_i(t) + f \right]. \tag{A.22}$$

By substituting the marginal productivity conditions into the profit definition and using (A.22), we obtain:

$$\Pi_i(t) = \left(\frac{P_{D,i}(t) \left[1 + s_P(t) \right]}{\mu_i(t)} \right) \left[\mu_i(t) Y_i(t) - \lambda \left[Y_i(t) + f \right] \right]. \tag{A.23}$$

Consequently, the zero profit condition is:

$$\mu_i(t) Y_i(t) = \lambda \left[Y_i(t) + f \right]. \tag{A.24}$$

4 Model Solution

In order to solve the general model, it is useful to first condense the static part of the model (equations (TA.6)-(TA.12) in Table A-1) as much as possible. By using (TA.6)-(TA.10), the change in output ($\tilde{Y}(t)$), employment ($\tilde{L}(t)$), the wage bill ($\tilde{W}(t) + \tilde{L}(t)$), the rental rate on land ($\tilde{R}_L(t)$), and the real exchange rate can be written in terms of the state variable ($\tilde{X}(t)$) and the policy variable ($\tilde{s}_P(t)$):

$$\begin{aligned}
\tilde{Y}(t) &= \eta\epsilon\tilde{L}(t) = (1-\phi)\left[\tilde{X}(t) - \tilde{s}_P(t) \right], & \text{(A.25)}\\
\tilde{R}_L(t) &= \tilde{W}(t) + \tilde{L}(t) = (1-\phi)\tilde{X}(t) + \phi\tilde{s}_P(t), & \text{(A.26)}\\
\tilde{E}(t) &= \frac{-(\phi - 1 + \theta_C)\tilde{X}(t) + (\phi - 1)\tilde{s}_P(t)}{(1-\theta_C)\left[\sigma_T + \theta_C(\sigma_A - 1) \right]} \equiv \Omega_{EX}\tilde{X}(t) + \Omega_{ES}\tilde{s}_P(t), & \text{(A.27)}
\end{aligned}$$

where ϕ is defined as:

$$\phi \equiv \frac{1 + \omega_{LL}}{1 + \omega_{LL}(1 - \eta\epsilon)} \geq 1. \tag{A.28}$$

Equations (A.9) and (A.12) furthermore imply the expression for the true price index, $\tilde{P}_U(t)$:

$$\tilde{P}_U(t) = \gamma(1-\theta_C)\tilde{E}(t) + (1-\gamma)\tilde{W}(t). \tag{A.29}$$

By differentiating (A.27) with respect to time (holding constant the parameters Ω_{EX} and Ω_{ES}), and substituting the result in (TA.3), we obtain the expression for the domestic real rate of interest:

$$r_F\tilde{r}(t) = \Omega_{EX}\dot{\tilde{X}}(t) + \Omega_{ES}\dot{\tilde{s}}_P(t). \tag{A.30}$$

By using (A.26), (A.27) and (A.30) in (TA.1), (TA.2), and (TA.4) the dynamic equations for net foreign assets, the value of land, and total consumption are obtained:

$$\dot{\tilde{F}}(t) = r_F\tilde{F}(t) - r_F\phi\tilde{X}(t) + r_F(\phi - 1)\tilde{s}_P(t), \tag{A.31}$$

$$\dot{\tilde{V}}(t) - \omega_K\Omega_{EX}\dot{\tilde{X}}(t) = r_F\tilde{V}(t) + r_F\omega_K(\phi - 1)\tilde{X}(t) - r_F\omega_K\phi\tilde{s}_P(t) + \omega_K\Omega_{ES}\dot{\tilde{s}}_P(t), \tag{A.32}$$

$$(1-\Omega_{EX})\dot{\tilde{X}}(t) = (r_F-\alpha)\tilde{X}(t) - [(r_F-\alpha)/\omega_K]\left[\tilde{F}(t)+\tilde{V}(t)+\tilde{B}(t)\right] + \Omega_{ES}\dot{\tilde{s}}_P(t). \quad (A.33)$$

By substituting (A.33) into (A.32), the system can be written in a single matrix equation as:

$$\dot{x}(t) = \Delta x(t) + \Gamma(t), \quad (A.34)$$

where $x(t)^T \equiv [\tilde{F}(t), \tilde{V}(t), \tilde{X}(t)]$. The Jacobian matrix of coefficients on the right-hand side, Δ, has typical element δ_{ij} and is defined as:

$$\Delta \equiv \begin{bmatrix} r_F & 0 & -\phi r_F \\ -\frac{(r_F-\alpha)\Omega_{EX}}{1-\Omega_{EX}} & r_F - \frac{(r_F-\alpha)\Omega_{EX}}{1-\Omega_{EX}} & \omega_K\left[r_F(\phi-1)+\frac{(r_F-\alpha)\Omega_{EX}}{1-\Omega_{EX}}\right] \\ -\frac{r_F-\alpha}{\omega_K(1-\Omega_{EX})} & -\frac{r_F-\alpha}{\omega_K(1-\Omega_{EX})} & \frac{r_F-\alpha}{1-\Omega_{EX}} \end{bmatrix}. \quad (A.35)$$

The vectors of forcing terms, $\Gamma(t)$, is given by:

$$\Gamma(t) \equiv \begin{bmatrix} \gamma_F(t) \\ \gamma_V(t) \\ \gamma_X(t) \end{bmatrix} \equiv \begin{bmatrix} r_F(\phi-1)\tilde{s}_P(t) \\ -r_F\omega_K\phi\tilde{s}_P + \left(\frac{\omega_K\Omega_{ES}}{1-\Omega_{EX}}\right)\dot{\tilde{s}}_P(t) - \left(\frac{(r_F-\alpha)\Omega_{EX}}{1-\Omega_{EX}}\right)\tilde{B}(t) \\ \left(\frac{\Omega_{ES}}{1-\Omega_{EX}}\right)\dot{\tilde{s}}_P(t) - \left(\frac{r_F-\alpha}{\omega_K(1-\Omega_{EX})}\right)\tilde{B}(t) \end{bmatrix}. \quad (A.36)$$

The determinant of Δ is:

$$|\Delta| = -\frac{\phi r_F^2(r_F-\alpha)(1-\omega_K)}{\omega_K(1-\Omega_{EX})}. \quad (A.37)$$

This expression can be further simplified for the general Cobb-Douglas case by noting that $\sigma_A = \sigma_T = 1$ implies $1-\Omega_{EX} = \phi/(1-\gamma_D)$. Equation (A.37) shows that stability requires the land share to be between zero and unity ($0 < \omega_K < 1$). Furthermore, for the general model stability requires that $\Omega_{EX} < 1$.

The characteristic equation of Δ, $f(s) \equiv \mid sI - \Delta \mid$, can be written as follows:

$$f(s) = (s-\delta_{11})g(s), \quad (A.38)$$

where $g(s)$ is a quadratic equation:

$$g(s) = s^2 - [\delta_{11} - \delta_{31}\omega_K(1-\Omega_{EX})]\, s + \delta_{11}\delta_{31}\phi(1-\omega_K). \quad (A.39)$$

Equation (A.38) shows that one unstable root equals the interest rate in the rest of the world, i.e. $r_1^* \equiv \delta_{11} = r_F > 0$. The other roots, r_2^* and $-h^*$, are the solutions to $g(s) = 0$ in (A.39). It is straightforward to verify for the general case that $r_2^* > 0$ and $-h^* < 0$ provided $0 < \omega_K < 1$ and $\Omega_{EX} < 1$. For the Cobb-Douglas case in the text equation (A.39) collapses to:

$$g(s) = s^2 - [2\delta_{11} - \alpha]\, s - \delta_{11}(\delta_{11}-\alpha)(1-\gamma_D)(1-\omega_K)/\omega_K, \quad (A.40)$$

from which the following expressions and inequalities for r_2^* and $-h^*$can be derived:

$$r_2^* = \frac{1}{2}(2\delta_{11}-\alpha)\left[1+\sqrt{1+\frac{4\delta_{11}(\delta_{11}-\alpha)(1-\gamma_D)(1-\omega_K)}{\omega_K(2\delta_{11}-\alpha)^2}}\right] > 2\delta_{11}-\alpha, \quad (A.41)$$

$$h^* = \frac{1}{2}(2\delta_{11}-\alpha)\left[\sqrt{1+\frac{4\delta_{11}(\delta_{11}-\alpha)(1-\gamma_D)(1-\omega_K)}{\omega_K(2\delta_{11}-\alpha)^2}}-1\right] > 0. \quad (A.42)$$

Note that (A.41) implies that $\partial r_2^*/\partial\phi = 0$, $\partial r_2^*/\partial\gamma_D < 0$, $\partial r_2^*/\partial\omega_K < 0$, and $\partial r_2^*/\partial s_P < 0$, whereas (A.42) implies that $\partial h^*/\partial\phi = 0$, $\partial h^*/\partial\gamma_D < 0$, $\partial h^*/\partial\omega_K < 0$, and $\partial h^*/\partial s_P < 0$.

4.1 Long-run results

The long-run effects on the state variables of permanent/unanticipated shocks in the product subsidy ($\tilde{s}_P(t) = \tilde{s}_P$, $\dot{\tilde{s}}_P(t) = 0$, $\forall t \geq 0$) or the level of government debt ($\tilde{B}(\infty)$) can be computed from the steady-state version of (A.34). After some manipulation the following expressions is obtained:

$$\begin{bmatrix} \tilde{F}(\infty) \\ \tilde{V}(\infty) \\ \tilde{X}(\infty) \end{bmatrix} \equiv \begin{bmatrix} -\phi\omega_K \\ \omega_K\left[\phi - 1 + \phi(1-\omega_K)\right] \\ \phi(1-\omega_K) - 1 \end{bmatrix} \frac{\tilde{s}_P}{\phi(1-\omega_K)} + \begin{bmatrix} -\phi \\ \omega_K(\phi - 1) \\ -1 \end{bmatrix} \frac{\tilde{B}(\infty)}{\phi(1-\omega_K)}. \tag{A.43}$$

In the absence of bond policy, $\tilde{B}(\infty) = 0$, and equation (A.43) provides the results in section 3 of the paper. The results of section 4.2 are obtained by setting $\tilde{B}(\infty) = -\omega_K \tilde{s}_P$ in (A.43).

4.2 Impact results

The impact results are obtained as follows. By taking the Laplace transform of (A.34) we obtain the following expression:

$$A(s) \begin{bmatrix} \mathcal{L}\{\tilde{F}, s\} \\ \mathcal{L}\{\tilde{V}, s\} \\ \mathcal{L}\{\tilde{X}, s\} \end{bmatrix} = \begin{bmatrix} \mathcal{L}\{\gamma_F, s\} \\ \tilde{V}(0) + \mathcal{L}\{\gamma_V, s\} \\ \tilde{X}(0) + \mathcal{L}\{\gamma_X, s\} \end{bmatrix}, \tag{A.44}$$

where we have used the fact that the stock of net foreign assets is predetermined (i.e. $\tilde{F}(0) = 0$), and where $A(s) \equiv sI - \Delta$, so that $\mid A(s) \mid \equiv (s - r_1^*)(s - r_2^*)(s + h^*)$. By pre-multiplying (A.44) by $\mathrm{adj}(A(r_i^*))$ (for $i = 1, 2$) we obtain the two initial conditions for the jumps in the value of land and total consumption:

$$\mathrm{adj}(A(r_i^*)A(r_i^*) \begin{bmatrix} \mathcal{L}\{\tilde{F}, r_i^*\} \\ \mathcal{L}\{\tilde{V}, r_i^*\} \\ \mathcal{L}\{\tilde{X}, r_i^*\} \end{bmatrix} \equiv \begin{bmatrix} [r_i^* - (\delta_{11} + \delta_{21})](r_i^* - \delta_{33}) - \delta_{23}\delta_{31} & \delta_{13}\delta_{31} & \delta_{13}[r_i^* - (\delta_{11} + \delta_{21})] \\ \delta_{21}(r_i^* - \delta_{33}) + \delta_{23}\delta_{31} & (r_i^* - \delta_{11})(r_i^* - \delta_{33}) - \delta_{13}\delta_{31} & \delta_{23}(r_i^* - \delta_{11}) + \delta_{13}\delta_{21} \\ \delta_{31}(r_i^* - \delta_{11}) & \delta_{31}(r_i^* - \delta_{11}) & (r_i^* - \delta_{11})[r_i^* - (\delta_{11} + \delta_{21})] \end{bmatrix} \times \tag{A.45}$$

$$\begin{bmatrix} \mathcal{L}\{\gamma_F, r_i^*\} \\ \tilde{V}(0) + \mathcal{L}\{\gamma_V, r_i^*\} \\ \tilde{X}(0) + \mathcal{L}\{\gamma_X, r_i^*\} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

for r_i^* $(i = 1, 2)$. Since the characteristic roots of Δ are distinct, $\mathrm{rank}(\mathrm{adj}(A(r_i^*))) = 1$ and there is exactly one independent equation per unstable root. Hence, (A.45) yields two independent equations in the two unknowns, $\tilde{V}(0)$ and $\tilde{X}(0)$. Obviously, as the r_i^*'s are roots of Δ, the system in (A.45) is singular, and some entries of $\mathrm{adj}(A(r_i^*))$ can be simplified substantially for each characteristic root. Indeed, after some manipulation it can be shown that:

$$\mathrm{adj}(A(r_1^*) = \begin{bmatrix} -\delta_{21}(\delta_{11} - \delta_{33}) - \delta_{23}\delta_{31} & \delta_{13}\delta_{31} & -\delta_{13}\delta_{21} \\ \delta_{21}(\delta_{11} - \delta_{33}) + \delta_{23}\delta_{31} & -\delta_{13}\delta_{31} & \delta_{13}\delta_{21} \\ 0 & 0 & 0 \end{bmatrix}, \tag{A.46}$$

$$\text{adj}(A(r_2^*) = \begin{bmatrix} \delta_{13}\delta_{31} & \delta_{13}\delta_{31} & \chi\delta_{13}\delta_{31} \\ \delta_{21}(r_2^* - \delta_{33}) + \delta_{23}\delta_{31} & \delta_{21}(r_2^* - \delta_{33}) + \delta_{23}\delta_{31} & \chi\left[\delta_{21}(r_2^* - \delta_{33}) + \delta_{23}\delta_{31}\right] \\ \delta_{31}(r_2^* - \delta_{11}) & \delta_{31}(r_2^* - \delta_{11}) & \chi\delta_{31}(r_2^* - \delta_{11}) \end{bmatrix}, \tag{A.47}$$

where $\chi \equiv [r_2^* - (\delta_{11} + \delta_{21})]/\delta_{31}$. By using (A.46)-(A.47) in (A.45) we obtain:

$$\begin{bmatrix} -\delta_{13}\delta_{31} & \delta_{13}\delta_{21} \\ \delta_{31} & r_2^* - \delta_{11} - \delta_{21} \end{bmatrix} \begin{bmatrix} \tilde{V}(0) \\ \tilde{X}(0) \end{bmatrix} \tag{A.48}$$

$$= \begin{bmatrix} -\left[\delta_{21}(\delta_{11} - \delta_{33}) + \delta_{23}\delta_{31}\right]\mathcal{L}\{\gamma_F, r_1^*\} + \delta_{13}\delta_{31}\mathcal{L}\{\gamma_V, r_1^*\} - \delta_{13}\delta_{21}\mathcal{L}\{\gamma_X, r_1^*\} \\ -\delta_{31}\mathcal{L}\{\gamma_F, r_2^*\} - \delta_{31}\mathcal{L}\{\gamma_V, r_2^*\} - \left[r_2^* - \delta_{11} - \delta_{21}\right]\mathcal{L}\{\gamma_X, r_2^*\} \end{bmatrix}.$$

All results reported in the paper are based on the assumption that the shocks to the product subsidy and government debt are permanent and unanticipated. This implies that the Laplace transforms for these shocks take the following form:

$$\mathcal{L}\{\tilde{s}_P, s\} = \frac{\tilde{s}_P}{s}, \quad \mathcal{L}\{\tilde{B}, s\} = \frac{\tilde{B}(0)}{s} = \frac{\tilde{B}(\infty)}{s}. \tag{A.49}$$

By using (A.36) and (A.49) in (A.48) the jumps in full consumption and the value of land can be computed:

$$\tilde{X}(0) = -\delta_{31}\left[\frac{\omega_K r_2^*\left[\phi + (\phi - 1)(1 - \Omega_{EX})\right] + \phi\delta_{11}\left[\phi(1 - \omega_K) - 1\right]}{\phi r_2^*(r_2^* - \delta_{11})}\right]\tilde{s}_P - \left(\frac{\delta_{31}}{r_2^*}\right)\tilde{B}(\infty). \tag{A.50}$$

$$\tilde{V}(0) = \left[\frac{\omega_K(2\phi - 1)}{\phi}\right]\tilde{s}_P + \omega_K\Omega_{EX}\left[\tilde{X}(0) - \left(\frac{\phi - 1}{\phi}\right)\tilde{s}_P\right]. \tag{A.51}$$

The expression for $\tilde{X}(0)$ can be written in a very simple form. Starting from equation (A.50) we obtain:

$$\begin{aligned} \tilde{X}(0) \quad = \quad & -\delta_{31}\left[\frac{\omega_K r_2^*\left[\phi + (\phi - 1)(1 - \Omega_{EX})\right] + \phi\delta_{11}\left[\phi(1 - \omega_K) - 1\right] - \phi\omega_K(r_2^* - \delta_{11})}{\phi r_2^*(r_2^* - \delta_{11})}\right]\tilde{s}_P \\ & -\delta_{31}\left[\frac{\phi\omega_K(r_2^* - \delta_{11})}{\phi r_2^*(r_2^* - \delta_{11})}\right]\tilde{s}_P - \left(\frac{\delta_{31}}{r_2^*}\right)\tilde{B}(\infty). \end{aligned}$$

$$= -\delta_{31}\left[\frac{\omega_K r_2^*\left(1 - \Omega_{EX}\right) + \phi\delta_{11}(1 - \omega_K)}{r_2^*(r_2^* - \delta_{11})}\right]\left(\frac{\phi - 1}{\phi}\right)\tilde{s}_P - \delta_{31}\left[\frac{\omega_K\tilde{s}_P + \tilde{B}(\infty)}{r_2^*}\right]$$

$$= \left(\frac{\phi - 1}{\phi}\right)\tilde{s}_P - \delta_{31}\left[\frac{\omega_K\tilde{s}_P + \tilde{B}(\infty)}{r_2^*}\right], \tag{A.52}$$

where in the final step we have made use of the fact that r_2^* is a root of the quadratic function $g(s)$ defined in (A.39):

$$-\delta_{31}\left[\frac{\omega_K r_2^*\left(1 - \Omega_{EX}\right) + \phi\delta_{11}(1 - \omega_K)}{r_2^*(r_2^* - \delta_{11})}\right] = 1 \Leftrightarrow$$

$$r_2^*(r_2^* - \delta_{11}) + \delta_{31}\left[\omega_K r_2^*\left(1 - \Omega_{EX}\right) + \phi\delta_{11}(1 - \omega_K)\right] = 0 \Leftrightarrow$$

$$(r_2^*)^2 - [\delta_{11} - \delta_{31}\omega_K (1 - \Omega_{EX})] r_2^* + \delta_{11}\delta_{31}\phi(1 - \omega_K) = 0 \Leftrightarrow g(r_2^*) = 0.$$

By using (A.52) in (A.51), the following expression for the jump in the value of land can be derived:

$$\tilde{V}(0) = \left(\frac{\omega_K(2\phi - 1)}{\phi}\right) \tilde{s}_P - \omega_K\delta_{31}\Omega_{EX} \left[\frac{\omega_K\tilde{s}_P + \tilde{B}(\infty)}{r_2^*}\right]. \tag{A.53}$$

4.2.1 Without bond policy

In section A.3 of the paper a number of inequalities for $\tilde{X}(0)$ and $\tilde{V}(0)$ are stated. The inequalities for $\tilde{X}(0)$ can be proved by using (A.52) with $\tilde{B}(\infty) = 0$ imposed. The first inequality, $\tilde{X}(0)/\tilde{s}_P > (\phi - 1)/\phi \geq 0$, follows immediately from (A.52) because $\delta_{31} < 0$. The second inequality, $\tilde{X}(0)/\tilde{s}_P < 1$, is proved as follows.

$$\frac{\tilde{X}(0)}{\tilde{s}_P} = \frac{\phi - 1}{\phi} - \frac{\delta_{31}\omega_K}{r_2^*} < 1 \Leftrightarrow r_2^* > -\delta_{31}\phi\omega_K. \tag{A.54}$$

For the Cobb-Douglas case in the paper $r_2^* > -\delta_{31}\phi\omega_K = (1 - \gamma_D)(\delta_{11} - \alpha)$ is guaranteed to hold as $0 \leq \gamma_D \leq 1$ and $r_2^* > 2\delta_{11} - \alpha$ (see Lemma 1). Hence, $\tilde{X}(0)/\tilde{s}_P < 1$.

The inequalities for $\tilde{V}(0)$ are proved as follows. We use (A.52) and (A.53), impose $\tilde{B}(\infty) = 0$, and obtain:

$$\frac{r_2^*\left[\tilde{V}(0)/\omega_K - \tilde{X}(0)\right]}{\tilde{s}_P} = [r_2^* + \delta_{31}\omega_K(1 - \Omega_{EX})] = [r_2^* - (\delta_{11} - \alpha)] > 0, \tag{A.55}$$

which shows that $\tilde{V}(0)/\tilde{s}_P > \omega_K\tilde{X}(0)/\tilde{s}_P > 0.\Box$

The long-run results for full consumption and the value of land in the absence of bond policy can be computed from equation (A.43) by setting $\tilde{B}(\infty) = 0$. The resulting expression for full consumption can be written as follows:

$$\tilde{X}(\infty) = \left(\frac{\phi - 1}{\phi}\right) \tilde{s}_P - \frac{\omega_K\tilde{s}_P}{\phi(1 - \omega_K)} < \left(\frac{\phi - 1}{\phi}\right) \tilde{s}_P, \tag{A.56}$$

which shows that $\tilde{X}(0)/\tilde{s}_P > \tilde{X}(\infty)/\tilde{s}_P$. Similarly, we can use (A.43) and (A.53) to compute the difference between the long-run and impact response in the value of land:

$$\tilde{V}(\infty) - \tilde{V}(0) = \omega_K \left[\frac{\omega_K(\phi - 1)}{\phi(1 - \omega_K)} - \frac{(\delta_{11} - \alpha)\Omega_{EX}}{(1 - \Omega_{EX})r_2^*}\right] \tilde{s}_P. \tag{A.57}$$

In view of the definition of Ω_{EX} in (A.27), $\Omega_{EX} < 0$ for the Cobb-Douglas case and also for the general case unless both σ_A and σ_T are very small. Hence, the expression in square brackets on the right-hand side is certainly positive for the Cobb-Douglas case and $\tilde{V}(\infty)/\tilde{s}_P > \tilde{V}(0)/\tilde{s}_P$. This completes the proofs concerning $\tilde{X}(0)$, $\tilde{X}(\infty)$, $\tilde{V}(0)$, and $\tilde{V}(\infty)$ in the absence of bond policy.$\Box$

4.3 Transition results

Since the shock administered at time $t = 0$ is permanent and unanticipated, the transition path of the state variables has the following form:

$$\begin{bmatrix} \tilde{F}(t) \\ \tilde{V}(t) \\ \tilde{X}(t) \end{bmatrix} = e^{-h^* t} \begin{bmatrix} 0 \\ \tilde{V}(0) \\ \tilde{X}(0) \end{bmatrix} + \left[1 - e^{-h^* t}\right] \begin{bmatrix} \tilde{F}(\infty) \\ \tilde{V}(\infty) \\ \tilde{X}(\infty) \end{bmatrix}, \tag{A.58}$$

where h^* is minus the stable root of Δ which represents the transition speed in the economy. These expressions coincide with the expression in section A.3 of the paper.

5 Welfare Analysis

The welfare implications of the production subsidy can be derived in the manner suggested by Judd (1982). The optimum utility level of the representative agent of vintage v at time t is denoted by $\Lambda(v,t)$:

$$\Lambda(v,t) \equiv \int_t^\infty \log\,[X(v,\tau)/P_U(\tau)]\exp[(\alpha+\beta)(t-\tau)]\,d\tau. \tag{A.59}$$

The Euler equation for the household, $\dot{X}(v,\tau) = [r(\tau)-\alpha]X(v,\tau)$, implies that:

$$X(v,\tau) = X(v,t)\exp\left[\int_t^\tau [r(\mu)-\alpha]d\mu\right],\quad \tau \geq t. \tag{A.60}$$

Substitution of this result in (A.59) yields:

$$\begin{aligned}\Lambda(v,t) &= \int_t^\infty \left[\log\,X(v,t) + \int_t^\tau [r(\mu)-\alpha]\,d\mu - \log\,P_U(\tau)\right]\exp[(\alpha+\beta)(t-\tau)]\,d\tau \\ &= \Lambda_X(v,t) + \Lambda_R(t) - \Lambda_D(t) \end{aligned} \tag{A.61}$$

where:

$$\Lambda_X(v,t) = \frac{\log\,X(v,t)}{\alpha+\beta}, \tag{A.62}$$

$$\Lambda_R(t) = \frac{1}{\alpha+\beta}\int_t^\infty [r(\tau)-\alpha]\exp[(\alpha+\beta)(t-\tau)]\,d\tau, \tag{A.63}$$

$$\Lambda_D(t) = \int_t^\infty \log\,P_U(\tau)\exp[(\alpha+\beta)(t-\tau)]\,d\tau. \tag{A.64}$$

The change in utility is calculated as $d\Lambda(v,t) = d\Lambda_X(v,t) + d\Lambda_R(t) - d\Lambda_D(t)$, with:

$$d\Lambda_X(v,t) = \frac{1}{\alpha+\beta}\frac{dX(v,t)}{X(v,t)} = \frac{\tilde{X}(v,t)}{\alpha+\beta}, \tag{A.65}$$

$$d\Lambda_R(t) = \left(\frac{r_F}{\alpha+\beta}\right)\int_t^\infty \tilde{r}(\tau)\exp[(\alpha+\beta)(t-\tau)]\,d\tau, \tag{A.66}$$

$$d\Lambda_D(t) = \int_t^\infty \tilde{P}_U(\tau)\exp[(\alpha+\beta)(t-\tau)]\,d\tau. \tag{A.67}$$

The Laplace transforms of $d\Lambda_R(t)$ and $d\Lambda_D(t)$ can be written in the following form:

$$\mathcal{L}\{d\Lambda_R,s\} = \left(\frac{r_F}{\alpha+\beta}\right)\frac{\mathcal{L}\{\tilde{r},\alpha+\beta\}-\mathcal{L}\{\tilde{r},s\}}{s-(\alpha+\beta)}, \tag{A.68}$$

$$\mathcal{L}\{d\Lambda_D,s\} = \frac{\mathcal{L}\{\tilde{P}_U,\alpha+\beta\}-\mathcal{L}\{\tilde{P}_U,s\}}{s-(\alpha+\beta)}. \tag{A.69}$$

5.1 Existing generations

Existing generations are born before the policy shock occurs and have a negative generations index, $v \leq 0$. For an individual we have that $X(v,0) = (\alpha+\beta)[A(v,0)+H(0)]$, so that:

$$\tilde{X}(v,0) = [1-\alpha_H(v)]\,\tilde{A}(v,0) + \alpha_H(v)\tilde{H}(0),\quad \alpha_H(v) \equiv \frac{H(0)}{A(v,0)+H(0)}, \tag{A.70}$$

where $\tilde{A}(v,0) \equiv dA(v,0)/A(v,0)$, $\tilde{X}(v,0) \equiv dX(v,0)/X(v,0)$, and $\tilde{H}(0) \equiv dH(0)/H(0)$. Aggregate total consumption satisfies $X(0) = (\alpha+\beta)[A(0)+H(0)]$, so that:

$$\tilde{X}(0) = [1-\omega_H]\,\tilde{A}(0) + \omega_H\tilde{H}(0),\quad \omega_H \equiv \frac{H(0)}{A(0)+H(0)}, \tag{A.71}$$

where $\tilde{A}(0) \equiv dA(0)/A(0)$. In the steady-state we have that $X(v,0) = X(v,v)\exp[-(r_F - \alpha)v]$, implying:

$$\begin{aligned}(\alpha+\beta)\left[A(v,0)+H(0)\right] &= (\alpha+\beta)H(0)\exp\left[-(r_F-\alpha)v\right] \\ \Rightarrow &\quad \alpha_H(v) \equiv \exp\left[(r_F-\alpha)v\right].\end{aligned} \tag{A.72}$$

Furthermore, we know that $\tilde{A}(v,0) = \tilde{A}(0)$ for $v < 0$, i.e. the rate of change in the value of individual assets equals the rate of change in the value of aggregate financial wealth. Combining these equations, (A.70) is written as:

$$\tilde{X}(v,0) = \tilde{A}(0) + \left[\alpha_H(v)/\omega_H\right]\left[\tilde{X}(0) - \tilde{A}(0)\right]. \tag{A.73}$$

Next, it follows from (A.69)-(A.68) that:

$$d\Lambda_R(0) = \lim_{s\to\infty} s\mathcal{L}\{d\Lambda_R, s\} = \left(\frac{r_F}{\alpha+\beta}\right)\mathcal{L}\{\tilde{r}, \alpha+\beta\}, \tag{A.74}$$

$$d\Lambda_D(0) = \lim_{s\to\infty} s\mathcal{L}\{d\Lambda_D, s\} = \mathcal{L}\{\tilde{P}_U, \alpha+\beta\}. \tag{A.75}$$

The effect on welfare for existing generations ($v \leqslant 0$) can thus be written as:

$$\begin{aligned}(\alpha+\beta)d\Lambda(v,0) &= \tilde{A}(0) + \left[\alpha_H(v)/\omega_H\right]\left[\tilde{X}(0)-\tilde{A}(0)\right] r_F\mathcal{L}\{\tilde{r},\alpha+\beta\} \\ &\quad -(\alpha+\beta)\mathcal{L}\{\tilde{P}_U,\alpha+\beta\},\end{aligned} \tag{A.76}$$

where the change in the value of financial assets that occurs at impact can be written as:

$$\tilde{A}(0) \equiv (1/\omega_K)\left[\tilde{V}(0) + \tilde{B}(0)\right]. \tag{A.77}$$

5.2 Future generations

The utility change for future generations is evaluated at birth, i.e. we compute $dU(v,v)$ for $v \geq 0$. First, we know that agents are born without financial wealth, $A(v,v) = 0$, so that:

$$X(v,v) = (\alpha+\beta)H(v) \Rightarrow \tilde{X}(v,v) = \tilde{H}(v). \tag{A.78}$$

From the aggregate counterpart, $X(v) = (\alpha+\beta)[V(v) + E(v)F(v) + B(v) + H(v)]$, an expression for $\tilde{H}(v)$ is obtained:

$$\tilde{H}(v) = (1/\omega_H)\left[\tilde{X}(v) - \left[(1-\omega_H)/\omega_K\right]\left(\tilde{V}(v)+\tilde{F}(v)+\tilde{B}(v)\right)\right]. \tag{A.79}$$

The change in welfare of future generations ($v \geqslant 0$) is rewritten as:

$$\begin{aligned}(\alpha+\beta)d\Lambda(v,v) &= \tilde{X}(v,v) + r_F\mathcal{L}^{-1}\left[\frac{\mathcal{L}\{\tilde{r},\alpha+\beta\} - \mathcal{L}\{\tilde{r},s\}}{s-(\alpha+\beta)}\right] \\ &- (\alpha+\beta)\mathcal{L}^{-1}\left[\frac{\mathcal{L}\{\tilde{P}_U,\alpha+\beta\} - \mathcal{L}\{\tilde{P}_U,s\}}{s-(\alpha+\beta)}\right],\end{aligned} \tag{A.80}$$

The Laplace transforms of $\tilde{r}(t)$ and $\tilde{P}_U(t)$ take the following form (recall that $\tilde{r}(\infty) = 0$):

$$\mathcal{L}\{\tilde{r}, s\} = \frac{\tilde{r}(0)}{s+h^*}, \tag{A.81}$$

$$\mathcal{L}\{\tilde{P}_U, s\} = \frac{\tilde{P}_U(0) - \tilde{P}_U(\infty)}{s+h^*} + \frac{\tilde{P}_U(\infty)}{s}. \tag{A.82}$$

This implies the following results:

$$\frac{\mathcal{L}\{\tilde{r},\alpha+\beta\}-\mathcal{L}\{\tilde{r},s\}}{s-(\alpha+\beta)}=\left(\frac{1}{s}\right)\mathcal{L}\{\tilde{r},\alpha+\beta\}-\left[\frac{1}{s}-\frac{1}{s+h^*}\right]\left[\frac{\tilde{r}(0)}{\alpha+\beta+h^*}\right], \tag{A.83}$$

$$\begin{aligned}\frac{\mathcal{L}\{\tilde{P}_U,\alpha+\beta\}-\mathcal{L}\{\tilde{P}_U,s\}}{s-(\alpha+\beta)} &= \left(\frac{1}{s}\right)\mathcal{L}\{\tilde{P}_U,\alpha+\beta\} \\ &\quad -\left[\frac{1}{s}-\frac{1}{s+h^*}\right]\left[\frac{\tilde{P}_U(0)-\tilde{P}_U(\infty)}{\alpha+\beta+h^*}\right].\end{aligned} \tag{A.84}$$

Hence, the terms in square brackets in (A.80) can be written as weighted averages of the initial and long-run effects of the respective variables. Since the paths of the variables themselves take the same form, utility can also be written in terms of weighted impact and long-run effects. Indeed, by using the solution paths for $\tilde{X}(t)$, $\tilde{V}(t)$, $\tilde{F}(t)$, as well as (A.83)-(A.84) in (A.80), we obtain the following expression for $d\Lambda(t,t)$:

$$d\Lambda(t,t)=e^{-h^*t}d\Lambda(0,0)+\left[1-e^{-h^*t}\right]d\Lambda(\infty,\infty),\quad t\geqslant 0, \tag{A.85}$$

where $d\Lambda(\infty,\infty)$ is given by:

$$\begin{aligned}(\alpha+\beta)d\Lambda(\infty,\infty) &= \frac{\tilde{X}(\infty)}{\omega_H}-\tilde{P}_U(\infty)-\frac{(1-\omega_H)\left[\tilde{V}(\infty)+\tilde{F}(\infty)+\tilde{B}(\infty)\right]}{\omega_K\omega_H} \\ &= \tilde{X}(\infty)-\tilde{P}_U(\infty),\end{aligned} \tag{A.86}$$

and where we have used the steady-state version of (TA.2)-(TA.3) in the final step.

6 Bond policy

6.1 A suitable bond policy removes transitional dynamics

The impact effects in the presence of bond policy are given in (A.52) and (A.53). For convenience the impact results are restated here:

$$\begin{aligned}\tilde{F}(0) &= 0, \\ \tilde{V}(0) &= \left(\frac{\omega_K(2\phi-1)}{\phi}\right)\tilde{s}_P-\omega_K\delta_{31}\Omega_{EX}\left[\frac{\omega_K\tilde{s}_P+\tilde{B}(\infty)}{r_2^*}\right], \\ \tilde{X}(0) &= \left(\frac{\phi-1}{\phi}\right)\tilde{s}_P-\delta_{31}\left[\frac{\omega_K\tilde{s}_P+\tilde{B}(\infty)}{r_2^*}\right].\end{aligned} \tag{A.87}$$

The long-run results with bond policy are given in (A.43) and can be rewritten in the following fashion:

$$\begin{aligned}\tilde{F}(\infty) &= -\left[\frac{\omega_K\tilde{s}_P+\tilde{B}(\infty)}{1-\omega_K}\right], \\ \tilde{V}(\infty) &= \left(\frac{\omega_K(2\phi-1)}{\phi}\right)\tilde{s}_P+\frac{\omega_K(\phi-1)\left[\omega_K\tilde{s}_P+\tilde{B}(\infty)\right]}{\phi(1-\omega_K)}, \\ \tilde{X}(\infty) &= \left(\frac{\phi-1}{\phi}\right)\tilde{s}_P-\left[\frac{\omega_K\tilde{s}_P+\tilde{B}(\infty)}{\phi(1-\omega_K)}\right].\end{aligned} \tag{A.88}$$

A comparison between (A.87) and (A.88) reveals that impact and long-run results coincide for the respective variables if the bond policy takes the form $\tilde{B}(0) = \tilde{B}(\infty) = -\omega_K \tilde{s}_P$, i.e. $\tilde{X}(0) = \tilde{X}(\infty)$, $\tilde{V}(0) = \tilde{V}(\infty)$, and $\tilde{F}(0) = \tilde{F}(\infty) = 0$ in that case. Since all transition in full consumption is eliminated, the same also holds for the other variables. Indeed, by substituting the constant value of $\tilde{X}(t) = [(\phi - 1)/\phi]\tilde{s}_P$ in equations (A.25)-(A.27) and (A.30) we obtain:

$$\begin{aligned} \tilde{Y}(t) &= \eta\epsilon\tilde{L}(t) = (1/\xi)\tilde{E}(t) = [(\phi-1)/\phi]\,\tilde{s}_P > 0, \;\; \tilde{r}(t) = \tilde{C}_F(t) = 0, \\ \tilde{R}_L(t) &= [(2\phi-1)/\phi]\,\tilde{s}_P > 0, \;\; \tilde{W}(t) = \frac{[1+\eta\epsilon\omega_{LL}]\,\tilde{s}_P}{1+\omega_{LL}} > 0, \end{aligned} \tag{A.89}$$

where $\xi \equiv 1/[\sigma_T - \theta_C(\sigma_A - 1)]$. Note that for the Cobb-Douglas case, $\xi = 1$.

6.2 A suitable bond policy removes intergenerational inequities

It is straightforward to show that a bond policy of the form $\tilde{B}(0) = \tilde{B}(\infty) = -\omega_K\tilde{s}_P$ not only removes transitional dynamics in the macroeconomic variables but also eliminates all intergenerational inequities. The welfare effect on existing generations is given in (A.76). Using (A.77) and (A.87) and imposing $\tilde{B}(\infty) = \tilde{B}(0) = -\omega_K\tilde{s}_P$ shows that the generation-specific term in (A.76) vanishes:

$$\begin{aligned} \tilde{X}(0) - \tilde{A}(0) &= \tilde{X}(0) - (1/\omega_K)\left[\tilde{V}(0) + \tilde{B}(0)\right] \\ &= \left(\frac{\phi-1}{\phi}\right)\tilde{s}_P - \left[\left(\frac{\omega_K(2\phi-1}{\phi}\right)\tilde{s}_P - \tilde{s}_P\right] = 0. \end{aligned} \tag{A.90}$$

Hence, all existing generations are affected by the shock in the same manner, i.e. $d\Lambda(v,0) = d\Lambda(0,0)$ for all $v \le 0$.

For future generations the welfare effect is given in (A.85). It is straightforward to show that $d\Lambda(v,v) = d\Lambda(0,0)$ for all $v \ge 0$. In view of (A.85), all that needs to be done is to show that $d\Lambda(0,0) = d\Lambda(\infty,\infty)$:

$$\begin{aligned} (\alpha+\beta)d\Lambda(0,0) &= \tilde{X}(0,0) + r_F\mathcal{L}\{\tilde{r}, \alpha+\beta\} - (\alpha+\beta)\mathcal{L}\{\tilde{P}_U, \alpha+\beta\} \\ &= \tilde{X}(0) - (\alpha+\beta)\mathcal{L}\{\tilde{P}_U, \alpha+\beta\} \\ &= \tilde{X}(\infty) - \tilde{P}_U(\infty) = (\alpha+\beta)d\Lambda(\infty,\infty), \end{aligned} \tag{A.91}$$

where we have used $\tilde{X}(0,0) = \tilde{X}(0)$ in the first step, $\tilde{r}(t) = 0$ in the second step, and $\tilde{P}_U(0) = \tilde{P}_U(\infty)$ as well as equation (A.82) in the third step.

To evaluate the common welfare effect on all generations it suffices to evaluate $d\Lambda(\infty,\infty)$. By using (A.86) and (A.88)-(A.89), and imposing $\tilde{B}(\infty) = \tilde{B}(0) = -\omega_K\tilde{s}_P$, we obtain the required expression for π:

$$\begin{aligned} (\alpha+\beta)\pi &\equiv \tilde{X}(\infty) - \tilde{P}_U(\infty) = \tilde{X}(\infty) - \gamma(1-\theta_C)\tilde{E}(\infty) - (1-\gamma)\tilde{W}(\infty) \\ &= [1-\gamma\xi(1-\theta_C)]\left(\frac{\eta\epsilon\omega_{LL}}{1+\omega_{LL}}\right)\tilde{s}_P - (1-\gamma)\left(\frac{1+\eta\epsilon\omega_{LL}}{1+\omega_{LL}}\right)\tilde{s}_P \\ &= \left(\frac{(1-\gamma)\tilde{s}_P}{\gamma(1+s_P)(1+\omega_{LL})}\right)[\eta\,(1-\gamma\xi(1-\theta_C)) - \gamma(1+s_P) - \eta(1-\gamma)] \\ &= \left(\frac{(1-\gamma)\tilde{s}_P}{(1+s_P)(1+\omega_{LL})}\right)[\eta\,(1-\xi(1-\theta_C)) - 1 - s_P]\,, \end{aligned} \tag{A.92}$$

where we have used $\omega_{LL} = (1-\gamma)/[\gamma\epsilon(1+s_P)]$ to simplify the expression and where we have used $\xi \equiv 1/[\sigma_T - \theta_C(\sigma_A - 1)]$ (see below (A.89)). Equation (29) in the text is obtained from (A.92) by setting $\sigma_T = \sigma_A = 1$, so that $\xi = 1$ and $\theta_C = \gamma_D$. Equation (30) in the text is obtained by setting $\pi = 0$ in (A.92) and solving for s_P.

7 Intergenerational welfare effects

In order to compute the intergenerational welfare effects in the most general case, it is useful to write the variables in deviation from their respective egalitarian-optimum levels, as was done for the state variables in equations (A.87)-(A.88) above. After some manipulation, it is possible to obtain the following expression for the impact effect on the domestic interest rate:

$$r_F \tilde{r}(0) = \frac{\delta_{21}\left[r_2^* - \delta_{21} + \delta_{31}\omega_K\right]\left[\omega_K \tilde{s}_P + \tilde{B}(\infty)\right]}{\omega_K r_2^*}, \tag{A.93}$$

where $r_2^* - \delta_{21} - \delta_{31}\omega_K = r_2^* - (\delta_{11} - \alpha) > 0$. In a similar fashion, the impact and long-run effects on the real exchange rate can be written as follows:

$$\tilde{E}(0) = \xi\left(\frac{\phi-1}{\phi}\right)\tilde{s}_P - \delta_{21}\left[\frac{\omega_K \tilde{s}_P + \tilde{B}(\infty)}{\omega_K r_2^*}\right], \tag{A.94}$$

$$\tilde{E}(\infty) = \xi\left(\frac{\phi-1}{\phi}\right)\tilde{s}_P - \Omega_{EX}\left[\frac{\omega_K \tilde{s}_P + \tilde{B}(\infty)}{\phi(1-\omega_K)}\right]. \tag{A.95}$$

A comparison of (A.94) and (A.95) reveals that $\mathrm{sgn}[\tilde{E}(\infty) - \tilde{E}(0)] = \mathrm{sgn}(-\Omega_{EX})$ in the absence of bond policy. In the normal case, with $\xi > 0$, $\Omega_{EX} < 0$ and the long-run effect on the real exchange rate exceeds the short-run effect (In the Cobb-Douglas case $\xi = 1$). The impact and long-run effects on the wage rate can be written as follows:

$$\tilde{W}(0) = \left[1 + \left(\frac{\eta\epsilon - 1}{\eta\epsilon}\right)\left(\frac{\phi-1}{\phi}\right)\right]\tilde{s}_P - \frac{\delta_{31}(\eta\epsilon-1)(1-\phi)}{\eta\epsilon}\left[\frac{\omega_K \tilde{s}_P + \tilde{B}(\infty)}{r_2^*}\right], \tag{A.96}$$

$$\tilde{W}(\infty) = \left[1 + \left(\frac{\eta\epsilon - 1}{\eta\epsilon}\right)\left(\frac{\phi-1}{\phi}\right)\right]\tilde{s}_P - \frac{(\eta\epsilon-1)(1-\phi)}{\eta\epsilon}\left[\frac{\omega_K \tilde{s}_P + \tilde{B}(\infty)}{\phi(1-\omega_K)}\right]. \tag{A.97}$$

A comparison of (A.96) and (A.97) reveals that $\mathrm{sgn}[\tilde{W}(\infty) - \tilde{W}(0)] = \mathrm{sgn}(\eta\epsilon - 1)$ in the absence of bond policy. Hence, with a weak diversity effect ($\eta\epsilon < 1$) or under perfect competition, the wage falls during transition, whereas the opposite holds with a strong diversity effect ($\eta\epsilon > 1$).

By using (A.94) and (A.96) in (A.29), the following expression for the impact effect on the cost-of-living index is obtained:

$$\begin{aligned}\tilde{P}_U(0) = {} & \left[\gamma\xi(1-\theta_C)\left(\frac{\phi-1}{\phi}\right) + (1-\gamma)\left[1 + \left(\frac{\eta\epsilon-1}{\eta\epsilon}\right)\left(\frac{\phi-1}{\phi}\right)\right]\right]\tilde{s}_P \\ & - \left[\gamma(1-\theta_C)\delta_{21} + (1-\gamma)\omega_K\frac{\delta_{31}(\eta\epsilon-1)(1-\phi)}{\eta\epsilon}\right]\left[\frac{\omega_K \tilde{s}_P + \tilde{B}(\infty)}{\omega_K r_2^*}\right].\end{aligned} \tag{A.98}$$

Not surprisingly, since both the impact results on the real exchange rate and the wage are ambiguous, it is not possible to unambiguously sign the impact effect on the cost-of-living index. The long-run effect on the cost-of-living index can be obtained by using (A.95) and (A.97) in

(A.29):

$$\begin{aligned}\tilde{P}_U(\infty) &= \left[\gamma\xi(1-\theta_C)\left(\frac{\phi-1}{\phi}\right)+(1-\gamma)\left[1+\left(\frac{\eta\epsilon-1}{\eta\epsilon}\right)\left(\frac{\phi-1}{\phi}\right)\right]\right]\tilde{s}_P \\ &\quad -\left[\gamma(1-\theta_C)\Omega_{EX}+(1-\gamma)\frac{(\eta\epsilon-1)(1-\phi)}{\eta\epsilon}\right]\left[\frac{\omega_K\tilde{s}_P+\tilde{B}(\infty)}{\phi(1-\omega_K)}\right].\end{aligned} \tag{A.99}$$

By deducting (A.98) from (A.99) and rewriting, we obtain the following expression:

$$\begin{aligned}&\tilde{P}_U(\infty)-\tilde{P}_U(0)=\left[\frac{1}{\phi(1-\omega_K)}-\frac{\delta_{31}}{r_2^*}\right]\times \\ &\left(-\gamma(1-\theta_C)\Omega_{EX}+(1-\gamma)\frac{(\eta\epsilon-1)(\phi-1)}{\eta\epsilon}\right)\left[\omega_K\tilde{s}_P+\tilde{B}(\infty)\right].\end{aligned} \tag{A.100}$$

The first term in square brackets on the right-hand side is positive but the sign of the term in round brackets is ambiguous. If $\Omega_{EX}<0$ and there are either strong diversity effects ($\eta\epsilon>1$), or labour supply is exogenous ($\phi=1$), the term in round brackets is positive so that the cost-of-living index increases over time, i.e. $\tilde{P}_U(\infty)>\tilde{P}_U(0)$ in those cases. In general, however, no unambiguous sign can be determined even for the Cobb-Douglas case.

By using (A.76)-(A.77) and (A.81)-(A.82), the welfare effect on extremely old existing generations can be written as follows:

$$\begin{aligned}(\alpha+\beta)d\Lambda(-\infty,0) &= (1/\omega_K)\left[\tilde{V}(0)+\tilde{B}(0)\right]+\frac{r_F\tilde{r}(0)}{\alpha+\beta+h^*}-\tilde{P}_U(\infty) \\ &\quad+\left(\frac{\alpha+\beta}{\alpha+\beta+h^*}\right)\left[\tilde{P}_U(\infty)-\tilde{P}_U(0)\right].\end{aligned} \tag{A.101}$$

By substituting $\tilde{V}(0)$ from (A.87), (A.93), and (A.99)-(A.100) into (A.101) and noting that $\tilde{B}(0)=\tilde{B}(\infty)$, the following expression for $d\Lambda(-\infty,0)$ is obtained:

$$(\alpha+\beta)d\Lambda(-\infty,0)=(\alpha+\beta)\pi+\Gamma_1\left[\frac{\omega_K\tilde{s}_P+\tilde{B}(\infty)}{r_2^*}\right], \tag{A.102}$$

where π is the common welfare effect on all generations in the presence of suitable bond policy (see (A.92)) and where Γ_1 is defined as:

$$\begin{aligned}\Gamma_1 &= \frac{r_2^*-\delta_{21}}{\omega_K}+\frac{\delta_{21}(r_2^*-\delta_{21}+\delta_{31}\omega_K)}{(\alpha+\beta+h^*)\omega_K}+\left[\gamma\xi(\phi-1+\theta_C)+(1-\gamma)\frac{(\eta\epsilon-1)(\phi-1)}{\eta\epsilon}\right] \\ &\quad\times\left[-r_2^*+\left(\frac{\alpha+\beta}{\alpha+\beta+h^*}\right)(r_2^*-\delta_{31}\phi(1-\omega_K))\right]\frac{1}{\phi(1-\omega_K)}.\end{aligned} \tag{A.103}$$

We demonstrate below that $\Gamma_1>0$ for the Cobb-Douglas case of the paper.

The welfare effect on newborns at the time of the shock can be written as follows:

$$(\alpha+\beta)d\Lambda(0,0)=(\alpha+\beta)d\Lambda(-\infty,0)+(1/\omega_H)\left[\tilde{X}(0)-(1/\omega_K)\left[\tilde{V}(0)+\tilde{B}(0)\right]\right]. \tag{A.104}$$

By substituting $\tilde{V}(0)$ and $\tilde{X}(0)$ from (A.87) and noting that $\tilde{B}(0)=\tilde{B}(\infty)$, the terms in square brackets on the right-hand side can be written as:

$$\left[\tilde{X}(0)-(1/\omega_K)\left[\tilde{V}(0)+\tilde{B}(0)\right]\right]=-\left(\frac{r_2^*+\delta_{31}\omega_K(1-\Omega_{EX})}{\omega_K}\right)\left[\frac{\omega_K\tilde{s}_P+\tilde{B}(\infty)}{r_2^*}\right]. \tag{A.105}$$

By substituting (A.105) in (A.104) and noting that $\delta_{31}\omega_K(1-\Omega_{EX}) = -(\delta_{11}-\alpha)$, we obtain the following expression for $d\Lambda(0,0)$:

$$(\alpha+\beta)d\Lambda(0,0) = (\alpha+\beta)d\Lambda(-\infty,0) - \left(\frac{r_2^* - (\delta_{11}-\alpha)}{\omega_H}\right)\left[\frac{\omega_K\tilde{s}_P + \tilde{B}(\infty)}{\omega_K r_2^*}\right], \tag{A.106}$$

which confirms that $d\Lambda(0,0) < d\Lambda(-\infty,0)$ in the absence of bond policy (since the term in round brackets is positive).

Equation (A.86) shows that the welfare effect on generations born in the new steady state can be written as follows:

$$(\alpha+\beta)d\Lambda(\infty,\infty) = \tilde{X}(\infty) - \tilde{P}_U(\infty). \tag{A.107}$$

By substituting $\tilde{X}(\infty)$ from (A.88) and using (A.99), $d\Lambda(\infty,\infty)$ can be written as follows:

$$(\alpha+\beta)d\Lambda(\infty,\infty) = (\alpha+\beta)\pi - \Gamma_2\left[\frac{\omega_K\tilde{s}_P + \tilde{B}(\infty)}{\phi(1-\omega_K)}\right], \tag{A.108}$$

where Γ_2 is defined as follows:

$$\Gamma_2 \equiv 1 + \gamma\xi(\phi - 1 + \theta_C) + (1-\gamma)\phi\left(\frac{\eta\epsilon - 1}{\eta\epsilon}\right)\left(\frac{\phi-1}{\phi}\right). \tag{A.109}$$

We now prove that Γ_2 is positive for the Cobb-Douglas case discussed in the paper. With Cobb-Douglas preferences, $\xi = 1$ and $\theta_C = \gamma_D$ so that Γ_2 reduces to:

$$\Gamma_2 \equiv 1 + \gamma(\phi - 1 + \gamma_D) + (1-\gamma)\phi\left(\frac{\eta\epsilon - 1}{\eta\epsilon}\right)\left(\frac{\phi-1}{\phi}\right). \tag{A.110}$$

Since $\phi \geq 1$, $\Gamma_2 > 0$ is implied if $\eta\epsilon > 1$, i.e. if there are strong diversity effects. Hence, to complete the proof we only need to consider the case with $\eta\epsilon < 1$. In that case ϕ satisfies the following inequalities:

$$1 \leq \phi \leq \frac{1}{1-\eta\epsilon}, \quad 0 \leq \frac{\phi-1}{\phi} \leq \eta\epsilon. \tag{A.111}$$

Using these properties allows us to deduce that $\Gamma_2 > 0$:

$$\begin{aligned} \Gamma_2 &\equiv 1 + \gamma(\phi - 1 + \gamma_D) - (1-\gamma)\phi\left(\frac{1-\eta\epsilon}{\eta\epsilon}\right)\left(\frac{\phi-1}{\phi}\right) \\ &> 1 + \gamma(\phi - 1 + \gamma_D) - (1-\gamma)\phi(1-\eta\epsilon) \\ &> 1 + \gamma(\phi - 1 + \gamma_D) - (1-\gamma) = \gamma(\phi + \gamma_D) > 0. \end{aligned} \tag{A.112}$$

Hence, $\Gamma_2 > 0$ for the Cobb-Douglas case.□

The proof of $\Gamma_1 > 0$ for the Cobb-Douglas case proceeds as follows. First, note that Γ_1 in (A.103) can be re-written by using the definition for Γ_2 in (A.109):

$$\begin{aligned} \Gamma_1 &= \frac{(\alpha+\beta+h^*)(r_2^* - \delta_{21}) + \delta_{21}\left[r_2^* + \delta_{31}\omega_K(1-\Omega_{EX})\right]}{(\alpha+\beta+h^*)\omega_K} \\ &+ \frac{\omega_K(1-\Gamma_2)\left[\left(\frac{r_2^* h^*}{\phi(1-\omega_K)}\right) + (\alpha+\beta)\delta_{31}\right]}{(\alpha+\beta+h^*)\omega_K} \\ &\equiv \frac{\Gamma_4}{(\alpha+\beta+h^*)\omega_K}. \end{aligned} \tag{A.113}$$

Equation (A.39) implies the following useful results for r_2^* and h^*:

$$r_2^* h^* = -\delta_{11}\delta_{31}\phi(1-\omega_K), \quad r_2^* - h^* = 2\delta_{11} - \alpha. \tag{A.114}$$

By using these results in (A.113), and noting that $r_2^* + \delta_{31}\omega_K(1-\Omega_{EX}) = r_2^* - \delta_{11} + \alpha$, Γ_4 can be written as:

$$\begin{aligned}
\Gamma_4 &= (\alpha+\beta+h^*)(r_2^* - \delta_{21}) + \delta_{21}(r_2^* - \delta_{11} + \alpha) + \delta_{31}\omega_K(\alpha+\beta-\delta_{11})(1-\Gamma_2) \\
&= (r_2^*)^2 + [\beta + 2(\alpha - \delta_{11})]\, r_2^* - \delta_{21}(\alpha+\beta-\delta_{11}) + \delta_{31}\omega_K(\alpha+\beta-\delta_{11})(1-\Gamma_2) \\
&= (r_2^*)^2 + [\beta + 2(\alpha - \delta_{11})]\, r_2^* + (\alpha-\delta_{11})(\alpha+\beta-\delta_{11}) - \delta_{31}\omega_K(\alpha+\beta-\delta_{11})\Gamma_2 \\
&= [r_2^* - (\delta_{11}-\alpha)]\,[r_2^* - (\delta_{11}-\alpha-\beta)] - \delta_{31}\omega_K(\alpha+\beta-\delta_{11})\Gamma_2 > 0,
\end{aligned} \tag{A.115}$$

where the sign follows from the fact that $\Gamma_2 > 0$, $\delta_{31} < 0$, $\delta_{11} < \alpha + \beta$ and $r_2^* > \delta_{11} - \alpha$ (see Lemma 1). Hence $\Gamma_1 > 0$ for the Cobb-Douglas case.□

8 The Pareto optimal product subsidy

The Pareto optimal product subsidy, $\hat{s}_P$, is obtained by setting $d\Lambda(\infty,\infty) = 0$:

$$\left(\frac{(\alpha+\beta)(1-\omega_K)}{\gamma\tilde{s}_P}\right) d\Lambda(\infty,\infty) = \frac{\epsilon(1-\gamma)[\eta\gamma_D - 1 - s_P]}{1-\gamma+\gamma\epsilon(1+s_P)} - (1+\gamma_D)\omega_K. \tag{A.116}$$

Solving the quadratic function for $1 + \hat{s}_P$:

$$1 + \hat{s}_P = \left(\frac{(1-\gamma)\,[1+\gamma_D(1-\epsilon)]}{2\gamma\epsilon(1+\gamma_D)(1-\epsilon)}\right)\left[\sqrt{1+\Omega_3} - 1\right] > 0, \tag{A.117}$$

where Ω_3 is a positive constant:

$$\Omega_3 \equiv \frac{4\eta\gamma\gamma_D(1+\gamma_D)\epsilon^2(1-\epsilon)}{(1-\gamma)\,[1+\gamma_D(1-\epsilon)]^2} > 0. \tag{A.118}$$

Recovering comparative static effects from this expression is extremely tedious and unnecessary. Instead, we use the information that (A.116) with $d\Lambda(\infty,\infty) = 0$ imposed is in fact a maximum value function. After some manipulations we can write the function determining the Pareto optimal product subsidy as:

$$\frac{d\Lambda(\infty,\infty)}{ds_P} \equiv f(\hat{s}_P, \eta, \gamma_D) = 0, \tag{A.119}$$

where $f(.)$ is a function determining $\hat{s}_P$:

$$\begin{aligned}
f(\hat{s}_P, \eta, \gamma_D) \equiv\ & \Omega_4[\epsilon(1-\gamma)\,[\eta\gamma_D - 1 - \hat{s}_P] \\
& -(1+\gamma_D)(1-\epsilon)(1+\hat{s}_P)\,[1-\gamma+\gamma\epsilon(1+\hat{s}_P)]],
\end{aligned} \tag{A.120}$$

where Ω_4 is a positive constant. The second-order condition for the maximization problem, $d\Lambda(\infty,\infty)/ds_P = 0$, is simply that $\partial f/\partial s_P < 0$ for $s_P = \hat{s}_P$. The comparative static effects on $\hat{s}_P$ of η and γ_D are now a simple application of the envelope theorem:

$$\frac{\partial \hat{s}_P}{\partial \eta} = \frac{\partial f/\partial \eta}{-[\partial f/\partial s_P]_{s_P=\hat{s}_P}} = \frac{\epsilon\gamma_D(1-\gamma)\Omega_4}{-[\partial f/\partial s_P]_{s_P=\hat{s}_P}} > 0, \tag{A.121}$$

$$\frac{\partial \hat{s}_P}{\partial \gamma_D} = \frac{\partial f/\partial \gamma_D}{-[\partial f/\partial s_P]_{s_P=\hat{s}_P}} = \frac{[\epsilon(1-\gamma)\eta - \omega_K\,[1-\gamma+\gamma\epsilon(1+\hat{s}_P)]]\,\Omega_4}{-[\partial f/\partial s_P]_{s_P=\hat{s}_P}} > 0, \tag{A.122}$$

where we prove that the numerator of (A.122) is positive by using the result $f(\hat{s}_P, \eta, \gamma_D) = 0$:

$$f(\hat{s}_P, \eta, \gamma_D) = 0 \quad \Leftrightarrow \quad \epsilon\eta(1-\gamma) - \omega_K \left[1 - \gamma + \gamma\epsilon(1+\hat{s}_P)\right] \tag{A.123}$$
$$= \epsilon(1-\gamma)\left[\eta(1-\gamma_D) + 1 + \hat{s}_P\right] + \gamma_D\omega_K\left[1-\gamma+\gamma\epsilon(1+\hat{s}_P)\right] > 0.$$

Table A-1: Log-Linearized Version of the General Model

$$\dot{\tilde{V}}(t) = r_F\tilde{V}(t) + r_F\omega_K\left[\tilde{r}(t) - \tilde{R}_L(t)\right] \tag{TA.1}$$

$$\dot{\tilde{X}}(t) = [r_F - \alpha]\,\tilde{X}(t) + r_F\tilde{r}(t) - [(r_F-\alpha)/\omega_K]\left[\tilde{V}(t) + \tilde{F}(t) + \tilde{B}(t)\right] \tag{TA.2}$$

$$\dot{\tilde{E}}(t) = r_F\tilde{r}(t) \tag{TA.3}$$

$$r_F^{-1}\dot{\tilde{F}}(t) = \tilde{F}(t) + (1-\theta_C)\left[(\sigma_T - 1)\tilde{E}(t) - \tilde{C}_F(t)\right] \tag{TA.4}$$

$$r_F^{-1}\tilde{B}(0) = \mathcal{L}\{\tilde{T}, r_F\} - (1+s_P)\left[\mathcal{L}\{\tilde{s}_P, r_F\} + \left(\frac{s_P}{1+s_P}\right)\mathcal{L}\{\tilde{Y}, r_F\}\right] \tag{TA.5}$$

$$\tilde{L}(t) = \tilde{Y}(t) + \tilde{s}_P(t) - \tilde{W}(t) \tag{TA.6}$$

$$\tilde{R}_L(t) = \tilde{Y}(t) + \tilde{s}_P(t) \tag{TA.7}$$

$$\tilde{Y}(t) = \theta_C\tilde{C}_D(t) + \sigma_T(1-\theta_C)\tilde{E}(t) \tag{TA.8}$$

$$\tilde{C}_D(t) = (1-\theta_C)(\sigma_A - 1)\tilde{E}(t) + \tilde{X}(t) \tag{TA.9}$$

$$\tilde{C}_F(t) = \tilde{C}_D(t) - \sigma_A\tilde{E}(t) \tag{TA.10}$$

$$\tilde{L}(t) = \omega_{LL}\left[\tilde{W}(t) - \tilde{X}(t)\right] \tag{TA.11}$$

$$\tilde{Y}(t) = \eta\epsilon\tilde{L}(t) \tag{TA.12}$$

Shares:
$\omega_{LL} \equiv (1-L)/L$, leisure/work ratio.
$\omega_K \equiv R_L/Y$, share of rental income in national income.
$\theta_C \equiv C_D/Y$, national income share of domestic consumption.
Notes:
(a) We have used the normalization $E = 1$ and $B = F = 0$ initially. The total (constant) stock of land equals $K = 1$.
(b) Relationship between shares: $\omega_K = (1-\epsilon)(1+s_P)$.