Using the Laplace transform for dynamic policy analysis

Ben J. Heijdra*

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Abstract

The purpose of this note is to demonstrate the usefulness of the Laplace transform methods used by engineers. Among other things we show how to perform comparative dynamics in simple low-dimensional saddle-point stable models, how to evaluate welfare effects, and how to deal with zero-roots and the resulting hysteresis.

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1 Introduction

The main purpose of this note is to demonstrate how useful Laplace transform techniques can be to (macro) economists. Whilst the technique is not much more difficult than the method of comparative statics–that most students are familiar with–it enables one to thoroughly study (the properties of) low-dimensional[1] dynamic models in an analytical fashion. In this note we will focus on macroeconomic applications but the usefulness of the Laplace transform methods extends to other fields.

The outline of this note is as follows. In section 2 we introduce the Laplace transform. We give the definitions, study the main properties, show some often-used transforms, and give some simple examples. In section 3 we provide an extended example of tax policy analysis in an overlapping-generations model. In section 4 we show how the Laplace transform methods can be used to evaluate the welfare effects of policy measures. The method allows the researcher to explicitly take into account the transitional dynamics that results from policy shocks. In section 5 we show how the Laplace transform method is a natural tool

*Department of Economics, University of Groningen, P.O. Box 800, 9700 AV Groningen, The Netherlands. Phone: +31-50-363-7303, Fax: +31-50-363-7207, E-mail: `b.j.heijdra@eco.rug.nl`. Useful comments were received from Leon Bettendorf.

[1] With "low dimensional" we mean that the characteristic polynomial of the Jacobian matrix of the system must be of order four or less. For such polynomials closed-form solutions for the roots are available. For higher-order polynomials Abel's Theorem proves that finite algebraic formulae do not exist for the roots. See the amusing historical overview of this issue in Turnbull (1988, pp. 114-5).

with which to study hysteretic models, i.e. models containing a characteristic roots equal to zero. In section 6 we show briefly how the so-called Z-transform, which is closely related to the Laplace transform, can be used to study discrete-time models. Finally, in section 7 we conclude.

2 The Laplace transform

The Laplace transform is a tool used extensively in engineering contexts and the best available source (that I know) is an engineering mathematics textbook by Kreyszig (1988). The Laplace transform is extremely useful for solving (systems of) differential equations. Intuitively, the method works in three steps: (i) the difficult problem is transformed into a simple problem, (ii) we use (matrix) algebra to solve the simple problem, and (iii) we transform back the solution obtained in step (ii) to obtain the ultimate solution of our hard problem. Instead of having to work with difficult operations in calculus (in step (i)) we work with algebraic operations on transforms. This is why the Laplace transform technique is called *operational calculus*.

The major advantage of the Laplace transform technique lies in the ease with which time-varying shocks can be studied. In economic terms this makes it very easy to identify the propagation mechanism that is contained in the economic model. As we shall demonstrate below this is important, for example, in models in the real business cycle (RBC) tradition.

Suppose that $f(t)$ is a function defined for $t \geq 0$. Then we can define the Laplace transform of that function as follows:[2]

$$\mathcal{L}\{f,s\} \equiv \int_0^\infty e^{-st} f(t)dt. \tag{1}$$

In economic terms $\mathcal{L}\{f,s\}$ is the discounted present value of the function $f(t)$, from present to the indefinite future, using s as the discount rate. Clearly, provided the integral on the right-hand side of (1) exists, $\mathcal{L}\{f,s\}$ is well-defined and can be seen as a function of s.

Example 1 *Suppose that $f(t) = 1$ for $t \geq 0$. What is $\mathcal{L}\{f,s\}$? We use the definition in (1) to get:*

$$\mathcal{L}\{f,s\} = \mathcal{L}\{1,s\} = \int_0^\infty 1 \times e^{-st} dt = -\frac{1}{s} e^{-st}\Big|_0^\infty = \frac{1}{s},$$

for $s > 0$. We have found our first Laplace transform, i.e. $\mathcal{L}\{1,s\} = 1/s$.

Despite the ease with which it was derived, the transform of unity, $\mathcal{L}\{1,s\}$, is an extremely useful one to remember. Let us now try to find a more challenging one.

[2] Some authors prefer to use the notation $F(s)$ for the Laplace transform of $f(t)$. Yet others use notation similar to ours but suppress the s argument and write $\mathcal{L}\{f\}$ for the Laplace transform of $f(t)$. We adopt our elaborate notation since we shall need to evaluate the transforms for particular values of s below.

$f(t)$	$\mathcal{L}\{f,s\}$	*valid for:*
1	$\frac{1}{s}$	$s > 0$
t	$\frac{1}{s^2}$	$s > 0$
$\frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n}$,	$n = 1, 2, ...;\ s > 0$
e^{at}	$\frac{1}{s-a}$,	$s > a$
te^{at}	$\frac{1}{(s-a)^2}$,	$s > a$
$\frac{t^{n-1}e^{at}}{(n-1)!}$	$\frac{1}{(s-a)^n}$,	$n = 1, 2, ...;\ s > a$
$\frac{e^{at}-e^{bt}}{a-b}$	$\frac{1}{(s-a)(s-b)}$	$s > a,\ s > b,\ a \neq b$
$\frac{ae^{at}-be^{bt}}{a-b}$	$\frac{1}{(s-a)(s-b)}$	$s > a,\ s > b,\ a \neq b$
$\mathcal{U}(t-a) \equiv \begin{cases} 0 & \text{for } 0 \leq t < a \\ 1 & \text{for } t > a \end{cases}$	$\frac{e^{-as}}{s}$	

Table 1: Commonly used Laplace transforms

Example 2 *Suppose that $f(t) = e^{at}$ for $t \geq 0$. What is $\mathcal{L}\{f,s\}$? We once again use the definition in (1) and get:*

$$\mathcal{L}\{f,s\} = \mathcal{L}\{e^{at},s\} = \int_0^\infty e^{at}e^{-st}dt = \int_0^\infty e^{-(s-a)t}dt = -\frac{1}{s-a}e^{-(s-a)t}\bigg|_0^\infty = \frac{1}{s-a},$$

provided $s > a$ (otherwise the integral does not exist and the Laplace transform is not defined).

So now we have found our second Laplace transform and in fact we already possess the two transforms used most often in economic contexts. Of course there are very many functions for which the silly work has been done already by others and the Laplace transforms are known. In Table 1 we show a list of commonly used transforms. Such a table is certainly quite valuable but even more useful are the *general properties* of the Laplace transform which allow us to work with these transforms in an algebraic fashion. Let us look at some of the main properties.

Property 1 *Linearity. The Laplace transform is a linear operator. Hence, if the Laplace transforms of $f(t)$ and $g(t)$ both exist then we have for any constants a and b that:*

$$\mathcal{L}\{af + bg, s\} = a\mathcal{L}\{f,s\} + b\mathcal{L}\{g,s\}. \tag{P1}$$

The proof is too obvious to worry about.

The usefulness of (P1) is easily demonstrated: it allows us to deduce more complex transforms from simple transforms. Suppose that we are given a Laplace transform and want to figure out the function in the time domain which is associated with it.

Example 3 *Suppose that $\mathcal{L}\{f,s\} = \frac{1}{(s-a)(s-b)}$, $a \neq b$. What is $f(t)$? We use the method of partial fractions to split up the Laplace transform:*

$$\frac{1}{(s-a)(s-b)} = \frac{1}{a-b}\left[\frac{1}{s-a} - \frac{1}{s-b}\right]. \tag{a}$$

Now we apply (P1) to equation (a)–which is in a format we know–and derive:

$$\mathcal{L}\{f,s\} = \frac{1}{a-b}\left[\frac{1}{s-a} - \frac{1}{s-b}\right] = \frac{1}{a-b}\left[\mathcal{L}\{e^{at},s\} - \mathcal{L}\{e^{bt},s\}\right], \tag{b}$$

where we have used Table 1 to get to the final expression. But (b) can now be inverted to get our answer:

$$f(t) = \frac{e^{at} - e^{bt}}{a-b}. \tag{c}$$

This entry is also found in Table 1.

But we have now performed an operation (inverting a Laplace transform) for which we have not yet established the formal validity. Clearly, going from (c) to (b) is valid but is it also allowed to go from (b) to (c), i.e. is the Laplace transform unique? The answer is "no" in general but "yes" for all cases of interest. Kreyszig (1988, p. 247) states the following sufficient condition for existence.

Property 2 *Existence. Let $f(t)$ be a function that is piecewise continuous on every finite interval in the range $t \geq 0$ and satisfies:*

$$|f(t)| \leq Me^{\gamma t},$$

for all $t \geq 0$ and for some constants γ and M. Then the Laplace transform exists for all $s > \gamma$.

With "piecewise continuous" we mean that, on a finite interval $a \leq t \leq b$, $f(t)$ is defined on that interval and is such that the interval can be subdivided into finitely many sub-intervals in each of which $f(t)$ is continuous and has finite limits (Kreyszig, 1988, p. 246). Figure 1 gives an example of a piecewise continuous function. The requirement mentioned in the property statement is that $f(t)$ is of exponential order γ as $t \to \infty$. Functions of exponential order cannot grow in absolute value more rapidly than $Me^{\gamma t}$ as t gets large. But since M and γ can be as large as desired the requirement is not much of a restriction (Spiegel, 1965, p. 2).

Example 4 *$f(t) = t^2$ is of exponential order 3 (for example) since $\left|t^2\right| = t^2 < e^{3t}$ for all $t \geq 0$. An example of a function that is not of exponential order is $f(t) = e^{t^3}$. This is because $\left|e^{-\gamma t}e^{t^3}\right| = e^{t(t^2-\gamma)}$ can become unbounded–and thus larger than any constant M–by increasing t.*

Armed with these results we derive the next properties. The first one says that discounting very heavily will wipe out the integral (and thus the Laplace transform) of any function of exponential order. The second one settles the uniqueness issue.

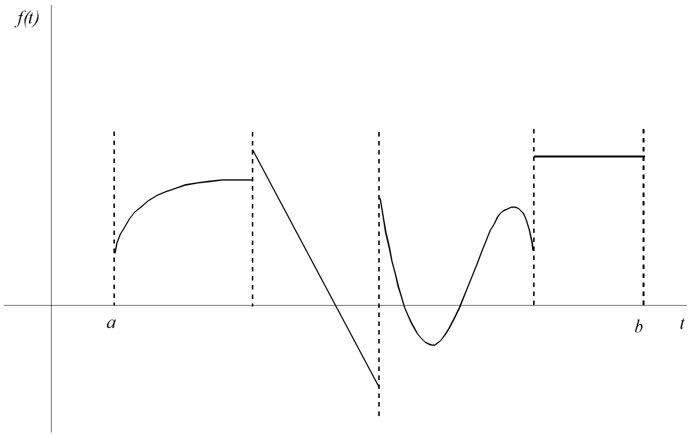

Figure 1: Piecewise continuous function

Property 3 *If $\mathcal{L}\{f, s\}$ is the Laplace transform of $f(t)$. Then:*

$$\lim_{s\to\infty} \mathcal{L}\{f, s\} = 0 \tag{P3}$$

Property 4 *Unique inversion [Lerch's theorem]. If we restrict ourselves to functions $f(t)$ which are piecewise continuous in every finite interval $0 \leq t \leq N$ and of exponential order for $t > N$, then the inverse Laplace transform of $\mathcal{L}\{f, s\}$, denoted by $\mathcal{L}^{-1}\{\mathcal{L}\{f, s\}\} = f(t)$, is unique.*

Let us now push on and study some more properties that will prove useful later on.

Property 5 *Transform of a derivative. If $f(t)$ is continuous for $0 \leq t \leq N$ and of exponential order γ for $t > N$ and $f'(t)$ is piecewise continuous for $0 \leq t \leq N$ then:*

$$\mathcal{L}\{f', s\} = s\mathcal{L}\{f, s\} - f(0), \tag{P5}$$

for $s > \gamma$.

Proof: Note that we state and prove the property for the simple case with $f(t)$ continuous for $t \geq 0$. Then we have by definition:[3]

$$\begin{aligned} \mathcal{L}\{f', s\} &= \int_0^\infty e^{-st} f'(t) dt \\ &= e^{-st} f(t)\big|_0^\infty + s\int_0^\infty e^{-st} f(t) dt \\ &= \lim_{t\to\infty} e^{-st} f(t) - f(0) + s\mathcal{L}\{f, s\}. \end{aligned}$$

[3] We use integration by parts, i.e. $\int u dv = uv - \int v du$, and set $u = e^{-st}$ and $v = f(t)$.

But for $s > \gamma$ the discounting by s dominates the exponential order of $f(t)$ so that $\lim_{t\to\infty} e^{-st} f(t) = 0$ and the result follows. □

Since this was so much fun, we can use (P5) repeatedly. We obtain:

$$\begin{aligned}\mathcal{L}\{f'', s\} &= s\mathcal{L}\{f', s\} - f'(0) \\ &= s\left[s\mathcal{L}\{f, s\} - f(0)\right] - f'(0).\end{aligned}$$

Similarly, we can deduce:

$$\mathcal{L}\{f''', s\} = s^3\mathcal{L}\{f, s\} - s^2 f(0) - s f'(0) - f''(0).$$

It does not take a genius to deduce the following property by induction.

Property 6 *Transform of the n-th derivative. If $f(t)$, $f'(t)$, ..., $f^{(n-1)}(t)$ are continuous for $0 \le t \le N$ and of exponential order γ for $t > N$ and $f^n(t)$ is piecewise continuous for $0 \le t \le N$ then:*

$$\mathcal{L}\{f^{(n)}, s\} = s^n\mathcal{L}\{f, s\} - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - s f^{(n-2)}(0) - f^{(n-1)}(0), \tag{P6}$$

for $s > \gamma$.

We can now illustrate the usefulness of the properties deduced so far and introduce the three-step procedure mentioned above by means of the following prototypical example.

2.1 Case study 1: Mickey Mouse case

Suppose we have the following differential equation:

$$\ddot{y}(t) + 4\dot{y}(t) + 3y(t) = 0, \tag{2}$$

which must be solved subject to the initial conditions:

$$y(0) = 3, \;\; \dot{y}(0) = 1, \tag{3}$$

where we have switched to the conventional economics notation for dynamical variables, i.e. $\dot{y}(t) \equiv dy(t)/dt = y'(t)$ and $\ddot{y}(t) \equiv d^2y(t)/dt^2 = y''(t)$.[4] Here goes the three-step procedure:

Step 1: Set up the subsidiary equation. By taking the Laplace transform of (2) and noting (P6) we get:

$$\begin{aligned}\mathcal{L}\{\ddot{y}, s\} + 4\mathcal{L}\{\dot{y}, s\} + 3\mathcal{L}\{y, s\} = 0 \;\Leftrightarrow \\ \left[s^2\mathcal{L}\{y, s\} - sy(0) - \dot{y}(0)\right] + 4\left[s\mathcal{L}\{y, s\} - y(0)\right] + 3\mathcal{L}\{y, s\} = 0 \;\Leftrightarrow \\ \left[s^2 + 4s + 3\right]\mathcal{L}\{y, s\} = (s+4)y(0) + \dot{y}(0).\end{aligned} \tag{4}$$

[4] We could have started with the Newtonian 'dot' notation from the start but would have run into notational trouble with (P6). How many dots can one fit on top of an italic f? Not many!

By substituting (3) in (4) we obtain the *subsidiary equation* of the differential equation including its initial conditions.

$$\left[s^2 + 4s + 3\right] \mathcal{L}\{y, s\} = 3s + 13. \tag{5}$$

Step 2: Solve the subsidiary equation. We now do the easy stuff of algebraically manipulating the expression (5) in s-space. We notice that the quadratic on the left-hand side of (5) can be written as $s^2 + 4s + 3 = (s + 1)(s + 3)$ so we can solve for $\mathcal{L}\{y, s\}$ quite easily:[5]

$$\begin{aligned}\mathcal{L}\{y, s\} &= \frac{3s + 13}{(s + 1)(s + 3)} = \frac{3(s + 1) + 10}{(s + 1)(s + 3)} \\ &= \frac{3}{s + 3} + \frac{10}{(s + 1)(s + 3)} \\ &= \frac{3}{s + 3} + \frac{10}{3 - 1}\left[\frac{1}{s + 1} - \frac{1}{s + 3}\right] \\ &= \frac{5}{s + 1} - \frac{2}{s + 3}.\end{aligned} \tag{6}$$

Step 3: Invert the transform to get the solution of the given problem. We have now written the (Laplace transform of the) solution in terms of known transforms. Inversion of (6) is thus straightforward and results in:

$$y(t) = \mathcal{L}^{-1}\left\{\mathcal{L}\{y, s\}\right\} = 5\mathcal{L}^{-1}\left\{\frac{1}{s + 1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s + 3}\right\} = 5e^{-t} - 2e^{-3t}. \tag{7}$$

Of course we could have obtained this solution also quite easily using the standard techniques so for this simple example the Laplace transform technique is not that useful. It has some value added but not a lot. The thing to note, however, is that the method is essentially unchanged for much more complex problems. We now study two such cases.

2.2 Case study 2: Time-varying forcing term

Assume that the differential equation (2) is replaced by:

$$\ddot{y}(t) + 4\dot{y}(t) + 3y(t) = g(t), \tag{8}$$

where $g(t)$ is some (piecewise continuous) *forcing function* which is time-dependent and has a unique Laplace transform $\mathcal{L}\{g, s\}$. The initial conditions continue to be as given in (3). Using the same procedure as before we derive the solution of the subsidiary equation in terms of the Laplace transforms:

$$\underbrace{\mathcal{L}\{y, s\}}_{\text{output}} = \underbrace{\frac{3s + 13}{(s + 1)(s + 3)}}_{\text{initial conditions}} + \underbrace{\frac{\mathcal{L}\{g, s\}}{(s + 1)(s + 3)}}_{\text{input}}. \tag{9}$$

[5] We show the trivial steps leading to the final result in order to demonstrate that the algebra involved in s-space is indeed trivial. In general, the work involved in step 2 of the procedure is always easier than tackling the problem directly in t-space.

The first term on the right-hand side is the same as before (see (6)) and results from the initial conditions of the problem. The second term on the right-hand side represents the influence of the time-varying forcing term. Two further things must be noted about equation (9). First, the expression is perfectly general. A whole class of shock terms can be used in (9) to solve for $y(t)$ after inversion. Second, it should be noted that all of the model's dynamic properties are contained in the quadratic function appearing in the denominator. In fact, $H(s) \equiv \frac{1}{(s+1)(s+3)}$ is often referred to as the *transfer function* in the engineering literature since it transfers the shock (the "input") to the variable of interest (the "output")–see for example Boyce and DiPrima (1992, p. 312). The inverse of $H(s)$, denoted by $h(t) \equiv \mathcal{L}^{-1}\{H(s)\}$, is called the *impulse reponse function* of the system.[6]

2.3 Case study 3: Systems of differential equations

The transform method is equally valuable for systems of differential equations. Suppose that the dynamic model is given in matrix form by:

$$\begin{bmatrix} \dot{K}(t) \\ \dot{Q}(t) \end{bmatrix} = \Delta \begin{bmatrix} K(t) \\ Q(t) \end{bmatrix} + \begin{bmatrix} g_K(t) \\ g_Q(t) \end{bmatrix}, \tag{10}$$

where Δ is the two-by-two Jacobian matrix with typical element δ_{ij}, and $g_i(t)$ are (potentially time-varying) shock terms. Note that a system like (10) occurs quite regularly in analytical low-dimensional macro models.

By taking the Laplace transform of (10), and noting property (P5) we get:

$$\begin{bmatrix} s\mathcal{L}\{K,s\} - K(0) \\ s\mathcal{L}\{Q,s\} - Q(0) \end{bmatrix} = \Delta \begin{bmatrix} \mathcal{L}\{K,s\} \\ \mathcal{L}\{Q,s\} \end{bmatrix} + \begin{bmatrix} \mathcal{L}\{g_K,s\} \\ \mathcal{L}\{g_Q,s\} \end{bmatrix} \Leftrightarrow$$

$$\Lambda(s) \begin{bmatrix} \mathcal{L}\{K,s\} \\ \mathcal{L}\{Q,s\} \end{bmatrix} = \begin{bmatrix} K(0) + \mathcal{L}\{g_K,s\} \\ Q(0) + \mathcal{L}\{g_Q,s\} \end{bmatrix}, \tag{11}$$

where $\Lambda(s) \equiv sI - \Delta$ is a two-by-two matrix depending on s and the elements of Δ. We know from matrix algebra that the inverse of this matrix, $\Lambda(s)^{-1}$, can be written as:

$$\Lambda(s)^{-1} = \frac{1}{|\Lambda(s)|} \text{adj}\Lambda(s), \tag{12}$$

where $\text{adj}\Lambda(s)$ is the adjoint matrix (i.e. the matrix of cofactors / signed minors) of $\Lambda(s)$ and

[6] Getting interesting impulse response functions out of a simple model is the Holy Grail for RBC-adepts. They almost uniformly use computer simulations to compute impulse response functions for calibrated models. As we show below it is often quite feasible to derive analytical expressions for the impulse response functions. This has the advantage that we can study precisely what are the critical parameters in the impulse response function. See also the plea by Campbell (1994) in this regard.

$|\Lambda(s)|$ is the determinant of $\Lambda(s)$.[7] For the simple two-by-two model $\text{adj}\Lambda(s)$ and $|\Lambda(s)|$ are:

$$\text{adj}\Lambda(s) \equiv \begin{bmatrix} s-\delta_{22} & \delta_{12} \\ \delta_{21} & s-\delta_{11} \end{bmatrix}, \tag{13}$$

and:

$$\begin{aligned} |\Lambda(s)| &= (s-\delta_{11})(s-\delta_{22}) - \delta_{12}\delta_{21} \\ &= s^2 - (\delta_{11}+\delta_{22})s + \delta_{11}\delta_{22} - \delta_{12}\delta_{21} \\ &= s^2 - s\,\text{tr}\Delta + |\Delta|, \end{aligned} \tag{14}$$

where $\text{tr}\Delta$ and $|\Delta|$ are, respectively, the trace (i.e. the sum of the diagonal elements) and the determinant of the matrix Δ. The quadratic equation in (14) can be factored as follows:

$$|\Lambda(s)| = (s-\lambda_1)(s-\lambda_2), \tag{15}$$

where λ_1 and λ_2 are the characteristic roots of the matrix Δ:

$$\lambda_{1,2} = \frac{\text{tr}\Delta \pm \sqrt{(\text{tr}\Delta)^2 - 4\,|\Delta|}}{2}. \tag{16}$$

Before going on we note–by comparing (14) and (15)–that for the two-by-two case we have:

$$\text{tr}(\Delta) = \lambda_1 + \lambda_2, \quad |\Delta| = \lambda_1\lambda_2, \tag{17}$$

i.e. the sum of the characteristic roots equals the trace of the Jacobian matrix Δ and the product of these roots equals the determinant of this matrix. This property is often very useful to deduce the signs of these roots. It is not difficult to see why this is so by looking at (16). We note that the roots are real (imaginary) if $(\text{tr}\Delta)^2 > (<)\ 4|\Delta|$ and that they are distinct provided $(\text{tr}\Delta)^2 \neq 4|\Delta|$. Also, if $\text{tr}\Delta > 0$ there must be at least one positive root. Finally, if $|\Delta| < 0$ there is exactly one positive (unstable) and one negative (stable) real characteristic root.[8]

Let us now consider the two cases encountered most often in the economics literature for which the roots are real and distinct, i.e. $(\text{tr}\Delta)^2 > 4|\Delta|$.

2.3.1 Both roots negative ($\lambda_1, \lambda_2 < 0$)

We can use (11), (12), and (15) to derive the following expression in Laplace transforms:

$$\begin{bmatrix} \mathcal{L}\{K,s\} \\ \mathcal{L}\{Q,s\} \end{bmatrix} = \frac{\text{adj}\Lambda(s) \begin{bmatrix} K(0) + \mathcal{L}\{g_K, s\} \\ Q(0) + \mathcal{L}\{g_Q, s\} \end{bmatrix}}{(s-\lambda_1)(s-\lambda_2)}, \tag{18}$$

[7] Students needing a brushup on matrix algebra should consult sources like–in increasing order of sophistication–Ayres (1974), Ortega (1987), and Lancaster and Tismenetsky (1985).

[8] Recall that these characteristic roots are going to show up in exponential functions, $e^{\lambda_i t}$, in the solution of the (system of) differential equation(s). If the root is positive (negative) $e^{\lambda_i t} \to \infty$ ($\to 0$) as $t \to \infty$ so positive (negative) roots are unstable (stable). The knife-edge case of a zero root is also stable as $e^{0t} = 1$ for all t. See section 5 below.

which is in the same format as equation (9), with $H(s) \equiv \text{adj}\Lambda(s)/[(s-\lambda_1)(s-\lambda_2)]$ acting as the transfer function. To solve the model for particular shocks it is useful to re-express the transfer function. We note that for the two-by-two case $\text{adj}\Lambda(s)$ has the following properties:

$$\begin{aligned} \text{adj}\Lambda(s) &= \text{adj}\Lambda(\lambda_i) + (s-\lambda_i)I, \quad (i=1,2), \\ I &= \frac{\text{adj}\Lambda(\lambda_1) - \text{adj}\Lambda(\lambda_2)}{\lambda_1 - \lambda_2}, \end{aligned} \tag{19}$$

where the second result follows from the first. We can now perform a partial fractions expansion of the transfer matrix:

$$\begin{aligned} \frac{\text{adj}\Lambda(s)}{(s-\lambda_1)(s-\lambda_2)} &= \frac{\text{adj}\Lambda(s)}{\lambda_1-\lambda_2}\left[\frac{1}{s-\lambda_1} - \frac{1}{s-\lambda_2}\right] \\ &= \frac{1}{\lambda_1-\lambda_2}\left[\frac{\text{adj}\Lambda(s)}{s-\lambda_1} - \frac{\text{adj}\Lambda(s)}{s-\lambda_2}\right] \\ &= \frac{1}{\lambda_1-\lambda_2}\left[I + \frac{\text{adj}\Lambda(\lambda_1)}{s-\lambda_1} - I - \frac{\text{adj}\Lambda(\lambda_2)}{s-\lambda_2}\right] \\ &= \frac{1}{\lambda_1-\lambda_2}\left[\frac{\text{adj}\Lambda(\lambda_1)}{s-\lambda_1} - \frac{\text{adj}\Lambda(\lambda_2)}{s-\lambda_2}\right]. \end{aligned} \tag{20}$$

By using (20) in (18) we obtain the following general expression in terms of the Laplace transforms:

$$\begin{bmatrix} \mathcal{L}\{K,s\} \\ \mathcal{L}\{Q,s\} \end{bmatrix} = \frac{1}{\lambda_1-\lambda_2}\left[\frac{\text{adj}\Lambda(\lambda_1)}{s-\lambda_1} - \frac{\text{adj}\Lambda(\lambda_2)}{s-\lambda_2}\right]\begin{bmatrix} K(0) + \mathcal{L}\{g_K,s\} \\ Q(0) + \mathcal{L}\{g_Q,s\} \end{bmatrix}. \tag{21}$$

Suppose that the shocks are step functions and satisfy $g_i(t) = g_i$ for $i = K, Q$ and $t \geq 0$. The Laplace transform for such step functions is $\mathcal{L}\{g_i, s\} = g_i/s$ which can be substituted in (21). After some manipulation we obtain the following result:

$$\begin{aligned} \begin{bmatrix} \mathcal{L}\{K,s\} \\ \mathcal{L}\{Q,s\} \end{bmatrix} &= \left[\frac{B}{s-\lambda_1} + \frac{I-B}{s-\lambda_2}\right]\begin{bmatrix} K(0) \\ Q(0) \end{bmatrix} \\ &\quad - \left[\frac{B}{\lambda_1}\left(\frac{-\lambda_1}{s(s-\lambda_1)}\right) + \frac{I-B}{\lambda_2}\left(\frac{-\lambda_2}{s(s-\lambda_2)}\right)\right]\begin{bmatrix} g_K \\ g_Q \end{bmatrix}, \end{aligned} \tag{22}$$

where $B \equiv \text{adj}\Lambda(\lambda_1)/(\lambda_1-\lambda_2)$ and $I - B \equiv -\text{adj}\Lambda(\lambda_2)/(\lambda_1-\lambda_2)$ are weighting matrices.[9] The expression is now in terms of known Laplace transforms so that inversion is child's play:

$$\begin{aligned} \begin{bmatrix} K(t) \\ Q(t) \end{bmatrix} &= \left[Be^{\lambda_1 t} + (I-B)e^{\lambda_2 t}\right]\begin{bmatrix} K(0) \\ Q(0) \end{bmatrix} \\ &\quad - \left[\frac{B}{\lambda_1}\left(1-e^{\lambda_1 t}\right) + \frac{I-B}{\lambda_2}\left(1-e^{\lambda_2 t}\right)\right]\begin{bmatrix} g_K \\ g_Q \end{bmatrix}. \end{aligned} \tag{23}$$

[9] These weighting matrices also satisfy:

$$\frac{B}{\lambda_1} + \frac{I-B}{\lambda_2} = \frac{\text{adj}\Lambda(0)}{-\lambda_1\lambda_2} = \frac{\text{adj}\Delta}{\lambda_1\lambda_2} = \Delta^{-1}.$$

These results are used below. Note that we have used the fact that $\text{adj}\Lambda(0) = \text{adj}(-\Delta) = (-1)^{n-1}\text{adj}\Delta$, where n is the order of Δ ($n = 2$ here). See Lancaster and Tismenetsky (1985, p. 43).

Equation (23) constitutes the full solution of the problem. It yields impact, transition, and long-run results of the shock. To check that we have done things correctly we verify that we can recover from (23) the initial conditions by setting $t = 0$ and the long-run steady state by letting $t \to \infty$. The first result is obvious: for $t = 0$ we have that $e^{\lambda_i t} = 1$ so that $K(t) = K(0)$ and $Q(t) = Q(0)$. Similarly, for $t \to \infty$, $e^{\lambda_i t} \to 0$ (since both roots are stable) and we get from (23):

$$\begin{aligned} \begin{bmatrix} K(\infty) \\ Q(\infty) \end{bmatrix} &= -\left[\frac{B}{\lambda_1} + \frac{I-B}{\lambda_2}\right] \begin{bmatrix} g_K \\ g_Q \end{bmatrix} = \frac{-\text{adj}\Lambda(0)}{-\lambda_1\lambda_2} \begin{bmatrix} g_K \\ g_Q \end{bmatrix} \\ &= \frac{\text{adj}\Delta}{-|\Delta|} \begin{bmatrix} g_K \\ g_Q \end{bmatrix} = -\Delta^{-1} \begin{bmatrix} g_K \\ g_Q \end{bmatrix}, \end{aligned} \tag{24}$$

which is the same solution we would have obtained by substituting the permanent shock in (10) and imposing the steady state. So at least the initial and ultimate effects check out!

We could have checked out our results also by working directly with the solution in terms of Laplace transforms [i.e. (21) in general and (22) for the particular shocks]. We need the following two properties to do so.

Property 7 *If the indicated limits exist then the initial-value theorem says:*

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} s\mathcal{L}\{f,s\} \tag{P7}$$

and the final-value theorem says:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\mathcal{L}\{f,s\} \tag{P8}$$

PROOF: The proofs are suggested by Spiegel (1965, p. 20). We know from property (P5) that:

$$\mathcal{L}\{f',s\} \equiv \int_0^\infty e^{-st} f'(t)dt = s\mathcal{L}\{f,s\} - f(0). \tag{a}$$

But if $f'(t)$ is piecewise continuous and of exponential order then it has a Laplace transform and we know from property (P3) that $\lim_{s\to\infty} \mathcal{L}\{f',s\} = 0$. Using this result in (a) yields:

$$0 = \lim_{s\to\infty} s\mathcal{L}\{f,s\} - f(0) \iff \lim_{s\to\infty} s\mathcal{L}\{f,s\} = f(0) = \lim_{t\to 0} f(t), \tag{b}$$

which proves (P7). The proof of (P8) works in a similar fashion. By using (a) we get:

$$\lim_{s\to 0}\int_0^\infty e^{-st} f'(t)dt = \lim_{s\to 0} s\mathcal{L}\{f,s\} - f(0). \tag{c}$$

The left-hand side of (c) can be rewritten as:

$$\begin{aligned} \lim_{s\to 0}\int_0^\infty e^{-st} f'(t)dt &= \int_0^\infty f'(t)dt = \lim_{T\to\infty}\int_0^T f'(t)dt = \lim_{T\to\infty}\int_0^T df(t) \\ &= \lim_{T\to\infty} f(T) - f(0). \end{aligned} \tag{d}$$

By combining (c) and (d) we obtain the required result. □

Applying Property (P7) directly to (22) we obtain:

$$\begin{aligned}
\lim_{s\to\infty}\begin{bmatrix} s\mathcal{L}\{K,s\} \\ s\mathcal{L}\{Q,s\}\end{bmatrix} &= \left[B\underbrace{\lim_{s\to\infty}\left(\frac{s}{s-\lambda_1}\right)}_{=1} + (I-B)\underbrace{\lim_{s\to\infty}\left(\frac{s}{s-\lambda_2}\right)}_{=1}\right]\begin{bmatrix} K(0) \\ Q(0)\end{bmatrix} \\
&- \left[\frac{B}{\lambda_1}\underbrace{\lim_{s\to\infty}\left(\frac{-\lambda_1 s}{s(s-\lambda_1)}\right)}_{=0} + \frac{I-B}{\lambda_2}\underbrace{\lim_{s\to\infty}\left(\frac{-\lambda_2 s}{s(s-\lambda_2)}\right)}_{=0}\right]\begin{bmatrix} g_K \\ g_Q\end{bmatrix} \\
&= [B+I-B]\begin{bmatrix} K(0) \\ Q(0)\end{bmatrix} = \begin{bmatrix} K(0) \\ Q(0)\end{bmatrix}.
\end{aligned}$$

Similarly, applying Property (P8) to (22) we get:

$$\begin{aligned}
\lim_{s\to 0}\begin{bmatrix} s\mathcal{L}\{K,s\} \\ s\mathcal{L}\{Q,s\}\end{bmatrix} &= \left[B\underbrace{\lim_{s\to 0}\left(\frac{s}{s-\lambda_1}\right)}_{=0} + (I-B)\underbrace{\lim_{s\to 0}\left(\frac{s}{s-\lambda_2}\right)}_{=0}\right]\begin{bmatrix} K(0) \\ Q(0)\end{bmatrix} \\
&- \left[\frac{B}{\lambda_1}\underbrace{\lim_{s\to 0}\left(\frac{-\lambda_1 s}{s(s-\lambda_1)}\right)}_{=1} + \frac{I-B}{\lambda_2}\underbrace{\lim_{s\to 0}\left(\frac{-\lambda_2 s}{s(s-\lambda_2)}\right)}_{=1}\right]\begin{bmatrix} g_K \\ g_Q\end{bmatrix} \\
&= -\left[\frac{B}{\lambda_1} + \frac{I-B}{\lambda_2}\right]\begin{bmatrix} g_K \\ g_Q\end{bmatrix} = \begin{bmatrix} K(\infty) \\ Q(\infty)\end{bmatrix}.
\end{aligned}$$

2.3.2 Roots alternate in sign ($\lambda_1 < 0 < \lambda_2$)

A situation which occurs quite regularly in dynamic macro models is one in which the Jacobian matrix Δ in (10) has one negative (stable) root and one positive (unstable) root. The way to check for such saddle-point stability is either by means of (16) or (17). From (16) we observe that if $|\Delta| < 0$ then we have distinct and real roots for sure since $\sqrt{(\text{tr}\Delta)^2 - 4|\Delta|} > 0$. Also, since $|\Delta| < \lambda_1\lambda_2 < 0$ it must be the case that $\lambda_1 < 0 < \lambda_2$. Of course we also see this directly from (17).

The beauty of the Laplace transform technique is now that (18) is still appropriate and just needs to be solved differently. Let us motivate the alternative solution method heuristically by writing (18) as follows:

$$(s-\lambda_1)\begin{bmatrix} \mathcal{L}\{K,s\} \\ \mathcal{L}\{Q,s\}\end{bmatrix} = \frac{\text{adj}\Lambda(s)\begin{bmatrix} K(0)+\mathcal{L}\{g_K,s\} \\ Q(0)+\mathcal{L}\{g_Q,s\}\end{bmatrix}}{s-\lambda_2}. \tag{25}$$

In a two-by-two saddle-point stable system there is one predetermined and one non-predetermined (or "jumping") variable so we need to supply only one initial condition (and not two as before). Let us assume that K is the predetermined variable (the value of which is determined in the past, e.g. a stock of human or physical capital, assets, etcetera) so that $K(0)$ is given. But then Q is the non-predetermined variable (e.g. a (shadow) price) so we must somehow figure out its initial condition.[10] It is clear from (25) how we should do this.

Note that the instability originates from the unstable root λ_2. For $s = \lambda_2$ we have that the denominator on the right-hand side of (25) is zero. The only way we can still obtain bounded (and thus economically sensible) solutions for $\mathcal{L}\{K,s\}$ and $\mathcal{L}\{Q,s\}$ is if the numerator on the right-hand side of (25) is also zero for $s = \lambda_2$, i.e. if:

$$\text{adj}\Lambda(\lambda_2)\begin{bmatrix} K(0)+\mathcal{L}\{g_K,\lambda_2\} \\ Q(0)+\mathcal{L}\{g_Q,\lambda_2\} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{26}$$

All except one of the variables appearing in (26) are determined so $Q(0)$ must be such that (26) holds. At first view it appears as if (26) represents two equations in one unknown but that is not the case. A theorem from matrix algebra says that, since $\Lambda(\lambda_2)$ is of rank 1 so is $\text{adj}\Lambda(\lambda_2)$.[11] So, in fact, we can use either row of (26) to compute $Q(0)$:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_2-\delta_{22} & \delta_{12} \\ \delta_{21} & \lambda_2-\delta_{11} \end{bmatrix}\begin{bmatrix} K(0)+\mathcal{L}\{g_K,\lambda_2\} \\ Q(0)+\mathcal{L}\{g_Q,\lambda_2\} \end{bmatrix} \Longrightarrow$$

$$Q(0) = -\mathcal{L}\{g_Q,\lambda_2\} - \left(\frac{\lambda_2-\delta_{22}}{\delta_{12}}\right)[K(0)+\mathcal{L}\{g_K,\lambda_2\}] \tag{27}$$

$$= -\mathcal{L}\{g_Q,\lambda_2\} - \left(\frac{\delta_{21}}{\lambda_2-\delta_{11}}\right)[K(0)+\mathcal{L}\{g_K,\lambda_2\}]. \tag{28}$$

We next use (19), (25), and (26) to get:

$$(s-\lambda_1)\begin{bmatrix} \mathcal{L}\{K,s\} \\ \mathcal{L}\{Q,s\} \end{bmatrix} = \frac{\text{adj}\Lambda(\lambda_2)\begin{bmatrix} K(0)+\mathcal{L}\{g_K,s\} \\ Q(0)+\mathcal{L}\{g_Q,s\} \end{bmatrix}}{s-\lambda_2} + \begin{bmatrix} K(0)+\mathcal{L}\{g_K,s\} \\ Q(0)+\mathcal{L}\{g_Q,s\} \end{bmatrix}$$

$$= \begin{bmatrix} K(0)+\mathcal{L}\{g_K,s\} \\ Q(0)+\mathcal{L}\{g_Q,s\} \end{bmatrix} + \text{adj}\Lambda(\lambda_2)\begin{bmatrix} \frac{\mathcal{L}\{g_K,s\}-\mathcal{L}\{g_K,\lambda_2\}}{s-\lambda_2} \\ \frac{\mathcal{L}\{g_Q,s\}-\mathcal{L}\{g_Q,\lambda_2\}}{s-\lambda_2} \end{bmatrix}, \tag{29}$$

where we have used (26) in the last step. Note that in (29) all effects of the unstable root have been incorporated and only the stable dynamics remains (represented by the term involving $s-\lambda_1$).

[10] Of course, economic theory suggests which variables are predetermined and which ones are not.

[11] In general, if the n-square matrix Δ has distinct eigenvalues its eigenvectors are linearly independent and the rank of $\Lambda(\lambda_i) \equiv \lambda_i I - \Delta$ is $n-1$ (Ayres, 1974, p. 150). Furthermore, for any n-square matrix A of rank $n-1$ we have that $\text{adj}A$ is of rank 1 (Ayres, 1974, p. 50).

Suppose again that the shocks satisfy $g_i(t) = g_i$ for $i = K, Q$ and $t \geq 0$ so that $\mathcal{L}\{g_i, s\} = g_i/s$ and:

$$\frac{\mathcal{L}\{g_i, s\} - \mathcal{L}\{g_i, \lambda_2\}}{s - \lambda_2} = \frac{\frac{g_i}{s} - \frac{g_i}{\lambda_2}}{s - \lambda_2} = -\frac{g_i}{s\lambda_2}.$$

By using these results in (29) we obtain the full solution of the saddle-point stable model:

$$\begin{aligned} (s - \lambda_1) \begin{bmatrix} \mathcal{L}\{K, s\} \\ \mathcal{L}\{Q, s\} \end{bmatrix} &= \begin{bmatrix} K(0) \\ Q(0) \end{bmatrix} - \frac{1}{\lambda_2} [\text{adj}\Lambda(\lambda_2) - \lambda_2 I] \begin{bmatrix} g_K \\ g_Q \end{bmatrix} \frac{1}{s} \Longleftrightarrow \\ \begin{bmatrix} \mathcal{L}\{K, s\} \\ \mathcal{L}\{Q, s\} \end{bmatrix} &= \begin{bmatrix} K(0) \\ Q(0) \end{bmatrix} \left(\frac{1}{s - \lambda_1} \right) - \frac{\text{adj}\Lambda(0)}{-\lambda_1 \lambda_2} \begin{bmatrix} g_K \\ g_Q \end{bmatrix} \left(\frac{-\lambda_1}{s(s - \lambda_1)} \right) \\ &= \begin{bmatrix} K(0) \\ Q(0) \end{bmatrix} \left(\frac{1}{s - \lambda_1} \right) + \begin{bmatrix} K(\infty) \\ Q(\infty) \end{bmatrix} \left(\frac{-\lambda_1}{s(s - \lambda_1)} \right), \end{aligned} \tag{30}$$

where we have used (19) and the result in footnote 9, and where $Q(0)$ is obtained by substituting the shock terms in either (27) or (28). By inverting (30) we obtain the solution in the time dimension.

$$\begin{bmatrix} K(t) \\ Q(t) \end{bmatrix} = \begin{bmatrix} K(0) \\ Q(0) \end{bmatrix} e^{\lambda_1 t} + \begin{bmatrix} K(\infty) \\ Q(\infty) \end{bmatrix} \left(1 - e^{\lambda_1 t} \right). \tag{31}$$

The key point to note is that the stable root determines the speed of transition between the respective impact and long-run results.

3 Dynamic tax policy in an OLG model

We now possess all the technical tools needed to perform our first comparative dynamic analysis. In this section we illustrate the methods involved with the aid of the Heijdra-Ligthart (2000) paper which shows how to analyze tax policy in a dynamic overlapping-generations (OLG) model. This model is of some interest as it nests various influential models as special cases, viz. the OLG model of Blanchard (1985) and the prototypical RBC model of Baxter and King (1993).

3.1 The model

The model is described extensively in Heijdra and Ligthart (2000). The defining equations and variable and parameter definitions are gathered for convenience in Table 2. Equation (T2.1) shows that net investment ($\dot{K}(t)$) equals output minus consumption (government consumption is deemed absent and $Y(t)$ represents **net** output, i.e. after allowing for depreciation of the capital stock). (T2.2) is the modified Keynes-Ramsey rule, i.e. the aggregate consumption Euler equation which differs from the Euler equation for individual households because of the turnover of generations. (T2.3) is the government budget constraint (there are no bonds). Transfers to the household sector add up to the total revenue of the taxes on capital, labour income, and consumption. (T2.4)-(T2.5) are the marginal productivity conditions for, respectively, labour and capital. Since capital is taxed at firm level, t_K appears in (T2.5). In contrast, since labour is not taxed at firm level (there is no payroll tax) there is no tax term appearing in (T2.4). (T2.6) shows how the marginal rate of substitution between consumption and leisure depends on the after-tax wage rate. In view of the intertemporal additivity of the utility functional, all the dynamics in the household choice problem is summarized by the consumption Euler equation and the static decision rule (T2.6) relates spending on leisure (and thus labour supply) to consumption and the tax wedge. Both the consumption tax and the labour income tax are component of this tax wedge. (T2.7) is the aggregate production function relating net output to the production factors labour and capital. Total factor productivity is exogenous and represented by Ω_0. Finally, (T2.8) is the factor price frontier, obtained by combining (T2.4), (T2.5), and (T2.7).

Table 2: The model in levels

$$\dot{K}(t) = Y(t) - C(t) \tag{T2.1}$$

$$\frac{\dot{C}(t)}{C(t)} = \left[r(t) - \alpha - \frac{\dot{t}_C(t)}{1+t_C(t)}\right] - \epsilon_C \beta(\alpha+\beta)\left[\frac{K(t)}{[1+t_C(t)]C(t)}\right] \tag{T2.2}$$

$$T(t) = t_K(t)\left[Y(t) - W(t)L(t)\right] + t_L(t)W(t)L(t) + t_C(t)C(t) \tag{T2.3}$$

$$W(t) = \epsilon_L \left(\frac{Y(t)}{L(t)}\right) \tag{T2.4}$$

$$\frac{r(t)}{1-t_K(t)} = (1-\epsilon_L)\left(\frac{Y(t)}{K(t)}\right) \tag{T2.5}$$

$$W(t)\left[1-L(t)\right] = \left(\frac{1-\epsilon_C}{\epsilon_C}\right)\left(\frac{1+t_C(t)}{1-t_L(t)}\right)C(t) \tag{T2.6}$$

$$Y(t) = \Omega_0 L(t)^{\epsilon_L} K(t)^{1-\epsilon_L} \tag{T2.7}$$

$$\Omega_0 = \left(\frac{W(t)}{\epsilon_L}\right)^{\epsilon_L}\left(\frac{r(t)}{(1-\epsilon_L)(1-t_K(t))}\right)^{1-\epsilon_L} \tag{T2.8}$$

Variables:		*Parameters:*	
$L(t)$	employment	α	rate of time preference
$C(t)$	aggregate consumption	β	birth rate
$Y(t)$	net aggregate output	Ω_0	productivity parameter
$W(t)$	wage rate	ϵ_L	production share of labour
$K(t)$	capital stock	ϵ_C	preference parameter
$r(t)$	rate of interest		
$T(t)$	lump-sum transfers	*Notation:*	
$t_C(t)$	consumption tax	$\dot{x}(t) \equiv dx(t)/dt$	
$t_L(t)$	labour income tax		
$t_K(t)$	capital tax		

3.2 Model properties

Before using the model contained in Table 2 for anything at all we must first study its properties. In particular, we must investigate existence, uniqueness, and stability of the equilibrium. With the advent of computer simulation models this step of the argumentation is rapidly becoming a "lost art" among macroeconomists. As we demonstrate in the remainder of this section, it is not only feasible but also insightful to study the analytical properties of the model before turning to linearisation and/or simulation. The first task at hand is the derivation of the phase diagram.

3.2.1 Employment as a function of the state variables

By using labour demand (T2.4), labour supply (T2.6), and the production function (T2.7)– and dropping the time index where no confusion is possible–we obtain an expression relating (labour-market-clearing) equilibrium employment to the state variables (C and K) and the exogenous variables:

$$(\Gamma(L) \equiv)\ (1-L)L^{\epsilon_L - 1} = \left(\frac{(1-\epsilon_C)(1+t_H)}{\epsilon_C \epsilon_L \Omega_0}\right) C K^{-(1-\epsilon_L)}, \tag{32}$$

where $t_H \equiv (t_C + t_L)/(1-t_L)$ is the tax wedge directly facing households, and $\Gamma(L)$ is a decreasing function in the feasible interval $L \in [0,1]$ with $\Gamma'(L) = -L^{\epsilon_L - 2}\left[(1-\epsilon_L)(1-L) + L\right] < 0$ and $\Gamma''(L) = (1-\epsilon_L)L^{\epsilon_L - 3}\left[2 - \epsilon_L(1-L)\right] > 0$. In summary, (32) shows that equilibrium employment depends negatively on consumption and positively on the capital stock.

3.2.2 Capital stock equilibrium

The capital stock equilibrium (CSE) locus represents points in (C, K)-space for which $\dot{K} = 0$ and thus $Y = C$. We note from (T2.4) and (T2.6) that:

$$\frac{1-L}{L} = \left(\frac{(1-\epsilon_C)(1+t_H)}{\epsilon_C \epsilon_L}\right)\left(\frac{C}{Y}\right), \tag{33}$$

so that L is constant along the CSE line (since $C/Y = 1$):

$$L^* = \frac{\epsilon_C \epsilon_L}{\epsilon_C \epsilon_L + (1-\epsilon_C)(1+t_H)}. \tag{34}$$

Note that if labour supply is exogenous ($\epsilon_C = 1$ so that $L = 1$) the consumption and labour income taxes do not affect equilibrium employment. In contrast, with endogenous labour supply ($0 < \epsilon_C < 1$), an increase in the tax wedge reduces equilibrium employment along the CSE curve.

By substituting (34) into (T2.7) and using (T2.1) in steady-state format, the CSE curve can be written as:

$$C = \Omega_0 \left(L^*\right)^{\epsilon_L} K^{1-\epsilon_L}, \tag{35}$$

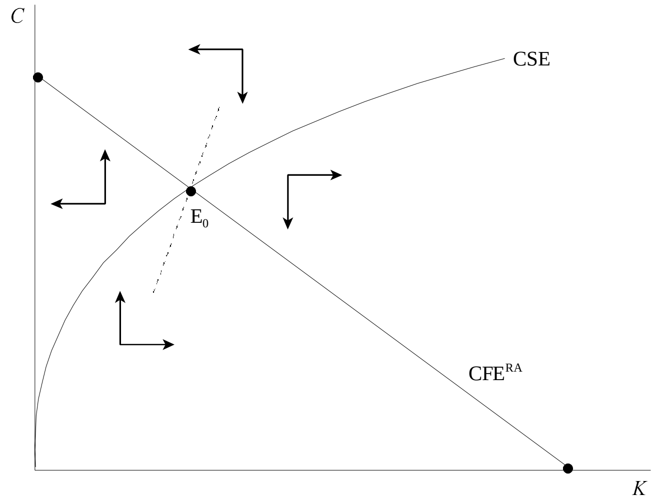

Figure 2: Phase diagram for the RA model

which is the concave function through the origin in Figures 2 and 3. *Ceteris paribus* the capital stock, an increase (decrease) in consumption decreases (increases) equilibrium employment (see (32)), output, and net investment. Hence, $\dot{K} < 0$ (> 0) for point above (below) the CSE line. This has been illustrated with horizontal arrows in Figures 2 and 3.

3.2.3 Consumption flow equilibrium

The consumption flow equilibrium (CFE) locus represents points in (C, K)-space for which the aggregate flow of consumption is in equilibrium ($\dot{C} = 0$). By using (T2.2) in steady-state, (33), and (T2.7) we can write the CFE locus as follows:

$$\beta(\alpha+\beta) = \left(\frac{\epsilon_L(1-t_L)}{1-\epsilon_C}\right)\left(\frac{1-L}{L}\right) y\left[(1-\epsilon_L)(1-t_K)y-\alpha\right] \tag{36}$$

$$y = \Omega_0\left(\frac{L}{K}\right)^{\epsilon_L}, \tag{37}$$

where $y \equiv Y/K$ is the output-capital ratio. Equations (36)-(37) define consumption flow equilibrium in (K, L)-space. In combination with (34) they yield an important result.

Proposition 1 *Unique steady state. A steady-state equilibrium of the model in Table 2 exists and is unique.*

PROOF: In the steady state $\dot{K} = \dot{C} = 0$ so that (by (34)) $L = L^*$. Then (36) defines a unique output-capital ratio y^* and (37) a unique capital stock K^*. The production function (T2.7) then yields a unique output level Y^* which equals consumption C^* in the steady state. All other variables are determined uniquely also. □

Representative-agent model Before turning to the most general version of the model, we first study the proto-typical representative-agent (RA) version of the model which is obtained by setting the birth-death rate equal to zero ($\beta = 0$). Equation (36) then features two solutions, i.e. a trivial one ($L = 1$, representing the horizontal intercept, $C = 0$, in Figure 2) and a non-trivial equilibrium output-capital ratio:

$$y = \bar{y} = \frac{\alpha}{(1-\epsilon_L)(1-t_K)}. \tag{38}$$

In view of (37) and (38) the corresponding capital-labour ratio is:

$$\frac{K}{L} = \left(\frac{\Omega_0}{\bar{y}}\right)^{1/\epsilon_L}. \tag{39}$$

By using (38)-(39) in (32) we obtain a simple expression for the CFE curve associated with the RA version of the model:

$$\begin{aligned} C &= \left(\frac{\epsilon_C \epsilon_L \Omega_0}{(1-\epsilon_C)(1+t_H)}\right)\left(\frac{K}{L}\right)^{1-\epsilon_L}(1-L) \\ &= \left(\frac{\epsilon_C \epsilon_L \Omega_0}{(1-\epsilon_C)(1+t_H)}\right)\left(\frac{\Omega_0}{\bar{y}}\right)^{(1-\epsilon_L)/\epsilon_L}\left[1-\left(\frac{\Omega_0}{\bar{y}}\right)^{-1/\epsilon_L}K\right] \\ &= \left(\frac{\epsilon_C \epsilon_L \Omega_0}{(1-\epsilon_C)(1+t_H)}\right)\left(\frac{\alpha}{(1-\epsilon_L)(1-t_K)\Omega_0}\right)\left[\left(\frac{\Omega_0}{\bar{y}}\right)^{1/\epsilon_L}-K\right] \\ &= \left(\frac{\alpha \epsilon_C \epsilon_L}{(1-\epsilon_C)(1-\epsilon_L)(1-t_K)(1+t_H)}\right)\left[\left(\frac{\Omega_0}{\bar{y}}\right)^{1/\epsilon_L}-K\right]. \end{aligned} \tag{40}$$

Hence, the CFE curve for the RA model is linear and downward sloping–see CFE^{RA} in Figure 2.

The dynamic forces for aggregate consumption follow from (T2.2) with $\beta = 0$ imposed, (T2.5), and (T2.7) :

$$\frac{\dot{C}}{C} = (1-\epsilon_L)(1-t_K)\left(\frac{L}{K}\right)^{\epsilon_L} - \alpha. \tag{41}$$

Ceteris paribus the capital stock, an increase (decrease) in consumption decreases (increases) equilibrium employment (see equation (32)) which increases (decreases) the capital labour ratio and decreases (increases) the interest rate. Hence, $\dot{C}/C < 0$ (> 0) for points above (below) the CFE^{RA} line. These effects have been illustrated with vertical arrows in Figure 2. It follows from the arrow configuration in Figure 2 that the unique equilibrium E_0 is saddle-point stable. In subsection 3.3 below this fact shall be demonstrated more formally.

Overlapping-generations model Let us now consider the most general version of the model for which the birth rate is positive ($\beta > 0$) and the economy is populated by disconnected overlapping generations of households. The derivation of the CFE line is now much more complicated as it can only be described *parametrically*, i.e. by varying L in the feasible interval $[0,1]$.

We first write (36) in a more convenient format:

$$\zeta_0 \equiv \frac{\beta(\alpha+\beta)(1-\epsilon_C)}{\epsilon_L(1-\epsilon_L)(1-t_L)(1-t_K)} = \left(\frac{1-L}{L}\right) y\left[y-\bar{y}\right], \tag{42}$$

where $\bar{y}$ is given in (38) and $\zeta_0 > 0$. Solving (42) for the positive (economically sensible) root yields the equilibrium output-capital ratio for the overlapping-generations (OLG) model as a function of L:

$$y = \bar{y}\left[\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{\zeta_0}{\bar{y}^2}\left(\frac{L}{1-L}\right)}\right]. \tag{43}$$

Using (43) in (37) yields an expression for the capital-labour ratio:

$$\left(\frac{K}{L}\right) = \left(\frac{\Omega_0}{\bar{y}}\right)^{1/\epsilon_L}\left[\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{\zeta_0}{\bar{y}^2}\left(\frac{L}{1-L}\right)}\right]^{-1/\epsilon_L}, \tag{44}$$

from which we derive the following limiting results:

$$\lim_{L\to 0}\left(\frac{K}{L}\right) = \left(\frac{\Omega_0}{\bar{y}}\right)^{1/\epsilon_L}, \quad \lim_{L\to 1}\left(\frac{K}{L}\right) = 0. \tag{45}$$

The labour market equilibrium condition (32) yields an expression for consumption:

$$C = \left(\frac{\epsilon_C\epsilon_L\Omega_0}{(1-\epsilon_C)(1+t_H)}\right)\left(\frac{K}{L}\right)^{1-\epsilon_L}(1-L), \tag{46}$$

from which we derive the following limiting results:

$$\begin{aligned}\lim_{L\to 0} C &= \left(\frac{\epsilon_C\epsilon_L\Omega_0}{(1-\epsilon_C)(1+t_H)}\right)\lim_{L\to 0}\left(\frac{K}{L}\right)^{1-\epsilon_L} \qquad (47)\\ &= \left(\frac{\epsilon_C\epsilon_L\Omega_0}{(1-\epsilon_C)(1+t_H)}\right)\left(\frac{\Omega_0}{\bar{y}}\right)^{(1-\epsilon_L)/\epsilon_L},\\ \lim_{L\to 1} C &= 0. \qquad (48)\end{aligned}$$

Hence, the CFE line for the OLG model has the same vertical intercept as CFE^{RA} as $L \to 0$ (compare (47) and (40)) and goes through the origin as $L \to 1$. These point have all been illustrated in Figure 3.

It is straightforward–though somewhat tedious–to prove that CFE^{OLG} is horizontal near the origin (where $L \approx 1$) and downward sloping and steeper than CFE^{RA} near the vertical intercept (where $L \approx 0$). Intuitively, CFE^{OLG} is very similar to the one for the standard Blanchard model with exogenous labour supply for values of L close to unity. Similarly, it is very similar to the RA model with endogenous labour supply for values of L close to zero.

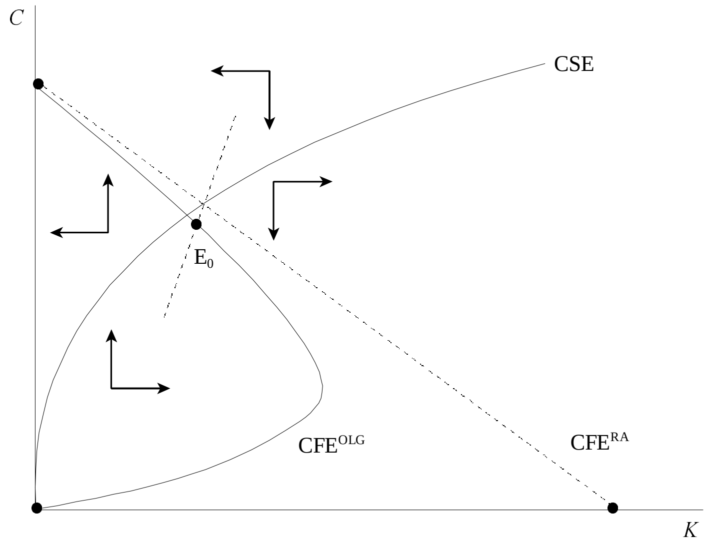

Figure 3: Phase diagram for the OLG model

Put differently, on the lower branch of the CFE^{OLG} curve the "generational-turnover effect" dominates whereas on the upper branch the "labour supply effect" dominates (see Heijdra (1999), Heijdra and Ligthart (2000), and below).

The dynamic forces at work can be studied by using (T2.2), (T2.5), and (T2.7):

$$\frac{\dot{C}}{C} = r(\underset{-}{C}, \underset{-}{K}) - \alpha - \left(\frac{\beta\epsilon_C(\alpha+\beta)}{1+t_C}\right)\left(\frac{K}{C}\right), \tag{49}$$

where $r(C, K)$ is short-hand notation for the dependence of the real interest rate on consumption and the capital stock. Simple intuitive arguments can be used to motivate the signs of the partial derivatives of the $r(C, K)$ function, which are denoted by r_C and r_K, respectively. Some simple graphs can clarify matters.

Consider Figure 4 which depicts the situation on the rental market for capital and the labour market. In the left-hand panel, the supply of capital is predetermined in the short run–say at K_0–and the demand for capital is downward sloping–due to diminishing returns to capital–and depends positively on the employment level–because the two factors are cooperative in production. The right-hand panel shows the situation on the labour market. There are diminishing returns to labour–so labour demand slopes downwards–and additional capital boosts labour demand. The labour supply curve follows from the optimal leisure-consumption choice (T2.6). It slopes upwards because (T2.6) in effect isolates the pure substitution effect of labour supply.[12]

[12] Normally, in static models of labour supply, the income and substitution effects work in opposite directions

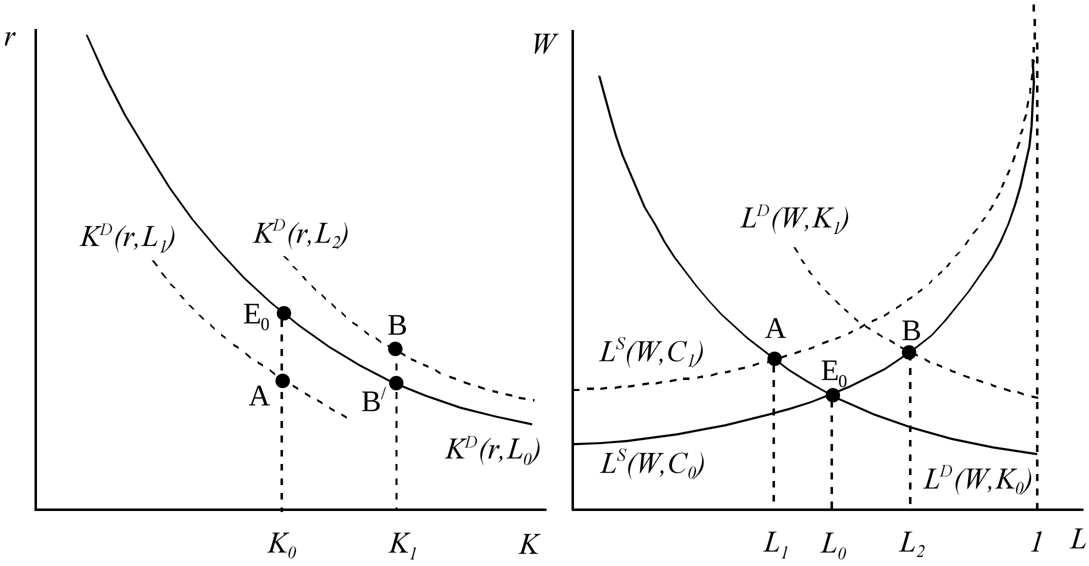

Figure 4: Factor markets

Let us now use Figure 4 to deduce the signs of r_C and r_K. Ceteris paribus the capital stock, an increase in consumption shifts labour supply to the left so that the wage rises and employment falls. The reduction in employment shifts the demand for capital to the left so that–for a given inelastic supply of capital–the real interest rate must fall to equilibrate the rental market for capital, i.e. $r_C < 0$. The thought experiment compares points E_0 and A in both panels.

An increase in capital supply–ceteris paribus consumption–has a *direct effect* which pushes the interest rate down (a movement along the initial capital demand schedule, $K^D(r, L_0)$ from E_0 to B′) and an *induced effect* operating via the labour market. The boost in K shifts the labour demand curve to the right, leading to an increase in wages and employment and thus–in the left-hand panel–an outward shift in the capital demand curve. Although this induced effect pushes the interest rate up somewhat, the direct effect dominates and $r_K < 0$.[13] The comparison is between points E_0 and B in the two panels of Figure 4.

We can now study the dynamical forces acting on aggregate consumption along the two branches of the CFE$^{\text{OLG}}$ curve. First consider a point on the lower branch of this curve for which $L \approx 1$. Holding capital constant, an increase in aggregate consumption leads to a small

thus rendering the slope of the labour supply curve ambiguous. Here we do not have this "problem" because the income effect is incorporated in C. Technically speaking, (T2.6) is a so-called *Frisch demand* for leisure. See also Judd (1987b).

[13] This follows directly from the factor price frontier (T2.8). The boost in the wage is associated with a higher capital labour ratio and thus relatively more abundant capital. This translates itself into a lower return to capital.

decrease in labour supply[14] and thus a small decrease in the interest rate. At the same time, however, the capital-consumption ratio falls so that aggregate consumption growth increases, i.e. $\dot{C}/C > 0$ for points above the lower branch of CFE$^{\text{OLG}}$:

$$\underbrace{\frac{\dot{C}}{C}}_{\uparrow} = \underbrace{r(C,K)}_{\downarrow} - \alpha - \left(\frac{\beta\epsilon_C(\alpha+\beta)}{1+t_C}\right)\underbrace{\left(\frac{K}{C}\right)}_{\downarrow\downarrow}. \qquad \text{(lower branch)}$$

Now consider a point on the upper branch of the CFE$^{\text{OLG}}$ curve for which $L \approx 0$. Ceteris paribus K, a given increase in C has a strong negative effect on labour supply and thus causes a large reduction in the interest rate which offsets the effect operating via the capital-consumption ratio, i.e. $\dot{C}/C < 0$ for points above the upper branch of CFE$^{\text{OLG}}$:

$$\underbrace{\frac{\dot{C}}{C}}_{\downarrow} = \underbrace{r(C,K)}_{\downarrow\downarrow} - \alpha - \left(\frac{\beta\epsilon_C(\alpha+\beta)}{1+t_C}\right)\underbrace{\left(\frac{K}{C}\right)}_{\downarrow}. \qquad \text{(upper branch)}$$

These dynamic effects have been illustrated with vertical arrows in Figure 4. It follows from the configuration of arrows that the unique equilibrium E_0 is saddle-point stable.

3.3 Relating the non-linear model to the linearized model

In the previous subsection we have demonstrated by graphical/analytical means that the model features a unique saddle-point stable steady-state equilibrium. The problem with the model of Table 2 is–of course–its nonlinearity. This property of the model makes it impossible to derive analytical results for the variables of interest. In order to be able to conduct comparative-dynamic exercises with the model, it must somehow be forced into the (linear) framework studied in section 2, i.e. it must be loglinearized.[15] The loglinearized model can then be solved analytically although the solutions are, of course, only valid for small (indeed, "infinitesimal") shocks.[16]

Whilst loglinearization is a "necessary evil", it is often very useful to assume that the economy is initially in the steady state. Though this assumption is not necessary, it does lead

[14] Note that we can use (T2.6) to derive

$$\frac{dL}{L} = \left(\frac{1-L}{L}\right)\left[\frac{dW}{W} - \frac{dC}{C}\right].$$

Hence, for $L \approx 1$ ($L \approx 0$) the labour supply curve in Figure 4 is very steep (very flat) and a given change in consumption shifts the curve by a little (a lot). This explains why the parameter $\omega_{LL} \equiv (1-L)/L$ plays a vital role in the analysis of the loglinearized model in the next section.

[15] Note that the same need for (log) linearization also exists in nonlinear static models; it is not a complication that is specific to dynamic models.

[16] Simulation studies reveal that the errors due to linearization are often not very large even when the shocks are substantial (Meijdam and Verhoeven, 1994). Figure 2 shows why this is likely to be the case in the RA version of the model. In that version the CFE line is linear and the CSE curve is log-linear. The OLG version of the model is "more nonlinear" and is conjectured to feature greater linearization errors.

to more clearcut results because all transitional dynamics can be attributed to the particular shock in that case. In contrast, if the economy starts outside the steady state, transitional dynamics will be due to both the shock and to the (non-equilibrium) initial conditions.

By loglinearizing the model around the initial steady state, the expressions in Table 3 are obtained. We use the following notational conventions. A deviation of a variable relative to the initial steady state is denoted by a tilde ('~'), i.e. $\tilde{x}(t) \equiv dx(t)/x$ for $x \in \{K, C, Y, L, W, r\}$. For the fiscal variables we adopt a slightly different notation by using $\tilde{t}_i(t) \equiv dt_i(t)/(1-t_i)$, for $i \in \{K, L\}$, $\tilde{t}_C(t) \equiv dt_C(t)/(1+t_C)$, and $\tilde{T}(t) \equiv dT(t)/Y$.[17] For the time rate of change of a variable we use a tilde and a dot: i.e. $\dot{\tilde{x}}(t) \equiv d\dot{x}(t)/x = \dot{x}(t)/x$ for $x \in \{K, C\}$, since $\dot{x}(0) = 0$ in the initial steady state. Finally, for the consumption tax we have $\dot{\tilde{t}}_C(t) \equiv d\dot{t}_C(t)/(1+t_C)$.

The dynamic part of the model consists of equations (T3.1)-(T3.2) and the static part of (T3.3)-(T3.8). By using labour demand (T3.4), labour supply (T3.6) and the aggregate production function (T3.7), we obtain a useful 'quasi-reduced form' expression for aggregate output:

$$\tilde{Y}(t) = \phi(1-\epsilon_L)\tilde{K}(t) - (\phi - 1)\left[\tilde{C}(t) + \tilde{t}_L(t) + \tilde{t}_C(t)\right], \tag{50}$$

where ϕ is a crucial parameter representing the labour supply effect:

$$1 \leq \phi \equiv \frac{1+\omega_{LL}}{1+\omega_{LL}(1-\epsilon_L)} \leq \frac{1}{1-\epsilon_L}. \tag{51}$$

By using (50) and (T3.5) in (T3.1)-(T3.2), we obtain the dynamical system for $\tilde{K}(t)$ and $\tilde{C}(t)$ in the required format:

$$\begin{bmatrix} \dot{\tilde{K}}(t) \\ \dot{\tilde{C}}(t) \end{bmatrix} = \Delta \begin{bmatrix} \tilde{K}(t) \\ \tilde{C}(t) \end{bmatrix} + \begin{bmatrix} g_K(t) \\ g_C(t) \end{bmatrix}, \tag{52}$$

where the Jacobian matrix, Δ, and the shock terms, $g_K(t)$ and $g_C(t)$, are given by:

$$\Delta \equiv \begin{bmatrix} \frac{r\phi}{1-t_K} & -\frac{r\phi}{(1-t_K)(1-\epsilon_L)} \\ -(r-\alpha) - r\left[1-\phi(1-\epsilon_L)\right] & (r-\alpha) - r(\phi-1) \end{bmatrix}, \tag{53}$$

$$\begin{bmatrix} g_K(t) \\ g_C(t) \end{bmatrix} \equiv \begin{bmatrix} \frac{-r(\phi-1)}{(1-t_K)(1-\epsilon_L)}\left[\tilde{t}_L(t) + \tilde{t}_C(t)\right], \\ -r(\phi-1)\left[\tilde{t}_L(t) + \tilde{t}_C(t)\right] + (r-\alpha)\tilde{t}_C(t) - r\tilde{t}_K(t) - \dot{\tilde{t}}_C(t) \end{bmatrix}. \tag{54}$$

After some manipulation we obtain the following expression for the determinant of the Jacobian matrix:

$$|\Delta| \equiv \lambda_1\lambda_2 = -\frac{r\phi\epsilon_L\left[2r-\alpha\right]}{(1-t_K)(1-\epsilon_L)} < 0, \tag{55}$$

where λ_1 and λ_2 are the characteristic roots. This confirms that the steady-state equilibrium (E_0 in Figures 2 and 3) is saddle-point stable. In the RA version of the model the steady state interest rate equals the rate of time preference, $r = \alpha$, whilst in the OLG version generational turnover ensures that $r > \alpha$. In either case $|\Delta| < 0$ so that there is one stable ($\lambda_1 < 0$) and one unstable characteristic root ($\lambda_2 > 0$).

[17] The advantage of this approach is that the expressions are well defined even if the variable is zero initially.

Table 3: The log-linearised model*

$$\dot{\tilde{K}}(t) = \left(\frac{r}{(1-t_K)(1-\epsilon_L)}\right)\left[\tilde{Y}(t) - \tilde{C}(t)\right] \tag{T3.1}$$

$$\dot{\tilde{C}}(t) = (r-\alpha)\left[\tilde{C}(t) + \tilde{t}_C(t) - \tilde{K}(t)\right] + r\tilde{r}(t) - \dot{\tilde{t}}_C(t) \tag{T3.2}$$

$$\begin{aligned}\tilde{T}(t) = (1+t_C)\left[\tilde{t}_C(t) + \left(\frac{t_C}{1+t_C}\right)\tilde{C}(t)\right] + \epsilon_L(1-t_L)\left[\tilde{t}_L(t) + \left(\frac{t_L}{1-t_L}\right)\tilde{Y}(t)\right] \\ + (1-\epsilon_L)(1-t_K)\left[\tilde{t}_K(t) + \left(\frac{t_K}{1-t_K}\right)\tilde{Y}(t)\right]\end{aligned} \tag{T3.3}$$

$$\tilde{Y}(t) - \tilde{L}(t) = \tilde{W}(t) \tag{T3.4}$$

$$\tilde{Y}(t) - \tilde{K}(t) = \tilde{t}_K(t) + \tilde{r}(t) \tag{T3.5}$$

$$\tilde{L}(t) = \omega_{LL}\left[\tilde{W}(t) - \tilde{t}_L(t) - \tilde{C}(t) - \tilde{t}_C(t)\right] \tag{T3.6}$$

$$\tilde{Y}(t) = \epsilon_L\tilde{L}(t) + (1-\epsilon_L)\tilde{K}(t) \tag{T3.7}$$

$$0 = \epsilon_L\tilde{W}(t) + (1-\epsilon_L)[\tilde{r}(t) + \tilde{t}_K(t)] \tag{T3.8}$$

Initial shares and parameters:

ϵ_L	$\equiv WL/Y$	Share of before-tax wage income in real net output.
ω_{LL}	$\equiv (1-L)/L$	Ratio of leisure to labour.
α		Pure rate of time preference
β		Probability of death

$*$ The following notational conventions are adopted. A deviation of a variable relative to the initial steady state is denoted by a tilde ('$\sim$'), e.g. $\tilde{K}(t) \equiv dK(t)/K$. For the fiscal variables tildes are defined as follows: $\tilde{t}_i(t) \equiv dt_i(t)/(1-t_i)$, for $i = K, L$, $\tilde{t}_C(t) \equiv dt_C(t)/(1+t_C)$ and $\tilde{T}(t) \equiv dT(t)/Y$. For the time rate of change of a variable we use a tilde and a dot: $\dot{\tilde{K}}(t) \equiv d\dot{K}(t)/K = \dot{K}(t)/K$, since $\dot{K}(0) = 0$ in the initial steady state, except $\dot{\tilde{t}}_C(t) \equiv d\dot{t}_C(t)/(1+t_C)$.

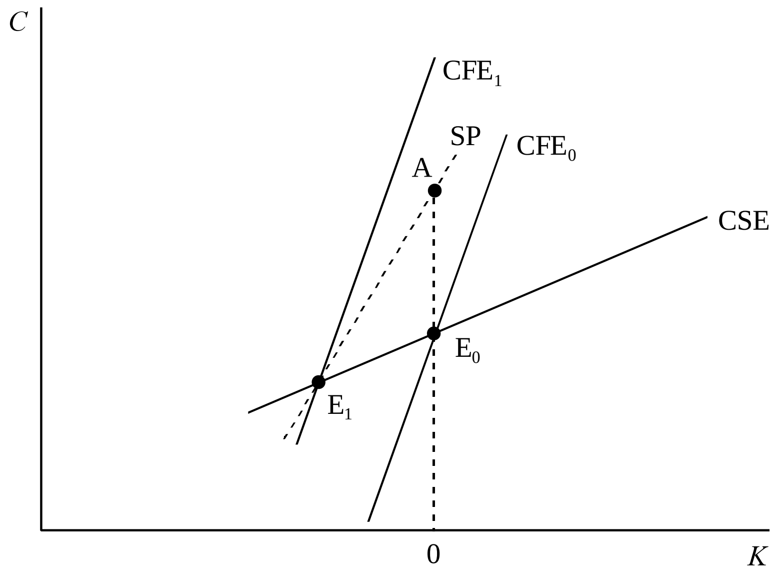

Figure 5: Capital taxation in the OLG model

3.4 Example of a tax shock

We are now in a position to study the comparative dynamic effects of tax shocks on the macroeconomic variables. We assume that the shock is unanticipated and permanent and restrict attention to an increase in the capital tax. The time at which the shock occurs is normalized to zero so the scenario studied here amounts to setting $\tilde{t}_L(t) = \tilde{t}_C(t) = \dot{\tilde{t}}_C(t) = 0$ and $\tilde{t}_K(t) = \tilde{t}_K > 0$ (all for $t \geq 0$) in (54). By using (27) and (54) and making the obvious substitutions we obtain the impact effect on consumption:

$$\begin{aligned}\tilde{C}(0) &= -\mathcal{L}\{g_C, \lambda_2\} - \left(\frac{\lambda_2 - \delta_{22}}{\delta_{12}}\right)\left[\tilde{K}(0) + \mathcal{L}\{g_K, \lambda_2\}\right] \\ &= \frac{r\tilde{t}_K}{\lambda_2} > 0, \end{aligned} \tag{56}$$

where we have used the fact that capital is predetermined ($\tilde{K}(0) = 0$) and note that the capital tax does not affect the position of the CSE curve (recall that employment is constant and independent of t_K along the CSE curve–see equation (34) above).

The effect of the shock can be illustrated with Figure 5 which is the (locally) loglinearized version of Figure 3 and assumes that the labour supply effect is relatively weak in the initial steady state so that the initial equilibrium occurs on the upward sloping section of CFE^{OLG}. The tax shock shifts the CFE curve to the left. The increase in the capital tax leads at impact to a fall in the demand for capital and a reduction in the interest rate. This makes current consumption more attractive compared to future consumption so that consumption rises on impact. Given that capital is predetermined at impact the effects on all other variables are directly related to the exogenous shock ($\tilde{t}_K$) and the induced impact effect on consumption ($\tilde{C}(0)$):

$$-\left(\frac{\epsilon_L}{1-\epsilon_L}\right)\tilde{W}(0)=\epsilon_L\tilde{L}(0)=\tilde{Y}(0)=-\left(\frac{r(\phi-1)}{\lambda_2}\right)\tilde{t}_K<0, \tag{57}$$

$$\tilde{r}(0)=-\left(\frac{\lambda_2+r(\phi-1)}{\lambda_2}\right)\tilde{t}_K<0. \tag{58}$$

In point A, the simultaneous increase in consumption and decrease in output crowds out net investment so that the capital stock falls over time ($\dot{\tilde{K}}(0)=0$). The economy gradually moves along the saddle path from point A to E_1 and both capital and consumption fall:

$$\begin{bmatrix}\tilde{K}(t)\\ \tilde{C}(t)\end{bmatrix}=\begin{bmatrix}0\\ \tilde{C}(0)\end{bmatrix}e^{\lambda_1 t}+\begin{bmatrix}K(\infty)\\ \tilde{C}(\infty)\end{bmatrix}\left(1-e^{\lambda_1 t}\right), \tag{59}$$

where $\tilde{K}(\infty)$ and $\tilde{C}(\infty)$ are given by:

$$\tilde{K}(\infty)=(1-\epsilon_L)^{-1}\tilde{C}(\infty)=-\left(\frac{r}{\epsilon_L\left[2r-\alpha\right]}\right)\tilde{t}_K<0. \tag{60}$$

In the long run consumption equals output ($\tilde{C}(\infty)=\tilde{Y}(\infty)$) and employment is constant ($\tilde{L}(\infty)=0$). From the labour demand equation we deduce that $\tilde{W}(\infty)=\tilde{Y}(\infty)$ and by using (60) and (T3.5) we obtain the long-run effect on the interest rate:

$$\tilde{r}(\infty)=-\left(\frac{r-\alpha}{2r-\alpha}\right)\tilde{t}_K<0. \tag{61}$$

With overlapping generations ($\beta>0$ so that $r>\alpha$) the after-tax return on capital remains below its initial steady-state equilibrium value so that capital bears part of the incidence of the tax. In stark contrast, in the absence of overlapping generations ($\beta=0$ so that $r=\alpha$) or in a small open economy capital escapes the entire long-run burden of capital taxation. In the former case the long-run after-tax interest rate is fixed by the exogenous rate of time preference (Judd (1985); Chamley (1985)) whereas in the latter case the world capital market fixes the after-tax rate of return since physical capital is perfectly mobile internationally.

3.5 Discussion

The example of this section is meant to demonstrate the ease with which comparative dynamic effects can be deduced by using relatively simple Laplace transform techniques. We focused our discussion on the capital tax but the analyses of Heijdra (1999), and Heijdra and Ligthart (2000) show that the methods are equally valuable to study other–more complicated–fiscal policy experiments.

In the next section we show yet another advantage of the Laplace transform techniques. As was first demonstrated by Judd (1982), these techniques are ideally suited to evaluate welfare effects in dynamic macro models exhibiting perfect foresight. The reason is that they make it quite feasible to take account of any transitional dynamics that may exist in such models. This is in a sense not that surprising in view of the fact that the popular additively separable utility function of an infinitely-lived agent is itself a Laplace transform.

4 Welfare evaluation

In most macro models the utility level enjoyed by an agent depends on the discounted integral of present and future flows of felicity. In a representative-agent (RA) model with exogenous labour supply, for example, life-time utility as of time $t = 0$ (the time of the shock) takes the form:

$$U(0) = \int_0^\infty \log C(\tau) e^{-\alpha\tau} d\tau, \tag{62}$$

where $U(0)$ is life-time utility, α is the rate of time preference, and $C(\tau)$ is consumption.[18] The first thing to note is–of course–that $U(0) = \mathcal{L}\{\log C, \alpha\}$, i.e. lifetime utility is the Laplace transform of felicity ($\log C(\tau)$) evaluated at $s = \alpha$.

The aim of this section is to see whether we can exploit the natural link that seems to exist between lifetime utility on the one hand and the Laplace transform of one or more macro variables on the other hand. We first establish the link for the RA model, where it turns out to be very direct indeed. We subsequently turn to the slightly more challenging case of the overlapping-generations (OLG) model.

4.1 Welfare effects in the RA model

What happens to $U(0)$ if a shock occurs (at time $t = 0$) which sets in motion transitional dynamics in one or more of the macro variables? One rather unsatisfactory solution would be to just ignore the transitional dynamics by merely looking at what happens in the steady state. Implicitly we would be assuming that C jumps immediately to its new steady-state value ($dC(\tau) = dC(\infty)$ for all $\tau \geq 0$) so that welfare would be deemed to have changed by:

$$\begin{aligned} dU(0) &= \int_0^\infty d\log C(\tau) e^{-\alpha\tau} d\tau = \int_0^\infty \frac{dC(\tau)}{C(\tau)} e^{-\alpha\tau} d\tau \\ &\approx \frac{dC(\infty)}{C} \int_0^\infty e^{-\alpha\tau} d\tau = \frac{\tilde{C}(\infty)}{\alpha}, \end{aligned} \tag{63}$$

where the tilde ("~") notation is explained at the bottom of Table 3. Not surprisingly, this procedure amounts to attributing the perpetuity felicity value of the shock to lifetime welfare. But from our earlier discussion in section 3 we know that transition to the ultimate steady state takes time (see (59)) so that the approach leading to (63) cannot be correct .

[18] To keep things as simple as possible we simplify the Heijdra-Ligthart (2000) model by assuming exogenous labour supply ($\epsilon_C = 1$ so that $L(\tau) = 1$). The resulting model extends the one by Bovenberg and Heijdra (1998) by also including taxes on consumption and labour.

Fortunately, now that we know the Laplace transform, a more satisfactory solution is easy to find. By totally differentiating (62) we obtain:

$$\begin{aligned} dU(0) &= \int_0^\infty d\log C(\tau) e^{-\alpha\tau} d\tau = \int_0^\infty \frac{dC(\tau)}{C(\tau)} e^{-\alpha\tau} d\tau \\ &\approx \int_0^\infty \frac{dC(\tau)}{C} e^{-\alpha\tau} d\tau = \int_0^\infty \tilde{C}(\tau) e^{-\alpha\tau} d\tau \\ &\equiv \mathcal{L}\{\tilde{C}, \alpha\}. \end{aligned} \tag{64}$$

Hence, the change in lifetime utility (due to the shock) is simply the Laplace transform of the induced changes in consumption. For the capital tax the transition path for $\tilde{C}(t)$ is given in (59) so that $dU(0)$ can be written as:

$$\alpha dU(0) = \left(\frac{\alpha}{\alpha+\mu}\right)\tilde{C}(0) + \left(\frac{\mu}{\alpha+\mu}\right)\tilde{C}(\infty), \tag{65}$$

where $\mu \equiv -\lambda_1 > 0$ is the absolute value of the stable characteristic root and $\tilde{C}(0)$ and $\tilde{C}(\infty)$ are given in, respectively, equations (56) and (60). Equation (65) says that the flow value of lifetime utility (the right-hand side) equals the weighted average of the initial and ultimate effect on consumption of the shock.[19] If transition would be instantaneous ($\mu \to \infty$) or if there were no transition ($\tilde{C}(0) = \tilde{C}(\infty)$) then the results in (63) and (65) would coincide but for any other case (65) is the relevant expression for evaluating welfare along the transition path.

By substituting (56) and (60) in (65) and simplifying we obtain the following alternative expression for $dU(0)$:[20]

$$\begin{aligned} \alpha dU(0) &= \left(\frac{\alpha}{\alpha+\mu}\right)\frac{\alpha\tilde{t}_K}{\lambda_2} - \left(\frac{\mu}{\alpha+\mu}\right)\left(\frac{1-\epsilon_L}{\epsilon_L}\right)\frac{\alpha\tilde{t}_K}{\alpha} \\ &= \left(\frac{\alpha}{\alpha+\mu}\right)\left[\frac{\alpha}{\lambda_2} - \frac{\alpha}{(1-t_K)\lambda_2}\right]\tilde{t}_K \\ &= -\left(\frac{\alpha}{\alpha+\mu}\right)\left(\frac{\alpha}{\lambda_2}\right)\left(\frac{t_K}{1-t_K}\right)\tilde{t}_K \Leftrightarrow \\ dU(0) &= -\left(\frac{\alpha t_K}{(\alpha+\mu)[\alpha+\mu(1-t_K)]}\right)\tilde{t}_K. \end{aligned} \tag{66}$$

Equation (66) shows the well-known result that, in the absence of other distortions, the marginal efficiency cost of capital taxation is positive if $t_K > 0$ but second-order small if $t_K = 0$ initially [What happens if capital is subsidized initially ($t_K < 0$)?].

[19] Note that the form of the shock itself determines $\mathcal{L}\{\tilde{C}, \alpha\}$ and thus the form of (65). For an anticipated shock in the capital tax, for example, forward looking agents would augment their behaviour, the transition path for $C(t)$ would contain additional terms, and (65) would be more complicated.

[20] In the first step we note that for the RA model $r = \alpha$, in the second and fourth steps we use, respectively, the determinant and trace results given in (17):

$$\mu\lambda_2 = \frac{\alpha^2\epsilon_L}{(1-t_K)(1-\epsilon_L)}, \quad \lambda_2 = \mu + \frac{\alpha}{1-t_K}.$$

4.2 Welfare in the OLG model

Matters are a little more complicated in the OLG version of the model because we must recognize the fact that current and future generations experience different welfare effects as a result of the shock. Existing generations at the time of the shock are different from each other because they differ in age and thus in their holdings of financial assets and consumption pattern. Future generations are different from each other because they have different birth dates.

Remaining lifetime utility for a representative agent born at time $v \leq t$ is denoted by $U(v, t)$:

$$U(v,t) \equiv \int_t^{\infty} \log C(v,\tau) e^{(\alpha+\beta)(t-\tau)} d\tau, \tag{67}$$

where $C(v, \tau)$ is consumption at time τ by an agent of generation v and β is the constant instantaneous probability of death of the agent (see Bovenberg and Heijdra (1998b) for details). Since the time of the shock is normalized to $t = 0$, the objective of this section is to determine $dU(v, 0)$ for *existing generations* (who have a non-positive generations index, $v \leq 0$) and $dU(t, t)$ for *future generations* (who will be born at time $t > 0$ and whose welfare is evaluated at birth).

Before studying these two cases we can simplify (67) somewhat by noting that all agents will choose consumption profiles satisfying the Euler equation, $\dot{C}(v, \tau)/C(v, \tau) = r(\tau) - \alpha$, or:

$$C(v,\tau) = C(v,t) e^{\int_t^{\tau} (r(\sigma)-\alpha) d\sigma}, \quad \tau \geq t. \tag{68}$$

By using (68) in (67) we obtain

$$\begin{aligned} U(v,t) &= \int_t^{\infty} \left[\log C(v,t) + \int_t^{\tau} (r(\sigma)-\alpha) d\sigma \right] e^{(\alpha+\beta)(t-\tau)} d\tau, \\ &= \frac{\log C(v,t)}{\alpha+\beta} + \Psi(t), \end{aligned} \tag{69}$$

where $\Psi(t)$ is defined as:[21]

$$\begin{aligned} \Psi(t) &\equiv \int_t^{\infty} \left[\int_t^{\tau} [r(\sigma)-\alpha] d\sigma \right] e^{(\alpha+\beta)(t-\tau)} d\tau \\ &= \int_t^{\infty} [r(\sigma)-\alpha] \left[\int_{\sigma}^{\infty} e^{(\alpha+\beta)(t-\tau)} d\tau \right] d\sigma \\ &= \int_t^{\infty} \left(\frac{r(\sigma)-\alpha}{\alpha+\beta} \right) e^{(\alpha+\beta)(t-\sigma)} d\sigma. \end{aligned} \tag{70}$$

[21] In going from the first to the second line we change the order of integration–see Spiegel (1974, pp. 180-181).

By totally differentiating (69) and (70) we obtain the fundamental expressions for intergenerational welfare analysis:

$$dU(v,t) = \frac{\tilde{C}(v,t)}{\alpha+\beta} + d\Psi(t), \tag{71}$$

$$d\Psi(t) = \left(\frac{r}{\alpha+\beta}\right)\int_t^{\infty} \tilde{r}(\mu)e^{-(\sigma-t)(\alpha+\beta)}d\sigma. \tag{72}$$

4.2.1 Existing generations ($v \leq 0$)

For existing generations (71) and (72) can be combined to:

$$(\alpha+\beta)dU(v,0) = \tilde{C}(v,0) + r\mathcal{L}\{\tilde{r},\alpha+\beta\}, \tag{73}$$

where the path of the interest rate (and thus $\mathcal{L}\{\tilde{r},\alpha+\beta\}$) has been determined in section 3 above for the capital tax shock. It remains to determine the generation-specific consumption effect $\tilde{C}(v,0)$. We first note that, for the utility function (67), consumption is proportional to total wealth for all agents, i.e. $C(v,0) = (\alpha+\beta)[H(0)+A(v,0)]$, where $H(0)$ and $A(v,0)$ are, respectively, human and financial wealth.[22] From this expression we note that:

$$\tilde{C}(v,0) \equiv \frac{dC(v,0)}{C(v,0^-)} = \omega_H(v)\left(\frac{dH(0)}{H}\right) + [1-\omega_H(v)]\left(\frac{dA(v,0)}{A(v,0^-)}\right), \tag{74}$$

where $\omega_H(v) \equiv H/[H+A(v,0^-)]$ and $C(v,0^-)$, H, and $A(v,0^-)$ are initial steady-state values for, respectively, consumption, human wealth and financial wealth of generation v.

In the second step we note that $C(0) = (\alpha+\beta)[H(0)+A(0)]$ which yields:

$$\tilde{C}(0) \equiv \frac{dC(0)}{C} = \omega_H\left(\frac{dH(0)}{H}\right) + (1-\omega_H)\left(\frac{dA(0)}{A}\right), \tag{75}$$

where $\omega_H \equiv H/[H+A]$. But capital is the only asset in this economy and both the size of the capital stock and its ownership are predetermined in the impact period ($t=0$), i.e. $dK(0) = dA(0) = 0$ and $dK(v,0) = dA(v,0) = 0$, so that (74) and (75) can be combined to obtain an important intermediate result linking $\tilde{C}(v,0)$ and $\tilde{C}(0)$:

$$\tilde{C}(v,0) = \left(\frac{\omega_H(v)}{\omega_H}\right)\tilde{C}(0). \tag{76}$$

The third and final step is to figure out an expression for $\omega_H(v)$ in terms of parameters of the model. Here our assumption–that the economy is initially in the steady state (and has been there for a long time)–comes to the rescue. Intuitively, in order to work out the distribution of financial assets over the existing population, we work backwards in time to all existing agents' birthdates and note that, provided the system has been in a steady state all along,

[22] Human wealth is age-independent because all agents are equally productive workers (wages are age-independent) and the government divides the revenue of the taxes equally over all agents (transfers are age-independent).

$C(v,0)$ satisfies: $C(v,0) = C(v,v)e^{-(r-\alpha)v}$. Recall that all generations start out without any financial assets ($A(v,v) = 0$ so that $C(v,v) = (\alpha+\beta)H$). But we now have two expressions for $C(v,0)$:

$$\underbrace{(\alpha+\beta)He^{-(r-\alpha)v}}_{C(v,0)} = \underbrace{(\alpha+\beta)[H + A(v,0^{-}]}_{C(v,0)} \Rightarrow$$

$$\omega_H(v) \equiv \frac{H}{H + A(v,0^{-})} = e^{(r-\alpha)v}. \tag{77}$$

The final expression makes intuitive sense. Very old agents have been around for a long time already and thus possess a lot of financial assets. For them human wealth is a trivial component of total wealth, i.e. as $v \to -\infty$ we have that $\omega_H(v) \to 0$ (since $r > \alpha$). In contrast, newborn agents at the time of the shock have no financial assets and thus rely solely on human wealth, i.e. as $v \to 0$ it follows that $\omega_H(v) \to 1$.

By combining (73), (76), and (77) we obtain the following expression for $dU(v,0)$:

$$(\alpha+\beta)dU(v,0) = \left(\frac{e^{(r-\alpha)v}}{\omega_H}\right)\tilde{C}(0) + r\mathcal{L}\{\tilde{r}, \alpha+\beta\}. \tag{78}$$

An equivalent–potentially more insightful–expression can be derived from (78):

$$dU(v,0) = dU(0,0)e^{(r-\alpha)v} + dU(-\infty,0)\left(1 - e^{(r-\alpha)v}\right), \tag{79}$$

i.e. the effect on a generation born v periods ago is the age-weighted average of the effect on a newborn generation and an extremely old generation.

4.2.2 Future generations ($t > 0$)

For future generations it is most convenient to evaluate the Laplace transform of the utility change. In view of (71) we get:

$$\mathcal{L}\{dU(t,t), s\} = \frac{\mathcal{L}\{\tilde{C}(t,t), s\}}{\alpha+\beta} + \mathcal{L}\{d\Psi(t), s\}. \tag{80}$$

All generations are born without any financial wealth so consumption of future generations at birth is given by $C(t,t) = (\alpha+\beta)H(t)$, or:

$$\tilde{C}(t,t) = \tilde{H}(t) \tag{81}$$

In the aggregate we have that $C(t) = (\alpha+\beta)[H(t) + A(t)]$ and $A(t) = K(t)$ which can be combined and linearized:

$$\tilde{C}(t) = \omega_H\tilde{H}(t) + (1-\omega_H)\tilde{K}(t). \tag{82}$$

By combining (81) and (82) we can eliminate $\tilde{H}(t)$ and express $\tilde{C}(t,t)$ in terms of $\tilde{C}(t)$ and $\tilde{K}(t)$ for which we have the solutions:

$$\tilde{C}(t,t) = \frac{\tilde{C}(t) - (1-\omega_H)\tilde{K}(t)}{\omega_H}. \tag{83}$$

It remains to find a convenient expression for $\mathcal{L}\{d\Psi(t), s\}$. We first note another useful property of the Laplace transform, namely the ease with which integrals can be evaluated.

Property 8 *Let $f(t)$ be defined as follows:*

$$f(t) \equiv \int_t^\infty g(\sigma) e^{a(t-\sigma)} d\sigma,$$

with $a > 0$, and assume that the integral exists. Then $\mathcal{L}\{f, s\}$ is given by:

$$\mathcal{L}\{f, s\} = \left[\frac{\mathcal{L}\{g, a\} - \mathcal{L}\{g, s\}}{s-a}\right]. \tag{P8}$$

PROOF: We note that $f(t)$ satisfies the differential equation $\dot{f}(t) = -g(t) + af(t)$ and that $f(0) = \mathcal{L}\{g, a\}$. Taking the Laplace transform of $\dot{f}(t)$ yields $(s-a)\mathcal{L}\{f, s\} = f(0) - \mathcal{L}\{g, s\}$. By substituting $f(0) = \mathcal{L}\{g, a\}$ and dividing by $(s-a)$ the required result is obtained. □

By applying property (P8) to (72) and setting $a = \alpha + \beta$ and $g(\sigma) = \tilde{r}(\sigma)$ we derive the expression for $\mathcal{L}\{d\Psi(t), s\}$:

$$\mathcal{L}\{d\Psi(t), s\} = \left(\frac{r}{\alpha+\beta}\right)\left[\frac{\mathcal{L}\{\tilde{r}, \alpha+\beta\} - \mathcal{L}\{\tilde{r}, s\}}{s-(\alpha+\beta)}\right]. \tag{84}$$

By using (83) and (84) in (80) we obtain an expression for the Laplace transform of future generations' utility change:

$$(\alpha+\beta)\mathcal{L}\{dU(t,t), s\} = \left(\frac{\mathcal{L}\{\tilde{C}, s\} - (1-\omega_H)\mathcal{L}\{\tilde{K}, s\}}{\omega_H}\right) + r\left(\frac{\mathcal{L}\{\tilde{r}, \alpha+\beta\} - \mathcal{L}\{\tilde{r}, s\}}{s-(\alpha+\beta)}\right). \tag{85}$$

The crucial thing to note about (85) is that $\mathcal{L}\{dU(t,t), s\}$ is now expressed in terms of (transforms of) the known solution paths for the macroeconomic variables $\tilde{C}(t)$, $\tilde{K}(t)$, and $\tilde{r}(t)$. By plugging these paths into (85) and inverting the resulting Laplace transform, we obtain the solution path for $dU(t,t)$. For the unanticipated and permanent capital tax shock the solution path is given in (59) and $dU(t,t)$ can be expressed in the following convenient format (comparable to (79)):

$$dU(t,t) = dU(0,0)e^{-\mu t} + dU(\infty,\infty)\left(1 - e^{-\mu t}\right), \tag{86}$$

where $dU(0,0)$ is obtained from (78) (by setting $v = 0$) and $dU(\infty,\infty)$–the effect of the shock on steady-state generations– is obtained by applying the final-value theorem (P8) to (85):

$$\begin{aligned}(\alpha+\beta)dU(\infty,\infty) &= (\alpha+\beta)\lim_{s\to 0} s\mathcal{L}\{dU(t,t), s\} \\ &= \left[\frac{\tilde{C}(\infty) - (1-\omega_H)\tilde{K}(\infty)}{\omega_H}\right] + \left(\frac{r}{\alpha+\beta}\right)\tilde{r}(\infty).\end{aligned} \tag{87}$$

Equation (86) shows that (for this particular shock) the effect of the generation born in period t is the weighted average of the effects on newborns at the time of the shock ($dU(0,0)$) and ultimate steady-state generations ($dU(\infty,\infty)$). The speed of convergence in the macroeconomic system ($\mu \equiv -\lambda_1 > 0$) is the crucial parameter determining the time-varying weight in (86).

Before going on it may be instructive to show how (86) is derived from (59) and (85). We first note from (T3.5) and (T3.7) that for the exogenous labour supply case the interest rate effect can be written as: $\tilde{r}(t) = -(\epsilon_L \tilde{K}(t) + \tilde{t}_K)$. This means that (84) can be written as:

$$\mathcal{L}\{d\Psi(t), s\} = -\left(\frac{r}{\alpha+\beta}\right)\left[\epsilon_L\left(\frac{\mathcal{L}\{\tilde{K},\alpha+\beta\} - \mathcal{L}\{\tilde{K},s\}}{s-(\alpha+\beta)}\right) + \frac{\tilde{t}_K}{s(\alpha+\beta)}\right]. \tag{88}$$

By using (59) we derive in a few steps:

$$\begin{aligned}
\mathcal{L}\{\tilde{K},\alpha+\beta\} - \mathcal{L}\{\tilde{K},s\} &= \left[\left(\frac{1}{\alpha+\beta} - \frac{1}{\alpha+\beta+\mu}\right) - \left(\frac{1}{s} - \frac{1}{s+\mu}\right)\right]\tilde{K}(\infty) \\
&= \left[\left(\frac{1}{\alpha+\beta} - \frac{1}{s}\right) - \left(\frac{1}{\alpha+\beta+\mu} - \frac{1}{s+\mu}\right)\right]\tilde{K}(\infty) \\
&= \left[\left(\frac{s-(\alpha+\beta)}{(\alpha+\beta)s}\right) - \left(\frac{s-(\alpha+\beta)}{(\alpha+\beta+\mu)(s+\mu)}\right)\right]\tilde{K}(\infty),
\end{aligned}$$

from which we derive:

$$\begin{aligned}
\frac{\mathcal{L}\{\tilde{K},\alpha+\beta\} - \mathcal{L}\{\tilde{K},s\}}{s-(\alpha+\beta)} &= \left[\frac{1}{s}\frac{1}{\alpha+\beta} - \frac{1}{s+\mu}\frac{1}{\alpha+\beta+\mu}\right]\tilde{K}(\infty) \\
&= \left[\frac{1}{s}\left(\frac{1}{\alpha+\beta} - \frac{1}{\alpha+\beta+\mu}\right) + \frac{1}{\alpha+\beta+\mu}\left(\frac{1}{s} - \frac{1}{s+\mu}\right)\right]\tilde{K}(\infty) \\
&= \frac{1}{s}\mathcal{L}\{\tilde{K},\alpha+\beta\} + \left(\frac{1}{s} - \frac{1}{s+\mu}\right)\left(\frac{\tilde{K}(\infty)}{\alpha+\beta+\mu}\right).
\end{aligned} \tag{89}$$

Simlarly, we can derive from (59) that:

$$\mathcal{L}\{\tilde{C},s\} - (1-\omega_H)\mathcal{L}\{\tilde{K},s\} = \frac{\tilde{C}(0)}{s+\mu} + \left(\frac{1}{s} - \frac{1}{s+\mu}\right)\left[\tilde{C}(\infty) - (1-\omega_H)\tilde{K}(\infty)\right]. \tag{90}$$

By substituting (88)-(90) in (85) and gathering terms we obtain:

$$\mathcal{L}\{dU(t,t), s\} = dU(0,0)\left(\frac{1}{s+\mu}\right) + dU(\infty,\infty)\left(\frac{1}{s} - \frac{1}{s+\mu}\right). \tag{91}$$

Since we recognize the Laplace transforms appearing on the right-hand side, inversion of (91) is straightforward and results in (86).

4.2.3 Discussion

We have shown how the welfare effects of a policy shock can be studied both for existing and future generations. For the particular shock studied, the welfare effects on current and future generations can be written as in (79) and (86), respectively. These expressions imply that the intergenerational welfare profile is fully characterized by the welfare effects on, respectively, "generation Methusalem" ($dU(-\infty, 0)$), newborns at the time of the shock ($dU(0, 0)$), and steady-state generations ($dU(\infty, \infty)$. Bovenberg and Heijdra (1999b, p. 15) prove that, for the capital tax shock, $dU(-\infty, 0) < 0$, $dU(0, 0) > 0$ (for low initial t_K), $dU(\infty, \infty) > 0$ (also for low initial t_K), and $dU(0, 0) > dU(\infty, \infty)$ (regardless of the initial tax rate). So, whereas in the RA model welfare unambiguously declines as a result of the tax shock (see equation (66)), the situation is considerably more complex in the OLG model. Some generations gain while others lose out. Tax incidence questions are thus much less straightforward–and arguably more relevant and interesting–in an OLG setting because the heterogeneity of agents causes efficiency and distributional effects to interact.[23]

5 Hysteretic models

We now consider a special class of models that have the *hysteresis* property. With hysteresis we mean a system whose steady state is not given, but can wander about and depends on the past path of the economy. Mathematically, this property implies that the Jacobian matrix of a continuous-time system has, apart from some "regular" (non-zero) eigenvalues, a zero eigenvalue.[24] Hysteretic systems are important in macroeconomics because they allow us to depart from the rigid framework of equilibrium, a-historical, economics. Put differently: history matters in such systems.

In the remainder of this section we show that the Laplace transform methods studied above can easily be applied in low-dimensional hysteretic models also. We restrict attention to the two cases encountered most frequently in the economics literature, namely two-dimensional models with both roots non-positive and non-negative, respectively.

5.1 Non-positive roots ($\lambda_1 < 0 = \lambda_2$)

Suppose that the matrix Δ in (10) satisfies $|\Delta| = \lambda_1\lambda_2 = 0$ and $\text{tr}\Delta = \lambda_1 + \lambda_2 < 0$ so that the system has a zero root and is hysteretic, i.e. $\lambda_1 = \text{tr}\Delta < 0$ and $\lambda_2 = 0$. Clearly, since

[23] In the OLG model public debt policy is useful because (1) it allows the government to practice tax smoothing and (2) because public debt can be used to redistribute gains and losses across generations. In the RA model, Ricardian equivalence holds and only the first role of debt policy remains.

[24] Note that in a discrete-time setting a model displays hysteresis if it contains a unit root. Amable et al. (1994) argue that it is inappropriate to equate zero-root (or unit-root) dynamics with 'true' hysteresis. Strong hysteresis is a much more general concept in their view and they suggest that zero-root dynamics at best captures some aspects of this concept.

$|\Delta| = 0$, the inverse matrix Δ^{-1} does not exist and we cannot compute the long-run results of a shock by imposing the steady state in (10) and inverting Δ. However, the derivations leading from (18) to (21) are all still valid even for $\lambda_2 = 0$, i.e. the general solution in Laplace transforms is:

$$\begin{bmatrix} \mathcal{L}\{K,s\} \\ \mathcal{L}\{Q,s\} \end{bmatrix} = \left[\frac{B}{s-\lambda_1} + \frac{I-B}{s}\right] \begin{bmatrix} K(0)+\mathcal{L}\{g_K,s\} \\ Q(0)+\mathcal{L}\{g_Q,s\} \end{bmatrix}. \tag{92}$$

where $B \equiv \text{adj}\Lambda(\lambda_1)/\lambda_1$ and $I - B \equiv -\text{adj}\Lambda(0)/\lambda_1$ are weighting matrices. Now assume that there is a *temporary* shock, i.e. $g_i(t) = g_i e^{-\xi_i t}$ for $i = K, Q$, $\xi_i > 0$, and $t \geq 0$. In a non-hysteretic model such a temporary shock has no effect in the long run as the system will eventually just return to its initial steady state which is uniquely determined by the long-run values of the shock terms.

In stark contrast, in a hysteretic model, a temporary shock does have permanent effects. In order to demonstrate this result we first substitute $\mathcal{L}\{g_i, s\} = g_i/(s+\xi_i)$ into (92):

$$\begin{bmatrix} \mathcal{L}\{K,s\} \\ \mathcal{L}\{Q,s\} \end{bmatrix} = \left[\frac{B}{s-\lambda_1} + \frac{I-B}{s}\right] \begin{bmatrix} K(0)+g_K/(s+\xi_K) \\ Q(0)+g_Q/(s+\xi_Q) \end{bmatrix}. \tag{93}$$

Equation (93) constitutes the full solution for $K(t)$ and $Q(t)$ once the (history-determined) initial conditions are plugged in. Using the final-value theorem (P8) we derive from (93):

$$\begin{aligned} \lim_{s\to 0} \begin{bmatrix} s\mathcal{L}\{K,s\} \\ s\mathcal{L}\{Q,s\} \end{bmatrix} &= \left[B \underbrace{\lim_{s\to 0}\left(\frac{s}{s-\lambda_1}\right)}_{=0} + (I-B)\underbrace{\lim_{s\to 0}\left(\frac{s}{s}\right)}_{=1} \right] \begin{bmatrix} K(0)+\lim_{s\to 0}\left(\frac{g_K}{s+\xi_K}\right) \\ Q(0)+\lim_{s\to 0}\left(\frac{g_Q}{s+\xi_Q}\right) \end{bmatrix} \\ &= \frac{\text{adj}\Delta}{\lambda_1} \begin{bmatrix} K(0)+g_K/\xi_K \\ Q(0)+g_Q/\xi_Q \end{bmatrix} = \begin{bmatrix} K(\infty) \\ Q(\infty) \end{bmatrix}, \end{aligned} \tag{94}$$

where we have used the fact that $\text{adj}\Lambda(0) = -\text{adj}\Delta$ in going from the first to the second line. Equation (94) shows that the hysteretic system does not return to its initial state following the temporary shock. It is not unstable, however, because it does settle down in a new "steady state" (for which $\dot{K}(\infty) = \dot{Q}(\infty) = 0$) but the position of this new steady state depends on the entire path of the shock terms, i.e. in our example on ξ_K and ξ_Q. The ultimate steady state is thus "path dependent" which explains why another term for hysteresis is *path dependency.*

Example 5 *Pegging the nominal interest rate. Giavazzi and Wyplosz (1985, p. 355) give a simple example of a hysteretic system. Consider the following simple macroeconomic model:*

$$m(t) - p(t) = ay(t) - bi_0 \tag{LM}$$

$$i_0 = r(t) + \dot{p}(t) \tag{Fisher}$$

$$y(t) = y_0^D(t) - \eta r(t) \tag{IS}$$

$$\dot{y}(t) = \theta\left[\bar{y}_0 - y(t)\right], \tag{AS}$$

where m, y, $\bar{y}$, and p are, respectively, the money supply, actual output, full employment output, and the price level (all in logarithms), r and i are the real and nominal interest rate, respectively, and y_0^D represents the exogenous elements of aggregate demand. The monetary authority uses monetary policy to peg the nominal interest rate (at $i(t) = i_0$) so the LM curve residually determines the money supply. By combining the Fisher relation with the IS curve we obtain $\dot{p}(t) = (1/\eta)[y(t) - y_0^D(t)] + i_0$. By differentiating this expression and the AS curve–keeping the other exogenous variables constant–we obtain the system in the required format:

$$\begin{bmatrix} d\dot{p}(t) \\ d\dot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1/\eta \\ 0 & -\theta \end{bmatrix} \begin{bmatrix} dp(t) \\ dy(t) \end{bmatrix} + \begin{bmatrix} -(1/\eta)dy_0^D(t) \\ 0 \end{bmatrix},$$

where the Jacobian matrix has characteristic roots $\lambda_1 = -\theta$ and $\lambda_2 = 0$ and it is assumed that both p and y are predetermined variables (so that $dp(0) = dy(0) = 0$). Now consider the effects of a temporary boost in aggregate demand, i.e. $dy_0^D(t) = e^{-\xi_D t}$ for $\xi_D > 0$ and $t \geq 0$. Using the methods developed in this subsection we derive:

$$\begin{bmatrix} \mathcal{L}\{dp, s\} \\ \mathcal{L}\{dy, s\} \end{bmatrix} = \begin{bmatrix} -1/(\eta\xi_D) \\ 0 \end{bmatrix} \left(\frac{1}{s} - \frac{1}{s + \xi_D} \right).$$

Despite the fact that the shock is purely transitory it has a permanent effect on the price level.

5.2 Non-negative roots ($\lambda_1 = 0 < \lambda_2$)

We now assume that Δ in (10) satisfies $|\Delta| = \lambda_1\lambda_2 = 0$ and $\text{tr}\Delta = \lambda_1 + \lambda_2 > 0$ so that $\lambda_1 = 0$ and $\lambda_2 = \text{tr}\Delta > 0$. For this hysteretic case the analysis in subsection 2.3.2 is relevant. The general solution in Laplace transforms is obtained by setting $\lambda_1 = 0$ in (29):

$$s \begin{bmatrix} \mathcal{L}\{K, s\} \\ \mathcal{L}\{Q, s\} \end{bmatrix} = \begin{bmatrix} K(0) + \mathcal{L}\{g_K, s\} \\ Q(0) + \mathcal{L}\{g_Q, s\} \end{bmatrix} + \text{adj}\Lambda(\lambda_2) \begin{bmatrix} \frac{\mathcal{L}\{g_K,s\} - \mathcal{L}\{g_K,\lambda_2\}}{s-\lambda_2} \\ \frac{\mathcal{L}\{g_Q,s\} - \mathcal{L}\{g_Q,\lambda_2\}}{s-\lambda_2} \end{bmatrix}. \tag{95}$$

Let us once again assume that the shock is temporary and has a Laplace transform $\mathcal{L}\{g_i, s\} = g_i/(s + \xi_i)$ for $i = K, Q$ so that:

$$\frac{\mathcal{L}\{g_i, s\} - \mathcal{L}\{g_i, \lambda_2\}}{s - \lambda_2} = \frac{-g_i}{(\lambda_2 + \xi_i)(s + \xi_i)}. \tag{96}$$

Equation (95) can then be rewritten as:

$$s \begin{bmatrix} \mathcal{L}\{K, s\} \\ \mathcal{L}\{Q, s\} \end{bmatrix} = \begin{bmatrix} K(0) + g_K/(s + \xi_K) \\ Q(0) + g_Q/(s + \xi_Q) \end{bmatrix} - \text{adj}\Lambda(\lambda_2) \begin{bmatrix} \frac{g_K}{(\lambda_2+\xi_K)(s+\xi_K)} \\ \frac{g_Q}{(\lambda_2+\xi_Q)(s+\xi_Q)} \end{bmatrix}, \tag{97}$$

where $Q(0)$ follows from either (27) or (28). By using the final-value theorem (P8) in (97) we derive the hysteretic result:[25]

$$\lim_{s\to 0} s \begin{bmatrix} \mathcal{L}\{K,s\} \\ \mathcal{L}\{Q,s\} \end{bmatrix} = \begin{bmatrix} K(0)+g_K/\xi_K \\ Q(0)+g_Q/\xi_Q \end{bmatrix} - \text{adj}\Lambda(\lambda_2) \begin{bmatrix} \frac{g_K}{\xi_K(\lambda_2+\xi_K)} \\ \frac{g_Q}{\xi_Q(\lambda_2+\xi_Q)} \end{bmatrix}$$

$$= \frac{\text{adj}\Delta}{\lambda_2} \begin{bmatrix} K(0)+g_K/\xi_K \\ Q(0)+g_Q/\xi_Q \end{bmatrix} = \begin{bmatrix} K(\infty) \\ Q(\infty) \end{bmatrix}. \tag{98}$$

As in the outright stable case (see (94)) parameters of the shock path determine the ultimate long-run result.

Example 6 *Consider the simple representative-agent model of a small open economy suggested by Blanchard (1985, p. 230). There is no capital and labour supply is exogenously fixed (at unity) so that output and the wage rate are exogenous. The model is:*

$$\dot{C}(t) = [r(t) - \alpha]\, C(t)$$
$$\dot{F}(t) = r(t)F(t) + W(t) - C(t),$$

where F is net foreign assets, and C, r, and W are, respectively, consumption, the exogenous interest rate, and the wage rate. As is well known, a steady state only exists in this model if the steady-state interest rate equals the rate of time preference, i.e. if $r(t) = \alpha$. After loglinearizing the model around an initial steady state we obtain:

$$\begin{bmatrix} \dot{\tilde{F}}(t) \\ \dot{\tilde{C}}(t) \end{bmatrix} = \begin{bmatrix} \alpha & -\alpha(1+\omega_F) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{F}(t) \\ \tilde{C}(t) \end{bmatrix} + \begin{bmatrix} \omega_F \\ 1 \end{bmatrix} \alpha\tilde{r}(t),$$

where $\omega_F \equiv \alpha F/Y = C/Y - 1$ is the initial share of foreign asset income in national output, $\dot{\tilde{F}}(t) \equiv \alpha d\dot{F}/Y$, and $\tilde{F}(t) \equiv \alpha dF/Y$. The Jacobian matrix on the right-hand side has characteristic roots $\lambda_1 = 0$ and $\lambda_2 = \alpha$ and it is assumed that F is the predetermined variable and C is the jumping variable. Now consider a temporary change in the world interest rate, $\tilde{r}(t) = e^{-\xi_R t}$ for $\xi_R > 0$ and $t \geq 0$. By using (27) and making the obvious substitutions we obtain the jump in consumption:

$$\tilde{C}(0) = -\frac{\alpha}{(\alpha+\xi_R)(1+\omega_F)} < 0.$$

In a similar fashion, the long-run results can be obtained by using (98):

$$\begin{bmatrix} \tilde{F}(\infty) \\ \tilde{C}(\infty) \end{bmatrix} = \begin{bmatrix} 0 & 1+\omega_F \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha\omega_F/\xi_R \\ \tilde{C}(0)+\alpha/\xi_R \end{bmatrix} = \begin{bmatrix} 1+\omega_F \\ 1 \end{bmatrix} \left(\frac{\alpha\,[\alpha+\omega_F(\alpha+\xi_R)]}{\xi_R(\alpha+\xi_R)(1+\omega_F)} \right).$$

In the impact period the household cuts back consumption to boost its savings. In the long run both consumption and net foreign assest are higher than in the initial steady state (provided $\omega_F > -\alpha/(\alpha+\xi_R)$ in the initial steady state).

[25] In going from the first to the second line we use (26), note that (19) implies $\lambda_2 I = \text{adj}\Lambda(\lambda_2) - \text{adj}\Lambda(0)$, and recall that $\text{adj}\Lambda(0) = -\text{adj}\Delta$.

6 Discrete-time models

Although continuous-time models are quite convenient to work with, economists often work with models formulated in discrete time. Most RBC models fall under this category as does the class of overlapping-generations models in the Samuelson (1958)-Diamond (1965) tradition. In this section we briefly introduce the z-transform method. This method plays the same role in discrete-time models that the Laplace transform method performs in continuous-time models. In order to avoid unnecessary duplication, only the basic elements of the z-transform are introduced. The student should be able to "translate" the insights obtained above to the discrete-time setting after reading this section. Extremely lucid expositions of the z-transform method are Ogata (1995) and Elaydi (1996). Meijdam and Verhoeven (1998) apply the techniques in an economic setting.

6.1 The z-transform

Suppose we have a discrete-time function, f_t, which satisfies $f_t = 0$ for $t = -1, -2, ...$ The (one-sided) z-transform of the function is then defined as follows:

$$\mathcal{Z}\{f_t, z\} \equiv \sum_{t=0}^{\infty} f_t z^{-t}, \tag{99}$$

where z is a complex variable.[26] Provided the sum on the right-hand side converges, $\mathcal{Z}\{f_t, z\}$ exists and can be seen as a function of z. The region of convergence is determined as follows. Suppose that f_t satisfies:

$$\lim_{t\to\infty} \left| \frac{f_{t+1}}{f_t} \right| = R. \tag{100}$$

Then the infinite sum in (99) converges provided:

$$\lim_{t\to\infty} \left| \frac{f_{t+1} z^{-(t+1)}}{f_t z^{-t}} \right| < 1, \tag{101}$$

and diverges if the inequality is reversed. Together, (100) and (101) imply that (99) converges– and $\mathcal{Z}\{f_t, z\}$ exists–in the region $|z| > R$ ("heavy discounting"). In the region $|z| < R$, on the other hand, discounting is "light" and $\mathcal{Z}\{f_t, z\}$ does not exist. R is referred to as the *radius of convergence* of $\mathcal{Z}\{f_t, z\}$.

Example 7 *Suppose that $f_t = 1$ for $t = 0, 1, 2, ...$ (and $f_t = 0$ otherwise). Then $\mathcal{Z}\{f_t, z\}$ is:*

$$\begin{aligned}\mathcal{Z}\{f_t, z\} \equiv \mathcal{Z}\{1, z\} &= \sum_{t=0}^{\infty} 1 \times z^{-t} = 1 + (1/z) + (1/z)^2 + ... \\ &= \frac{1}{1 - 1/z} = \frac{z}{z-1},\end{aligned}$$

[26] By comparing (1) and (99) we cannot help but notice the close relation that exists between the Laplace transform and the z-transform. Indeed, assuming that $f(t)$ in (1) is continuous we obtain by discretizing $\mathcal{L}\{f, s\} = \sum_{t=0}^{\infty} e^{-st} f_t$. By setting $z = e^s$ we obtain (99). See also Elaydi (1996, p. 254).

f_t	$\mathcal{Z}\{f,z\}$	*valid for:*
1	$\frac{z}{z-1}$	$\lvert z\rvert > 1$
t	$\frac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
a^t	$\frac{z}{z-a}$,	$\lvert z\rvert > \lvert a\rvert$
a^{t-1}	$\frac{1}{z-a}$,	$\lvert z\rvert > \lvert a\rvert$
ta^{t-1}	$\frac{z}{(z-a)^2}$,	$\lvert z\rvert > \lvert a\rvert$
$\frac{a^t-b^t}{a-b}$	$\frac{z}{(z-a)(z-b)}$	$\lvert z\rvert > \lvert a\rvert$, $\lvert z\rvert > \lvert b\rvert$, $a \neq b$

Table 4: Commonly used z-transforms

provided $|z| > 1$.

Now a slightly harder one:

Example 8 *Suppose that* $f_t = a^t$ *for* $t = 0, 1, 2, ...$ *(and* $f_t = 0$ *otherwise). Then* $\mathcal{Z}\{f_t, z\}$ *is:*

$$\mathcal{Z}\{f_t, z\} \equiv \mathcal{Z}\{a^t, z\} = \sum_{t=0}^{\infty} a^t z^{-t} = 1 + (a/z) + (a/z)^2 + ...$$
$$= \frac{1}{1 - a/z} = \frac{z}{z - a},$$

provided $|z| > |a|$.

In Table 4 we ave gathered some often-used z-transforms. The student should verify that both the form of each transform and its associated radius of convergence are correct.

The z-transform has a number of properties which allow us to perform algebraic calculations with them. The most important of these are the following. Notice that in each case we assume that f_t possesses a z-transform and that $f_t = 0$ for $t = -1, -2, ...$

Property 9 *Multiplication by a constant. If* $\mathcal{Z}\{f, s\}$ *is the* z*-transform of* f_t *then* $\mathcal{Z}\{af, z\} = a\mathcal{Z}\{f, z\}$.

Property 10 *If* f_t *and* g_t *both have a* z*-transform then we have for any constants* a *and* b *that:*

$$\mathcal{Z}\{af + bg, z\} = a\mathcal{Z}\{f, z\} + b\mathcal{Z}\{g, z\} \tag{P9}$$

Property 11 *Left-shifting.*

$$\mathcal{Z}\{f_{t+1}, z\} = z\mathcal{Z}\{f_t, z\} - zf_0 \tag{P10}$$
$$\mathcal{Z}\{f_{t+2}, z\} = z\mathcal{Z}\{f_{t+1}, z\} - zf_1 = z^2\mathcal{Z}\{f_t, z\} - z^2 f_0 - zf_1 \tag{P11}$$
$$...$$
$$\mathcal{Z}\{f_{t+k}, z\} = z^k \left[\mathcal{Z}\{f_t, z\} - \sum_{r=0}^{k-1} z^{k-r} f_r\right] \tag{P12}$$

Property 12 *Initial-value and final-value theorems:*

$$\lim_{|z|\to\infty} \mathcal{Z}\{f_t, z\} = f_0 \tag{P13}$$

$$\lim_{z\to 1}(z-1)\mathcal{Z}\{f_t, z\} = \lim_{t\to\infty} f_t \tag{P14}$$

6.2 Revisiting Mickey Mouse

Suppose we wish to solve the following difference equation:

$$x_{t+2} + 3x_{t+1} + 2x_t = 0,\ x_0 = 0,\ x_1 = 1. \tag{102}$$

By using properties (P10) and (P11) we obtain the subsidiary equation in a few steps:

$$\begin{aligned} 0 &= \left[z^2\mathcal{Z}\{x_t, z\} - z^2 x_0 - z x_1\right] + 3\left[z\mathcal{Z}\{x_t, z\} - z x_0\right] + 2\mathcal{Z}\{x_t, z\} \Leftrightarrow \\ \left(z^2 + 3z + 2\right)\mathcal{Z}\{x_t, z\} &= z^2 x_0 + z x_1 + 3 z x_0 = z \Leftrightarrow \\ \mathcal{Z}\{x_t, z\} &= \frac{z}{(z+1)(z+2)} = \frac{z}{z+1} - \frac{z}{z+2}. \end{aligned} \tag{103}$$

Inverting (103) yields the solution in the time domain:

$$x_t = (-1)^t - (-2)^t, \tag{104}$$

for $t = 0, 1, 2, ...$

This example is–of course–rather unexciting apart from the fact that it gives us a hint as to the stability properties of difference equations. Asymptotic stability of (a system of) difference equations is obtained if the roots lie inside the unit circle, i.e. terms like $\frac{z}{z+a}$ are (un) stable if $|a| < 1$ ($|a| > 1$).

6.3 Revisiting the saddle path model

We now consider the following system of difference equations (in analogy with (10)):

$$\begin{bmatrix} K_{t+1} - K_t \\ Q_{t+1} - Q_t \end{bmatrix} = \Delta \begin{bmatrix} K_t \\ Q_t \end{bmatrix} + \begin{bmatrix} g_{K,t} \\ g_{Q,t} \end{bmatrix}, \tag{105}$$

where $g_{K,t}$ and $g_{Q,t}$ are shock terms (possessing a z-transform) and Δ has typical element δ_{ij}. Taking the z-transform of (105) yields:

$$\Lambda(z) \begin{bmatrix} \mathcal{Z}\{K_t, z\} \\ \mathcal{Z}\{Q_t, z\} \end{bmatrix} = \begin{bmatrix} zK_0 + \mathcal{Z}\{g_K, z\} \\ zQ_0 + \mathcal{Z}\{g_Q, z\} \end{bmatrix}, \tag{106}$$

where $\Lambda(z-1) \equiv (z-1)I - \Delta$. We assume that the characteristic roots of Δ are both real and that $-1 < \lambda_1 < 0$ and $0 < \lambda_2 < 1$.[27] As before, K_t is deemed to be predetermined (so

[27] We write the system in a form which emphasizes the close analogy with (10). Of course, we can also re-express (105) as:

$$\begin{bmatrix} K_{t+1} \\ Q_{t+1} \end{bmatrix} = \Delta^* \begin{bmatrix} K_t \\ Q_t \end{bmatrix} + \begin{bmatrix} g_{K,t} \\ g_{Q,t} \end{bmatrix},$$

that K_0 is given) whilst Q_t is a non-predetermined variable (so that Q_0 can jump). Following the steps in subsection 2.3.2 we derive the expression for Q_0:

$$Q_0 = -\frac{\mathcal{Z}\{g_Q, 1+\lambda_2\}}{1+\lambda_2} - \left(\frac{\lambda_2 - \delta_{22}}{\delta_{12}}\right)\left[K_0 + \frac{\mathcal{Z}\{g_K, 1+\lambda_2\}}{1+\lambda_2}\right] \tag{107}$$

$$= -\frac{\mathcal{Z}\{g_Q, 1+\lambda_2\}}{1+\lambda_2} - \left(\frac{\delta_{21}}{\lambda_2 - \delta_{11}}\right)\left[K_0 + \frac{\mathcal{Z}\{g_K, 1+\lambda_2\}}{1+\lambda_2}\right]. \tag{108}$$

Simlarly, the general expression for the solution can be written as:

$$[z-(1+\lambda_1)]\begin{bmatrix} \mathcal{Z}\{K,z\} \\ \mathcal{Z}\{Q,z\} \end{bmatrix} = \begin{bmatrix} zK_0 + \mathcal{Z}\{g_K,z\} \\ zQ_0 + \mathcal{Z}\{g_Q,z\} \end{bmatrix} + \frac{\operatorname{adj}\Lambda(\lambda_2)\begin{bmatrix} \mathcal{Z}\{g_K,z\} - \left(\frac{z}{1+\lambda_2}\right)\mathcal{Z}\{g_K,1+\lambda_2\} \\ \mathcal{Z}\{g_Q,z\} - \left(\frac{z}{1+\lambda_2}\right)\mathcal{Z}\{g_Q,1+\lambda_2\} \end{bmatrix}}{z-(1+\lambda_2)}, \tag{109}$$

where the analogy with (29) should be obvious.

6.4 Discussion

One of the advantages of working in discrete time is the ease with which stochastic shocks can be incorporated in the model. Readers are referred to Whiteman (1983) and Uhlig (1999) for further information.

7 Guide to the literature

The reader with no background in the area of differential equations does well to consult an introductory text such as Chiang (1984), especially chapters 13-18. The most accessible sources to the Laplace transform method are to be found in the engineering literature. Kreyszig (1988, ch. 5) and Boyce and DiPrima (1992, ch. 6) are particularly illuminating. A good encyclopedic source on Laplace transforms is Spiegel (1965). Judd (1982, 1985, 1987a-b) was the first to apply the method to saddle-point stable perfect foresight models, and to note the close link with welfare evaluations along the transition path. Unfortunately Judd does not write that clearly. Further contributions were made by Bovenberg (1993, 1994) and by Bovenberg and Heijdra (1998b). Papers that use the Laplace transform extensively are Aoki (1986), Baldwin (1992), Bettendorf and Heijdra (1996, 1999), Bovenberg and Heijdra (1998a), Broer and Heijdra (1996), Heijdra and Meijdam (1997), and Heijdra and Van der Horst (2000). Further papers dealing with the zero-root problem are Giavazzi and Wyplosz (1985) and Amable et al. (1994).

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where $\Delta^* \equiv I + \Delta$. Saddle-point stability then requires the characteristic roots of Δ^* to be between 0 and 1 in absolute value.

For higher dimensional models, for which only numerical simulations are feasible, general solution methods are provided (for the discrete-time case) by Blanchard and Kahn (1980) and (for the continuous time case) by Buiter (1984).

Questions

Question 1
Verify that (23) is indeed the solution to (10) for the outright stable case. *Hint*: differentiate (23) with respect to time and show that the resulting expression can be rewritten as (10). Note that $\Lambda(\lambda_i)\text{adj}\Lambda(\lambda_i) = 0$ [why?].

Question 2 [Bettendorf and Heijdra (1996, 1999)]
Consider the system $\dot{X}(t) = \Delta X(t) + G(t)$, where $X(t) \equiv [X_1(t), X_2(t), X_3(t)]^T$ and $G(t) \equiv [g_1(t), g_2(t), g_3(t)]^T$. Assume that the three-by-three matrix Δ has distinct real eigenvalues of which two are unstable and one is stable, i.e. $\lambda_1 < 0 < \lambda_2, \lambda_3$. $X_1(t)$ is a predetermined variable whilst $X_2(t)$ and $X_3(t)$ are non-predetermined. Define $\text{adj}\Lambda(s) \equiv sI - \Delta$.

(a) Show that $\text{adj}\Lambda(s)$ can be written as:

$$\text{adj}\Lambda(s) = (s-\lambda_2)(s-\lambda_3)I + \left(\frac{s-\lambda_3}{\lambda_2-\lambda_3}\right)\text{adj}\Lambda(\lambda_2) - \left(\frac{s-\lambda_2}{\lambda_2-\lambda_3}\right)\text{adj}\Lambda(\lambda_3).$$

(b) Derive the expressions for the impact jumps, $X_2(0)$ and $X_3(0)$.
(c) Show that the solution of the system (in Laplace transforms) is:

$$(s-\lambda_1)\begin{bmatrix} \mathcal{L}\{X_1,s\} \\ \mathcal{L}\{X_2,s\} \\ \mathcal{L}\{X_3,s\} \end{bmatrix} = \begin{bmatrix} \mathcal{L}\{g_1,s\} \\ X_2(0)+\mathcal{L}\{g_2,s\} \\ X_3(0)+\mathcal{L}\{g_3,s\} \end{bmatrix} + \left(\frac{\text{adj}\Lambda(\lambda_2)}{\lambda_2-\lambda_3}\right)\begin{bmatrix} \frac{\mathcal{L}\{g_1,s\}-\mathcal{L}\{g_1,\lambda_2\}}{s-\lambda_2} \\ \frac{\mathcal{L}\{g_2,s\}-\mathcal{L}\{g_2,\lambda_2\}}{s-\lambda_2} \\ \frac{\mathcal{L}\{g_3,s\}-\mathcal{L}\{g_3,\lambda_2\}}{s-\lambda_2} \end{bmatrix}$$
$$- \left(\frac{\text{adj}\Lambda(\lambda_3)}{\lambda_2-\lambda_3}\right)\begin{bmatrix} \frac{\mathcal{L}\{g_1,s\}-\mathcal{L}\{g_1,\lambda_3\}}{s-\lambda_3} \\ \frac{\mathcal{L}\{g_2,s\}-\mathcal{L}\{g_2,\lambda_3\}}{s-\lambda_3} \\ \frac{\mathcal{L}\{g_3,s\}-\mathcal{L}\{g_3,\lambda_3\}}{s-\lambda_3} \end{bmatrix}.$$

(d) Assume that the shock terms satisfy $g_i(t) = g_i$, for $t \geq 0$ and $i = 1, 2, 3$. Derive the solution in the time domain.

Question 3
Prove that CFE^{OLG} in Figure 3 is horizontal near the origin and downward sloping and steeper than CFE^{RA} near the vertical intercept. *Hint*: use (44) and (46) and take limits for $L \to 0$ and $L \to 1$.

Question 4 [Sen and Turnovsky (1990)]
Loglinearize the model of Sen and Turnovsky (1990) around an initial steady state. Assume for simplicity that the felicity function is loglinear in consumption and leisure and that the

production function is Cobb-Douglas. Show that the model exhibits hysteresis. Use the Laplace transform techniques to study the effects of a temporary investment subsidy.

Question 5 [Giovannini (1988)]
Giovannini (1988) extends the Blanchard (1985) model to the open economy. Compute the impact and long-run effects of a fiscal spending shock for a small open economy populated by an infinitely-lived representative agent. (The model is given on page 1753 and the birth rate is zero). Interpret Giovannini's remarks in the first paragraph on page 1755 concerning underdeterminacy of the steady state. *Hint*: use the Laplace transform techniques and make sure you have already answered question 2 above.

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