A.1 Introduction

This paper contains some supplementary material to Heijdra and Romp (2008b). The interested reader is also referred to our other papers, viz. Heijdra and Romp (2008a, forthcoming).

A.2 Optimal household choices

We solve the household’s optimization program as follows. The lifetime utility function (1) can be written as:

$$\Lambda(v, t) \equiv e^{\theta u + M(u)} \int_t^\infty U(\bar{c}(v, \tau)) e^{-[\theta(\tau-v)+M(\tau-v)]} d\tau$$

$$- e^{\theta u + M(u)} \int_t^{\bar{v}+R(v)} D(\tau - v) e^{-[\theta(\tau-v)+M(\tau-v)]} d\tau,$$  

(A.1)

and the lifetime budget constraint (7) is:

$$e^{\theta u + M(u)} \int_t^\infty \bar{c}(v, \tau) e^{-[\theta(\tau-v)+M(\tau-v)]} d\tau = \bar{a}(v, t) + \bar{L}(v, t, R(v)).$$  

(A.2)
The Lagrangian can be written as:

\[
\mathcal{L} = e^{\theta u + M(u)} \int_t^\infty U(\bar{c}(v, \tau)) e^{\theta (\tau - v)} e^{[\theta (\tau - v) + M(\tau - v)]} d\tau
\]

\[ - e^{\theta u + M(u)} \int_t^{v + R(v)} D(\tau - v) e^{[\theta (\tau - v) + M(\tau - v)]} d\tau \]

\[ + \lambda(t) \cdot \left[ \bar{a}(v, t) + \bar{p}(v, t, R(v)) - e^{\theta u + M(u)} \int_t^\infty \bar{c}(v, \tau) e^{-[\theta (\tau - v) + M(\tau - v)]} d\tau \right], \]

and the first-order conditions for optimal consumption are \( \frac{\partial \mathcal{L}}{\partial \bar{c}(v, \tau)} = 0 \) for all \( \tau \), or:

\[
U'(\bar{c}(v, \tau)) e^{(\theta - \bar{\sigma})(r - \bar{\theta})} = \lambda(t). \tag{A.3}
\]

Differentiating this expression with respect to \( \tau \) and noting (8) we find the Euler equation:

\[
\frac{\partial \bar{c}(v, \tau)}{\partial \bar{c}(v, \tau)} = \sigma [r - \bar{\theta}]. \tag{A.4}
\]

Solving (A.4) we find equation (10) in the paper. Substituting (10) into (A.2) we find (9):

\[
\bar{c}(v, t) e^{\theta u + M(u)} \int_t^\infty e^{\sigma (r - \bar{\theta})(\tau - t)} e^{[\theta (\tau - v) + M(\tau - v)]} d\tau = \ldots
\]

\[
\bar{c}(v, t) \cdot \left[ e^{\theta u + M(u)} \int_t^\infty e^{[\theta (\tau - v) + M(\tau - v)]} d\tau \right] = \bar{a}(v, t) + \bar{p}(v, t, R(v)), \tag{A.5}
\]

where we have used the fact that \( u = t - v \) and \( r^* \equiv r - \sigma (r - \bar{\theta}) \). By rewriting the integral somewhat (by setting \( s = \tau - v \) and integrating over \( s \in [u, \infty) \)) we find that the term in square brackets is equal to \( \Delta (u, r^*) \).

Equation (13) is derived as follows. Substituting (10) into (12) and noting (8) we find:

\[
\bar{\Lambda}(v, t) = e^{\theta u + M(u)} \left[ \int_t^\infty \left( \bar{c}(v, t) e^{\theta (r - \bar{\theta})(\tau - t)} \right)^{1 - 1/\sigma} \cdot e^{-[\theta (\tau - v) + M(\tau - v)]} d\tau \right]
\]

\[ - \int_t^{v + R(v)} D(\tau - v) e^{-[\theta (\tau - v) + M(\tau - v)]} d\tau. \tag{A.6}
\]

Differentiating with respect to \( R(v) \) we find:

\[
\frac{d\bar{\Lambda}(v, t)}{dR(v)} = e^{\theta u + M(u)} \left[ \int_t^\infty \left( \bar{c}(v, t) e^{\theta (r - \bar{\theta})(\tau - t)} \right)^{-1/\sigma} \frac{d\bar{c}(v, t)}{dR(v)} e^{\theta (r - \bar{\theta})(\tau - t) - \theta (\tau - v) - M(\tau - v)} d\tau \right]
\]

\[ - D(R(v)) e^{-[\theta R(v) + M(R(v))]}. \tag{A.7}
\]

It follows from (9) that:

\[
\frac{d\bar{c}(v, t)}{dR(v)} = \frac{1}{\Delta(u, r^*)} \frac{d\bar{p}(v, t, R(v))}{dR(v)}. \tag{A.8}
\]
Using (A.8) in (A.7) and simplifying we find:

\[
\frac{d\bar{\Lambda}(v,t)}{dR(v)} = \bar{c}(v,t)^{-1/\sigma} \cdot \frac{d\bar{\Pi}(v,t,R(v))}{dR(v)} \left[ e^{\theta_R + M(u)} e^{(1-\sigma)(r-\theta)u} \frac{\Delta(u,r^*)}{\int_t^\infty e^{-[r^*(\tau-v)+M\tau(v)]} d\tau} \right] - e^{\theta_R + M(u)} D(R(v)) e^{-[\theta_R(v)+M(R(v))]}.
\]  

(A.9)

By noting the definition of \(\Delta(u,r^*)\) we find that the term in square brackets on the right-hand side equals unity. Writing \(U'(\bar{c}(v,t)) = \bar{c}(v,t)^{-1/\sigma}\) we thus find that (A.9) reduces to:

\[
\frac{d\bar{\Lambda}(v,t)}{dR(v)} = U'(\bar{c}(v,t)) \cdot \frac{d\bar{\Pi}(v,t,R(v))}{dR(v)} - e^{\theta_R + M(u)} D(R(v)) e^{-[\theta_R(v)+M(R(v))]}. \tag{A.10}
\]

The optimal retirement age is such that \(d\bar{\Lambda}(v,t)/dR(v) = 0:\)

\[
e^{-[\theta_R+M(u)]} U'(\bar{c}(v,t)) \cdot \frac{d\bar{\Pi}(v,t,R(v))}{dR(v)} = D(R(v)) e^{-[\theta_R(v)+M(R(v))]}.
\]  

(A.11)

This is equation (13) in the paper.

To show dynamic consistency we must show that the agent chooses the same \(R(v)\) at every age. We note that:

\[
\frac{d\bar{\Pi}(v,t,R(v))}{dR(v)} = e^{\theta_R + M(u)} \cdot \frac{d\bar{\Pi}(v,v,R(v))}{dR(v)},
\]

\[
U'(\bar{c}(v,t)) = U'(\bar{c}(v,v)) \cdot e^{-(r-\theta)u}.
\]

Substituting these expressions in (A.11) gives:

\[
D(R(v)) e^{-[\theta_R(v)+M(R(v))]} = e^{-[\theta_R(v)+M(u)]} e^{-(r-\theta)u} U'(\bar{c}(v,v)) \cdot e^{\theta_R + M(u)} \cdot \frac{d\bar{\Pi}(v,v,R(v))}{dR(v)}
\]

\[
= U'(\bar{c}(v,v)) \cdot \frac{d\bar{\Pi}(v,v,R(v))}{dR(v)},
\]  

(A.12)

which is the first-order condition for a newborn agent’s retirement choice.

### A.3 Aggregate equilibrium

The macroeconomic equilibrium is defined as follows.

1. Conditional on the parameters of the fiscal and pension systems, individual households choose consumption \(\bar{c}(v,t)\) according to (9), where \(\bar{\Pi}(v,v,R(v))\) is given in (5) and the age-dependent wage is given in (24). Working individuals set the optimum retirement age such that (13) is satisfied. Households are indifferent between holding (claims on) physical capital, \(\bar{k}(v,t)\), government bonds, \(\bar{d}(v,t)\), and net foreign assets, \(\bar{f}(v,t)\), i.e. \(\bar{a}(v,t) = \bar{k}(v,t) + \bar{d}(v,t) + \bar{f}(v,t)\).
2. Aggregate variables are defined as \( c(t) \equiv \int_{-\infty}^{t} \bar{c}(v,t) l(v,t) dv, a(t) \equiv \int_{-\infty}^{t} \bar{a}(v,t) l(v,t) dv, \)
\( h(t) \equiv \int_{-\infty}^{t} l(v,t) E(t-v) 1(t-v, R(v)) dv, k(t) \equiv \int_{-\infty}^{t} \bar{k}(v,t) l(v,t) dv, d(t) \equiv \int_{-\infty}^{t} \bar{d}(v,t) l(v,t) dv, \) and \( f(t) \equiv \int_{-\infty}^{t} \bar{f}(v,t) l(v,t) dv. \) The population weights, \( l(v,t), \)
are implicitly defined in (16).

3. Feasibility constraints are: \( \bar{k}(v,t) \geq 0, k(t) > 0, \bar{c}(v,t) > 0, c(t) > 0, h(t) > 0, \) and \( R(v) > 0. \)


\[ d(t) = (r - n(t)) d(t) + q(t), \]
with \( q(t) \equiv \int_{-\infty}^{t} \bar{q}(v,t) l(v,t) dv \) and:
\[ \bar{q}(v,t) = \begin{cases} \bar{t}_{L}(t) \bar{w}(t-v) + \bar{z}(t) & \text{if } t-v \leq R(v) \\ \bar{z}(t) - \bar{p}(v,t, R(v)) & \text{if } t-v > R(v) \end{cases}. \] (A.13)

(b) The intertemporal budget constraint is given by:
\[ d(t) = \int_{t}^{\infty} q(\tau) e^{-[r(\tau-t)+N(\tau-t)]} d\tau, \] (A.14)
where \( N(\tau-t) \) is implicitly determined in equation (17).

In section 6 of the paper we calibrate an initial steady state which satisfies all equilibrium conditions stated above. We also study four different scenarios. In each case, the economy is at a steady state at time \( t = 0 \) but is hit by a stepwise demographic shock consisting of either a baby bust or a longevity boost. Following the shock, the non-predetermined (or “jumping”) variables immediate react. These variables are consumption, lifetime income, and the retirement decision of pre-shock workers. Post-shock retirees stay retired as labour market exit is an absorbing state. At impact, the predetermined variables \( \bar{a}(v,0), \bar{d}(v,0), \bar{k}(v,0), a(0), d(0), k(0), \) and \( L(0) \) all stay constant. These variables form the initial conditions for the dynamic system.

In the first two scenarios, the government reacts to the particular demographic shock by announcing a stepwise change in the lump-sum tax to be implemented at some future time \( T_{R}, \) where \( T_{R} \) stands for the time of reform. The future tax change is such that the government intertemporal budget constraint is satisfied at all times. The level of public debt fluctuates during transition but ultimately settles down at a constant per capita level.

In the third scenario, the government reacts to the baby boost by announcing a stepwise (tax smoothing) increase in the labour income tax, \( t_{L}, \) at some future time \( T_{R}. \) Again, \( t_{L}(\tau) \) for \( \tau \geq T_{R} \) is set at such a level that (A.14) holds at all times. Finally, in the fourth scenario
the government reacts by announcing an stepwise change in the EEA to take place at time $T_R$. In this scenario all taxes are held constant and $R_E (\tau)$ for $\tau \geq T_R$ is set to satisfy (A.14).

Computational details on how to impose the government budget constraint in the presence of transitional dynamics can be found in Romp (2007, pp. 122–125). We ensure that the optimization routine singles out the globally optimal solution for the retirement date. In our solution algorithm we first divide the age domain in various blocks such that the first derivative of the lifetime income function $\ell' (R)$ is continuous. We then check for optimality of corner solutions and interior solution for every block separately and pick that solution that provides maximum utility. Our specific disutility and effort functions (constants) rule out the possibility of more than one interior solution per block. In this manner we find the globally optimal solution.

A.4 Technological change

Exogenous labour-augmenting technological change can easily be incorporated in the model. We assume that:

$$\gamma \equiv \frac{\dot{A}_Y (t)}{A_Y (t)} ,$$

(A.15)

where the technological growth rate, $\gamma$, is assumed to be time-invariant. Equations (20)–(21) in the paper are changed to:

$$r + \delta = \epsilon \left( \frac{A_Y (t) h (t)}{k (t)} \right)^{1-\epsilon} ,$$

(A.16)

$$w (t) = (1 - \epsilon) A_Y (t) \left( \frac{A_Y (t) h (t)}{k (t)} \right)^{-\epsilon} .$$

(A.17)

The constant word interest rate fixes the ratio $A_Y (t) h (t) / k (t)$ so that the wage rate can be written as:

$$w (t) = \omega_0 A_Y (t) ,$$

(A.18)

where $\omega_0 \equiv (1 - \epsilon) \left( \frac{\epsilon}{1+\epsilon} \right)^{\epsilon/(1-\epsilon)}$ is a constant. Since $A_Y (t)$ grows at the exponential rate $\gamma$, the same holds for the wage rate:

$$w (\tau) = w (t) \cdot e^{\gamma (\tau-t)} .$$

(A.19)

Equation (24) in the paper is thus changed to:

$$\bar{w} (v, t) = w (t) \cdot E (t - v) ,$$

(A.20)

i.e. the individual agent’s wage has both a time component and an age component.

As is well known from the RBC literature, in the presence of technological change, certain restrictions must be imposed on preferences and on the tax- and pension system in order to
allow for a meaningful steady state. See, for example, King, Plosser and Rebelo (2002, p. 94-95) for details relating to the intensive-margin labour supply decision. In the context of our model, we model the labour force participation ratio. This means that we must ensure that the retirement decision is not affected by the level of the wage rate, i.e. a change in the wage rate must induce exactly offsetting income- and substitution effects. The simplest possible way to achieve this, is to respecifying the lifetime utility function (A.1) to:

\[
\Lambda(v, t) = e^{\theta u + M(u)} \int_t^\infty \ln \bar{c}(v, \tau) e^{-[\theta (\tau - v) + M(\tau - v)]} d\tau - e^{\theta u + M(u)} \int_t^{v + R(v)} D(\tau - v) e^{-[\theta (\tau - v) + M(\tau - v)]} d\tau.
\] (A.21)

Felicity is logarithmic in consumption. Unlike Boucekkine et al. (2002, p. 346), who use a linear felicity function, we don’t need to scale disutility of labour by \( A_Y(v) \) because (A.21) is of the form required for the income and substitution effects of wage changes to exactly offset each other.

In order to allow for a meaningful steady state, the tax and pension system must also be changed. In particular, we index both the pension payment and the lump-sum tax to the index of labour-augmenting technological change, i.e. the lump-sum tax equals \( A_Y(\tau) \tilde{z}(\tau) \) whilst the pension system is given by:

\[
\bar{p}(v, \tau, R(v)) = \begin{cases} 
0 & \text{if } \tau - v < R_E \\
A_Y(\tau) \cdot B(R(v)) & \text{if } \tau - v \geq R_E.
\end{cases}
\] (A.22)

Equation (3) is changed to:

\[
\hat{a}(v, \tau) = [r + m(\tau - v)] \bar{a}(v, \tau) + I(\tau - v, R(v)) w(\tau) E(\tau - v) [1 - t_L(\tau)] + [1 - I(\tau - v, R(v))] \bar{p}(v, \tau, R(v)) - \bar{c}(v, \tau) - A_Y(\tau) \tilde{z}(\tau).
\] (A.23)

The lifetime budget constraint is still as in (7) but the definitions for lifetime income and social security wealth are both changed. Lifetime income is:

\[
\Pi(v, t, R(v)) = e^{\theta u + M(u)} \left[ \int_t^{v + R(v)} w(\tau) E(\tau - v) e^{-[r(\tau - v) + M(\tau - v)]} d\tau + SSW(v, t, R(v)) \right]
- \int_t^\infty \tilde{z}(\tau) A_Y(\tau) e^{-[r(\tau - v) + M(\tau - v)]} d\tau + SSW(v, t, R(v))
= e^{\theta u + M(u)} A_Y(t) \left[ \int_t^{v + R(v)} \omega_0 E(\tau - v) e^{-[r_0(\tau - v) + M(\tau - v)]} d\tau \right.
- \int_t^\infty \tilde{z}(\tau) e^{-[r_0(\tau - v) + M(\tau - v)]} d\tau + SSW(v, t, R(v)).
\] (A.24)
where $r_g \equiv r - \gamma$ is the growth-corrected interest rate. Social security wealth is:

$$SSW(v, t, R(v)) = e^{ru+M(u)} \left[ \int_{\max(R(v), R_E)}^{\infty} \bar{p}(v, \tau, R(v)) e^{-[r(\tau-v)+M(\tau-v)]} d\tau ight]$$

\[ - \int_{t}^{v+R(v)} t_L(\tau) w(\tau) E(\tau-v)e^{-[r(\tau-v)+M(\tau-v)]} d\tau \]

\[ = e^{ru+M(u)} A_Y(t) \left[ \int_{\max(R(v), R_E)}^{\infty} B(R(v)) e^{-[r_g(\tau-v)+M(\tau-v)]} d\tau ight] \]

\[ - \int_{t}^{v+R(v)} \omega_0 t_L(\tau) E(\tau-v)e^{-[r_g(\tau-v)+M(\tau-v)]} d\tau \]. \quad (A.25)

The Lagrangian for a household of vintage $v$ is:

$$\mathcal{L} \equiv e^{\theta u+M(u)} \int_{t}^{\infty} \ln \bar{e}(v, \tau) e^{-\theta(\tau-v)+M(\tau-v)} d\tau$$

\[ - e^{\theta u+M(u)} \int_{t}^{v+R(v)} D(\tau-v) e^{-\theta(\tau-v)+M(\tau-v)} d\tau \]

\[ + \lambda(t) \cdot \left[ \bar{a}(v, t) + \Pi(v, t, R(v)) - e^{\theta u+M(u)} \int_{t}^{\infty} \bar{e}(v, \tau) e^{-\theta(\tau-v)+M(\tau-v)} d\tau \right]. \]

The first-order conditions for consumption are still given by (A.3) and the Euler equation by (A.4) (with $\sigma = 1$ imposed). It follows that consumption at time $t$ is given by:

$$\Delta(u, \theta) \cdot \bar{e}(v, t) = \bar{a}(v, t) + \Pi(v, t, R(v)). \quad (A.26)$$

The concentrated lifetime utility function is now given by:

$$\bar{\Lambda}(v, t) \equiv e^{\theta u+M(u)} \left[ \int_{t}^{\infty} \ln \bar{e}(v, t) e^{(r-\theta)(\tau-t)} \cdot e^{-\theta(\tau-v)+M(\tau-v)} d\tau \right]$$

\[ - \int_{t}^{v+R(v)} D(\tau-v) e^{-\theta(\tau-v)+M(\tau-v)} d\tau \]. \quad (A.27)

Retracing the steps leading to (A.11) we find:

$$e^{-\theta u+M(u)} \frac{1}{\bar{e}(v, t)} \cdot \frac{d\Pi(v, t, R(v))}{dR(v)} = D(R(v)) e^{-\theta R(v)+M(R(v))}. \quad (A.28)$$

It follows from (A.24) that:

$$\frac{d\Pi(v, t, R(v))}{dR(v)} = \lambda(t) E(R(v)) e^{\nu[u-R(v)]+M(u)-M(R(v))} + \frac{dSSW(v, t, R(v))}{dR(v)}. \quad (A.29)$$

Also, we find from (A.25) that, for $R(v) \geq R_E$:

$$\frac{dSSW(v, t, R(v))}{dR(v)} = e^{r_g u+M(u)} A_Y(t) \cdot \left[ \int_{R(v)}^{\infty} B'(R(v)) e^{-r_g(\tau-v)+M(\tau-v)} d\tau \right]$$

\[ - B(R(v)) e^{-r_g R(v)+M(R(v))} \]

\[ - t_L(v + R(v)) \omega_0 E(R(v)) e^{-r_g R(v)+M(R(v))} \]. \quad (A.30)
Similarly, for $R(v) < R_E$ we get:

$$\frac{dSSW(v,t,R(v))}{dR(v)} = e^{\gamma u + M(u)} A_Y(t) \left[ \int_{R_E}^{\infty} B'(R(v)) e^{-\tau (\tau - v) + M(\tau - v)} d\tau \right]$$

$$- t_L(v + R(v)) \omega_0 E(R(v)) e^{-[\gamma R(v) + M(R(v))]}. \quad (A.31)$$

In either case, we conclude that $\frac{dSSW(v,t,R(v))}{dR(v)}$ is proportional to $A_Y(t)$ but is otherwise only age dependent. Hence, provided taxes ($\hat{z}$ and $t_L$) and the parameters of the pension system (the $B(\cdot)$ function) are time-invariant, equation (A.28) yields a time-invariant solution for the optimal retirement age, $R(v)$. This follows from the fact that the following two quantities and ratios only depend on age (and not on time) in the steady state:

$$\frac{1}{A_Y(t)} \frac{d\bar{I}(v,t,R(v))}{dR(v)} = \text{age dependent}$$

$$\frac{A_Y(t)}{\bar{c}(v,t)} = \frac{A_Y(0)}{\bar{c}(u,0)} e^{\gamma t} = \frac{A_Y(0)}{\bar{c}(u,0)} = \text{age dependent}$$

The only notable difference with the analysis in the paper is that the growth-corrected interest rate, $r_g$, features in the expressions (A.29)–(A.31). But since our objective is not to perform a comparative static exercise with respect to the productivity growth rate, we restrict the analysis in the paper to the case with $\gamma = 0$ so that $r_g = r$.\textsuperscript{1}

### A.5 Transformed retirement age

In this section we provide further details on retirement-age transformations. Burbidge and Robb (1980, p. 424) use a linear space transformation, i.e. instead of using the retirement age directly, they reformulate their model in terms of years or retirement, $T - R$, where $T$ is the fixed planning horizon. Two things are worth noting. First, their linear transformation does not solve the problem of non-convex indifference curves—see below. Second, in our model, $T$ is a stochastic variable and the expected planning horizon at birth is given by $\Delta(0,0)$, where $\Delta(u, \lambda)$ is defined in (11) in the paper. Transforming our model in terms of expected years of retirement (from the perspective of birth), $\Delta(0,0) - R$ suffers from the same defects.

The basic point is that a linear transformation does not guarantee well-behaved indifference curves. Linear transformations simply cannot solve convexity/concavity problems because they are shape-preserving. This can easily be demonstrated in the context of our model. The steady-state concentrated utility function is given by (25) in the text. We write it as $\bar{\Lambda}(u, \bar{I}, R)$ but hold $u$ constant. To determine the slope and curvature of the indifference

\textsuperscript{1}The interested reader is referred to Hu (1995) for a study of the effects of productivity growth on retirement.
curves, we need the following building blocks:

\[ \bar{\Lambda}_\Pi \equiv \frac{\partial \bar{\Lambda}}{\partial \bar{\Pi}} = \left[ \frac{\bar{a}(u) + \bar{l}_i}{\Delta(u, r^*)} \right]^{-1/\sigma} > 0, \tag{A.32} \]

\[ \bar{\Lambda}_R \equiv \frac{\partial \bar{\Lambda}}{\partial R} = -D(R)e^{\delta(u-R)+M(u)-M(R)} < 0, \tag{A.33} \]

\[ \bar{\Lambda}_{\Pi,\Pi} \equiv \frac{\partial^2 \bar{\Lambda}}{\partial \Pi \partial \Pi} = -\frac{1}{\sigma \left[ \bar{a}(u) + \bar{l}_i \right]} \bar{\Lambda}_{ii} < 0, \tag{A.34} \]

\[ \bar{\Lambda}_{\Pi,R} \equiv \frac{\partial^2 \bar{\Lambda}}{\partial \Pi \partial R} = 0, \tag{A.35} \]

\[ \bar{\Lambda}_{R,R} \equiv \frac{\partial^2 \bar{\Lambda}}{\partial R^2} = \left[ \frac{D'(R)}{D(R)} - \theta - m(R) \right] \bar{\Lambda}_R \gtrless 0. \tag{A.36} \]

The slope of the indifference curve in \((R, \bar{\Pi})\)-space is thus:

\[ \frac{d\bar{\Pi}}{dR} \bigg|_{\lambda_0} \equiv -\frac{\bar{\Lambda}_R}{\bar{\Lambda}_\Pi} = D(R)e^{\delta(u-R)+M(u)-M(R)} \left[ \frac{\bar{a}(u) + \bar{\Pi}}{\Delta(u, r^*)} \right]^{-1/\sigma} > 0. \tag{A.37} \]

Hence, the indifference curves are always upward sloping.

To compute the curvature of the indifference curve in \((R, \bar{\Pi})\)-space we must take into account the dependency of \(\bar{\Pi}\) on \(R\) along a given indifference curve. After some manipulation, we find:

\[ \frac{d^2\bar{\Pi}}{dR^2} \bigg|_{\lambda_0} \equiv -\frac{d \left[ \bar{\Lambda}_R / \bar{\Lambda}_\Pi \right]}{dR} = -\frac{1}{\bar{\Lambda}_R^2} \left[ \bar{\Lambda}_{\Pi,R,R} - \bar{\Lambda}_R \bar{\Lambda}_{\Pi,\Pi} \frac{d\bar{\Pi}}{dR} \bigg|_{\lambda_0} \right] \]

\[ = \frac{d\bar{\Pi}}{dR} \bigg|_{\lambda_0} \left[ \frac{1}{\sigma \left[ \bar{a}(u) + \bar{l}_i \right]} \frac{d\bar{\Pi}}{dR} \bigg|_{\lambda_0} + \frac{D'(R)}{D(R)} - \theta - m(R) \right] \gtrless 0. \tag{A.38} \]

Equation (A.38) is a rather intractable expression, and the sign is ambiguous in general. However, numerical simulations reveal that for realistic parameter values the indifference curves are either concave in \(R\) or S-shaped (i.e., convex for small \(R\) and concave for large \(R\)). Similar results can be derived for the specification used by Burbidge and Robb (1980, p. 425), so their assumption that the indifference curves are convex in the relevant region is problematic.

The key point to note is that a linear transformation of the retirement age is unhelpful. Hence, transforming our model in terms of expected years of retirement, \(\Delta (0, 0) - R\), is not useful either.

### A.6 Comparative statics, I

The comparative static effect of increased longevity on retirement—stated in (36) in the text—is computed as follows. First, we combine equations (29) and (31), set \(u = \bar{a}(u) = 0\), and write
the first-order condition for the optimal transformed retirement age, $R^*$, as:

$$
\Gamma (R^*, \psi_m) \equiv \tilde{w} (R^*) - D (R^*) e^{(r-\theta) R^*} \left[ \frac{\overline{y} (0, R^*, \psi_m)}{\Delta (0, r^*, \psi_m)} \right]^{1/\sigma} = 0, \quad (A.39)
$$

where the second-order condition for utility maximization implies that $\partial \Gamma / \partial R^* < 0$, and where $\overline{y} (0, R^*)$ is given by:

$$
\overline{y} (0, R^*, \psi_m) \equiv \int_0^{R^*} \tilde{w}(s) e^{-[r s + M(s) \psi_m]} ds - \bar{z} \Delta (0, r, \psi_m). \quad (A.40)
$$

In equation (A.40), lifetime income depends on the mortality parameter $\psi_m$ because both wage income and poll taxes are annuitized using the actuarially fair annuity rate of interest, $r + m (s, \psi_m)$. In addition, in equation (A.39) the mortality parameter affects the marginal propensity to consume out of total wealth. It follows from (A.39)–(A.40) that:

$$
\frac{dR^*}{d\psi_m} = \frac{\partial \Gamma / \partial \psi_m}{\partial \Gamma / \partial R^*} = \omega_1 \left[ \frac{\partial \Delta (0, r^*, \psi_m) / \partial \psi_m}{\Delta (0, r^*, \psi_m)} - \frac{\partial \overline{y} (0, R^*, \psi_m) / \partial \psi_m}{\overline{y} (0, R^*, \psi_m)} \right] \equiv 0, \quad (A.41)
$$

where $\omega_1 \equiv \frac{\varphi(R^*)}{\sigma \partial \Gamma / \partial R^*}$ is a positive constant.

### A.7 Replacement rates and implicit tax rates

In Section 4 of the paper we mention data on replacement rates and implicit tax rates that were gathered from the various chapters in Gruber and Wise (1999). (We use these data in Section A.11 below.) For convenience we present an overview of these data here. The figures refer to data taken from Tables 1.4, 2.2, 3.5, 5.4, 6.1, 7.1, 8.7, 9.2, 10.4, and 11.1. Note that we report the retirement age in the first column of Tables A1. and A.2. In contrast, Gruber and Wise (1999) report the last age of active employment. Our entries for age 60 are thus equivalent to their entries for age 59.

### A.8 Comparative statics, II

In this section we derive the comparative static effects reported in Table 1 in the text. By combining equations (29) and the lower branch in (39), and setting $u = \tilde{a}(u) = 0$, we obtain the first-order condition which implicitly defines a unique solution for $R^*$:

$$
\Gamma (R^*) \equiv (1 - t_L) \tilde{w}(R^*) + B'(R^*) \Delta (R^*, r) - B(R^*) - e^{(r-\theta) R^*} D (R^*) \left[ \frac{\overline{y} (0, R^*)}{\Delta (0, r^*)} \right]^{1/\sigma} = 0, \quad (A.42)
$$

where $\overline{y} (0, R^*)$ is given by:

$$
\overline{y} (0, R^*) \equiv \int_0^{R^*} \tilde{w}(s) e^{-[r s + M(s)]} ds - \bar{z} \Delta (0, r, \psi_m) + SSW (0, R^*). \quad (A.43)
$$
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*Source:* Gruber and Wise (1999)

Table A.1: Replacement rates in nine OECD countries

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*Source:* Gruber and Wise (1999)

Table A.2: Implicit tax rates in nine OECD countries
The second-order condition of utility maximization implies that $-\zeta_0 \equiv \partial \Gamma / \partial R^* < 0$, and we define the following partial derivative:

$$\zeta_1 \equiv \left| \frac{\partial \Gamma (R^*)}{\partial \bar{t}} \right| \equiv e^{(r-\theta)R^*} \frac{D(R^*)}{\sigma \Delta(0, r^*)} \left[ \frac{\bar{t}(0, R^*)}{\Delta(0, r^*)} \right]^{(1-\sigma)/\sigma} > 0. \quad (A.44)$$

The comparative static effects can now be obtained by straightforward application of Cramer’s rule, i.e. $dR^*/d\bar{z} = (1/\zeta_0) \partial \Gamma / \partial \bar{z}$, $dR^*/dt_L = (1/\zeta_0) \partial \Gamma / \partial t_L$, $dR^*/dB (R) = (1/\zeta_0) \partial \Gamma / \partial B (R)$, and $dR^*/dB' (R) = (1/\zeta_0) \partial \Gamma / \partial B' (R)$.

### A.9 Gradual demographic shocks

In the paper we mention the transitional effects of gradual demographic shocks. In Figures A.1–A.4 we illustrate the transition paths for both gradual shocks (solid) and stepwise shocks (dashed). For the gradual shock, we change the demographic parameters slowly over a period of 20 years, using a linear interpolation between the initial parameters and the ultimate parameters. The stepwise shocks are illustrated in Figure 4 in the paper. As we mention in the paper, transition is much slower for the gradual shocks. Qualitatively, however, the transition paths are very similar. For that reason we focus on stepwise shocks in the paper.

### A.10 Combined demographic shock

In the paper we study the demographic shocks separately. We did this mainly for expositional purposes. Of course, one of the key features of the demographic transition during the last half century or so has been the combined baby bust and longevity boost. Our quantitative model can easily be used to illustrate the effects of such a combined shock. We give exactly the same shocks to the birth rate and the parameters of the mortality function as in Table 2 in the paper. In Figures A.5–A.7 illustrates the transition paths for the various macroeconomic variables. The cases differ only in the method employed by the government to balance its budget. In Figure A.5 the lump-sum tax is adjusted, in A.6 the labour income tax, and in A.7 the EEA. The welfare effects of the three cases are illustrated in Figure A.8.

### A.11 Defense of Assumption 3

Our defense for Assumption 3 proceeds as follows. In the literature (e.g. Gruber and Wise, 1999, 2004; OECD, 2005), retirement incentives are typically summarized with the EEA, the NRA, the replacement rate, the pattern of benefit accrual, and the implicit tax rate. Using these incentive indicators, it is possible to derive the shape and slope of the lifetime income function.
Figure A.1: Aggregate effect of a baby bust (ε balances the budget)
Figure A.2: Welfare effects
Figure A.3: Aggregate effect of reduced adult mortality ($\bar{z}$ balances the budget)
Figure A.4: Aggregate effect of reduced adult mortality (EEA balances the budget)
Figure A.5: Aggregate effect of a combined demographic shock ($\xi$ balances the budget)
Figure A.6: Aggregate effect of a combined demographic shock ($t_L$ balances the budget)
Figure A.7: Aggregate effect of a combined demographic shock (EEA balances the budget)
Figure A.8: Welfare effects of a combined demographic shock
The replacement rate is defined as the ratio of the retirement benefit to net wages. In terms of our theoretical framework, the replacement rate \( RR \) for someone retiring at or after the EEA is given by:

\[
RR(R) \equiv \frac{B(R)}{(1 - t_L)\bar{w}(R)} \quad \text{for } R \geq R_E. \tag{A.45}
\]

This replacement rate differs greatly between countries, but also between ages. As can be seen from Table A.1, the replacement rate for France starts at 92% at the EEA (59 years) and slowly increases to 96% thereafter. In contrast, in Canada the replacement rate starts at 18% at age 59, after which it increases to 91% for an 69 year old.

The benefit accrual is the nominal change in social security wealth if one postpones retirement by one year (i.e., it is \( \partial SSW / \partial R \) in terms of our model). The benefit accrual depends on the age of the individual. To compare accrual levels, we can either hold constant the age at which social security wealth is evaluated or evaluate social security wealth at the actual retirement age. Both methods have their advantages and drawbacks. The first makes it easier to track social security wealth over time, the second allows for easier comparison of retirement incentives at the retirement age. In this paper we will use the second definition because it allows for easier mathematics in the transformed retirement age \( S \)-space. By differentiating equation (38) with respect to \( R \) and evaluating at age \( u = R \) we obtain:

\[
ACC(R) \equiv \frac{\partial SSW(u, R)}{\partial R} \bigg|_{u=R} = \begin{cases} 
B'(R)\Pi(R, R_E, \infty, r) - t_L\bar{w}(R) & \text{for } R < R_E \\
B'(R)\Delta(R, r) - t_L\bar{w}(R) - B(R) & \text{for } R > R_E
\end{cases} \tag{A.46}
\]

The level of benefit accrual is closely connected to the slope of the lifetime income function (as a function of \( S \)). Indeed, by combining (39) and (A.46) we obtain:

\[
\frac{d\bar{I}}{dS} = ACC(R) + \bar{w}(R). \tag{A.47}
\]

In this context, actuarial adjustment of the pension benefit is called fair if and only if \( ACC(R) = 0 \) for all \( R \), i.e. \( t_L\bar{w}(R) = B'(R)\Pi(R, R_E, \infty, r) = B'(R)\Delta(R, r) - B(R) \).\(^2\)

The benefit accrual depends on the monetary units in which social retirement benefits are measured and the age at which the social security wealth is evaluated. Most studies standardize the benefit accrual either with the level of social security wealth or with the present value of net wages. The first measure is the accrual rate, the second measure is the implicit subsidy.

The negative of the implicit subsidy is the implicit tax rate (IT), measuring the additional tax rate one ‘implicitly’ faces over and above the normal taxes. A negative accrual is an additional tax on labour and discourages work. Conversely, a positive accrual is an implicit tax.

\(^2\)As Gruber and Wise put it, under a less than actuarially fair system, “once benefits are available, a person who continues to work for an additional year will typically receive less in social security benefits over his lifetime than if he quit work and started to receive benefits at the first opportunity” (2005, p. 5). See stylized fact (SF4).
subsidy on labour and encourages the individual to work an additional year. Since we evaluate the accrued level at the retirement age, we should not discount the net wage rate, so the implicit tax can be written in terms of our model as:

\[ IT(R) \equiv -\frac{ACC(R)}{(1 - t_L)\bar{w}(R)}. \] (A.48)

By substituting this expression for the implicit tax rate into equation (A.47), we can write the slope of the lifetime income function as:

\[ \frac{d\bar{I}}{dS} = (1 - t_L)\bar{w}(R) \left[ \frac{1}{1 - t_L} - IT(R) \right]. \] (A.49)

Under the maintained assumption that gross wages are constant for higher ages (typically in the range 55–70), equation (A.49) can be used to compute the shape of the lifetime income function. Dividing lifetime income by net wages gives a ‘standardized’ measure of lifetime income which is more easily comparable between countries. The only caveat is that we do not have data on the relevant labour income tax, so we cannot estimate \( \frac{1}{1 - t_L} \). This is not a problem, however, because we are only interested in the curvature of the lifetime income function (its convexity or concavity). The term \( \frac{1}{1 - t_L} \) only influences the slope of the lifetime income profile, but it has no effect on its curvature. To get an idea of the shape of the lifetime income profile we proceed as if there are no labour income taxes. Since the fraction \( \frac{1}{1 - t_L} \) has a lower bound of 1 (because in reality taxes are positive), we thus obtain a conservative estimate for lifetime income.

Figure A.9 shows the lifetime income profiles for nine OECD countries, as we computed them using the implicit tax rates published in Gruber and Wise (1999). For convenience these tax rates have also been reported in Table A.2. The lifetime income profiles are normalized at age 54 to enable comparison between countries. The graphs contain two horizontal axes. The main (lower) horizontal axis measures the transformed retirement age, \( S \), whilst the secondary (upper) axis shows the corresponding values for the untransformed retirement age, \( R \). The effect of the non-monotonic scaling is clearly visible.

Figure A.9(a) characterizes the retirement systems in continental Europe. Lifetime income profiles are increasing in the retirement age, more or less concave and usually have a clear kink at the EEA (which is 60 years in most countries, but only 55 in Italy) or at the NRA (65 years). A notable exception is formed by the Netherlands. Its profile has a sharp spike at age 59 and decreases until the NRA of 65. The pension system in the Netherlands is such that there exists an implicit tax of more than 141% of net earnings. The pension benefits someone receives hardly increases if someone retires after age 59, but one still has to pay contributions to the pension system. Moreover, replacement rates are very high due to the

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Some contributors to Gruber and Wise (1999, 2004) do not provide information concerning the age at which they evaluate the present value of social security wealth. This is not a problem, however, provided we do not use the accrual levels, but the accrual rates or implicit tax rates.
usually generous, but mandatory, company pension systems. It is not surprising that most employees in the Netherlands retire at age 60.\footnote{The graph is based on retirement schemes as they existed in the late 1980s. More recent figures published in Gruber and Wise (2004) provide a qualitatively similar picture.}

Figure A.9(b) characterizes the retirement systems in Canada, Japan, the United Kingdom, and the United States. A feature of these systems is the rather low implicit tax rates. A low implicit tax is a symptom of either (i) a near-actuarially fair system, or (ii) a rather poorly developed pension system.\footnote{The former is the case in Canada and the United States, whereas the latter is relevant for the United Kingdom. Japan is a somewhat mixed case.} As a result of the small implicit tax rates, the wage effect in the lifetime income function (A.49) is dominant and the standardized lifetime income profiles are roughly the same in these four countries.

Although Figure A.9 only shows conservative estimates for the lifetime income profiles, it does give an accurate picture concerning the shape of these profiles. Apart from Spain and the Netherlands, the income profiles are concave and may feature a kink at the EEA. Even though the profile for the Netherlands is not concave, there is a pronounced kink at age 60 which precludes individuals from working beyond that age.

### A.12 Steady-state population composition

In Figure A.10 we illustrate the effects on the age composition of the population of the baby bust (panel (a)) and the longevity boost (panel (b)). As we mention in Section 6.1 of the paper, the baby bust makes the population older, i.e. mass of the distribution is shifted from younger to older ages. Similarly, reduced adult mortality also makes the population older. The population (semi-)pyramid is squeezed for ages up to about 65, but thickened for higher ages.

### A.13 Age profile of wealth components

In Figure A.11 we visualize the scaled steady-state age profiles of financial assets and lifetime income. Panel (a) shows that the profile for assets is inverse U-shaped and reaches a peak at $u = R_E$. After retirement, the agent slowly decumulates his assets. Panel (b) shows that there is a slight kink in the profile for lifetime income at $u = R_E$. 

23
Source: Gruber and Wise (1999) and own calculations.

Figure A.9: Lifetime income profiles for nine OECD countries (lower bound)
Figure A.10: Steady-state population composition

References


Figure A.11: Individual steady-state wealth profiles

