The investment tax credit under monopolistic competition: Mathematical appendix

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This appendix contains the derivations for the most important results discussed in the text. Equations without the 'A' prefix can be found in the text itself.

1 Derivation of equations (17)-(18)

Since the production factors are mobile across firms, hiring production factors is simply a rental decision, where the rental charges on labour and capital are denoted by $R_L(\tau)$ and $R_K(\tau)$, respectively:

$$R_L(\tau) \equiv W_C(\tau)P_C(\tau), \quad R_K(\tau) \equiv P_I(\tau)(1-\tau_I(\tau))\left[R(\tau)+\delta-\frac{\dot{P}_I(\tau)}{P_I(\tau)}+\frac{\dot{\tau}_I(\tau)}{1-\tau_I(\tau)}\right].$$

Each firm thus faces the same rental charge on the production factors. Cost minimization, given the technology (10), ensures that the cost function for all firms in the differentiated sector is defined as:

$$TC\left[Y_j(\tau),R_K(\tau),R_L(\tau)\right]=\left[Y_j(\tau)+f\right]^{1/\lambda}F^*\left[R_K(\tau),R_L(\tau)\right],$$

where $F^*[.]$ represents the dual unit cost function corresponding to the gross production function $F[.]$, given in equation (10):

$$F^*\left[R_K(\tau),R_L(\tau)\right]\equiv\left(\frac{R_K(\tau)}{1-\epsilon_L}\right)^{1-\epsilon_L}\left(\frac{R_L(\tau)}{\epsilon_L}\right)^{\epsilon_L}.$$

In view of the definition of the cost function, the following expression for the marginal cost of each firm can be derived:

$$MC_j(\tau)\equiv\frac{1}{\lambda\eta_j(\tau)}\frac{TC_j(\tau)}{Y_j(\tau)}. \tag{A.1}$$

Each firm sets its price equal to a markup, $\mu_j(\tau)$, times marginal cost:

$$P_j(\tau)=\mu_j(\tau)MC_j(\tau). \tag{A.2}$$

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By using (A.1)-(A.2), the profit rate, $\pi_j(\tau)$, can be written as:

$$\begin{aligned}\pi_j(\tau) &\equiv \frac{P_j(\tau)Y_j(\tau) - TC_j(\tau)}{TC_j(\tau)} = \frac{P_j(\tau)Y_j(\tau) - \left[\lambda\eta_j(\tau)/\mu_j(\tau)\right] P_j(\tau)Y_j(\tau)}{TC_j(\tau)} \\ &= \frac{\mu_j(\tau) - \lambda\eta_j(\tau)}{\lambda\eta_j(\tau)}.\end{aligned}$$

which coincides with equation (17) in the text. The zero profit condition follows from (18) in a straightforward fashion.□

2 Proof of Theorem 1

We first construct the phase diagram in $(C_I(t), K(t))$ space under the assumption of free exit/entry and a constant markup. Some useful relationships can be derived from (T1.4)-(T1.5):

$$\begin{aligned}Y(t) + N(t)^{\chi_I} f &= Y(t)\left(\frac{\mu}{\lambda}\right) = N(t)^{\chi_I} f\left(\frac{\mu}{\mu-\lambda}\right), \\ r_I(t) &= (1-\epsilon_L)\frac{Y(t)}{K(t)}\frac{1}{1-\tau_I(t)} - \delta - \frac{\dot{\tau}_I(t)}{1-\tau_I(t)}, \\ Y(t) &= \left(\frac{\lambda f}{\mu-\lambda}\right)\left(\frac{\mu-\lambda}{\mu f}\right)^{\chi_I/\lambda} K(t)^{\chi_I(1-\epsilon_L)} \equiv \Omega K(t)^{1-\phi},\end{aligned} \tag{A.3}$$

where $0 < \phi \equiv 1 - \chi_I(1-\epsilon_L) < 1$. From (A.3) we can determine the behaviour of the investment rate of interest, $r_I(t)$, as a function of $K(t)$. Under the assumption that $0 < \phi < 1$, we obtain:

$$\lim_{K(t)\downarrow 0} r_I(t) = \lim_{K(t)\downarrow 0}\left(\frac{(1-\epsilon_L)\Omega}{1-\tau_I(t)}\right) K(t)^{-\phi} - \delta - \frac{\dot{\tau}_I(t)}{1-\tau_I(t)} = \infty, \tag{A.4}$$

$$\lim_{K(t)\to\infty} r_I(t) = -\left[\delta + \frac{\dot{\tau}_I(t)}{1-\tau_I(t)}\right], \quad \frac{\partial r_I(t)}{\partial K(t)} = -\left(\frac{\phi(1-\epsilon_L)\Omega}{1-\tau_I}\right) K(t)^{-(1+\phi)} < 0. \tag{A.5}$$

Real consumption in terms of the investment price index is $C_I(t)$ and real debt is $B_I(t)$. The equilibrium relations $\dot{K}(t) = 0$ and $\dot{C}_I(t) = 0$ may be written (for given $B_I(0)$ and τ_I) as:

$$C_I(t) = Y(t) - \delta K(t) = \Omega K(t)^{1-\phi} - \delta K(t), \quad (\dot{K}(t) = 0 \text{ line}), \tag{A.6}$$

$$C_I(t) = \frac{\beta(\alpha+\beta)\left[(1-\tau_I)K(t) + B_I(0)\right]}{r_I(t) - \alpha}, \quad (\dot{C}_I(t) = 0 \text{ line}). \tag{A.7}$$

The behaviour of (A.6)-(A.7) around $K(t) = 0$ is then:

$$C_I(t)\big|_{\dot{K}(t)=0} \propto K(t)^{1-\phi},$$

$$C_I(t)\big|_{\dot{C}_I(t)=0} \propto \begin{cases} K(t)^{1+\phi} & \text{for } B_I(0) = 0 \\ K(t)^{\phi} & \text{for } B_I(0) > 0 \end{cases}$$

The slopes of equations (A.6) and (A.7) for $K(t) > 0$ are:

$$\left.\frac{\partial C_I(t)}{\partial K(t)}\right|_{\dot{K}(t)=0} = (1-\phi)\Omega K(t)^{-\phi} - \delta = (1-\phi)\left(\frac{Y(t)}{K(t)}\right) - \delta, \tag{A.8}$$

and

$$\left.\frac{\partial C_I(t)}{\partial K(t)}\right|_{\dot{C}_I(t)=0} = \frac{C_I(t)}{K(t)}\left[\frac{(1-\tau_I)K(t)}{(1-\tau_I)K(t) + B_I(0)} - \frac{K(t)}{r_I(t)-\alpha}\frac{\partial r_I(t)}{\partial K(t)}\right] > 0.$$

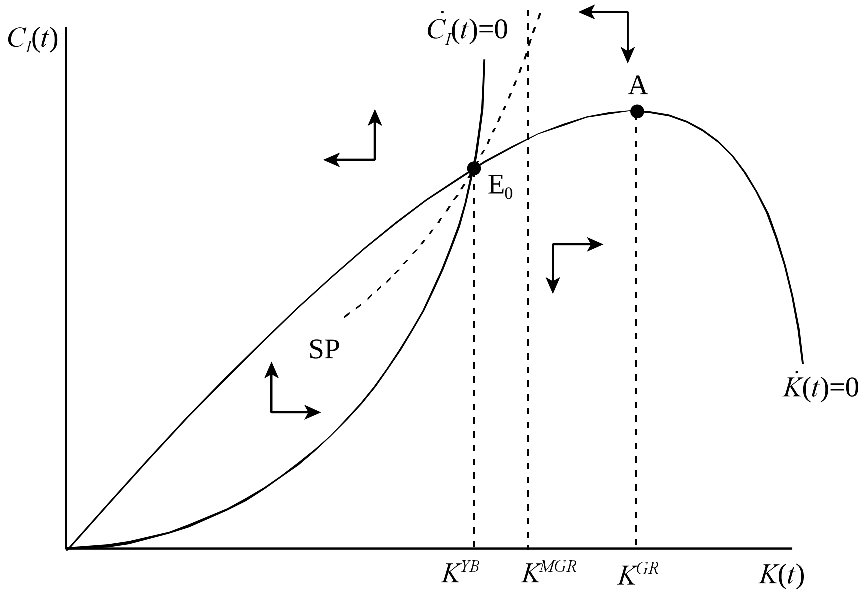

Figure 1: Phase diagram

It follows from (A.8) that the golden rule capital stock K^{GR} is such that:

$$\left.\frac{\partial C_I(t)}{\partial K(t)}\right|_{\dot{K}(t)=0} = 0 \Rightarrow \left(\frac{Y(t)}{K(t)}\right)^{GR} = \frac{\delta}{1-\phi}. \tag{A.9}$$

The first expression in equation (A.3) may be rewritten as:

$$(1-\epsilon_L)\frac{Y(t)}{K(t)} = (1-\tau_I)\left[r_I(t) + \delta + \frac{\dot{\tau}_I(t)}{1-\tau_I(t)}\right]. \tag{A.10}$$

It follows from (A.9)-(A.10) that (for $\dot{\tau}_I(t) = 0$) the golden rule rate of interest, r_I^{GR}, can be written as:

$$\begin{aligned} (1-\epsilon_L)\left(\frac{Y(t)}{K(t)}\right)^{GR} &= (1-\tau_I)\left[r_I^{GR} + \delta\right] = \frac{(1-\epsilon_L)\delta}{1-\phi} \\ &\Rightarrow \; r_I^{GR} = -\frac{\delta\left[(1-\tau_I)\chi_I - 1\right]}{\chi_I(1-\tau_I)}. \end{aligned} \tag{A.11}$$

Equation (A.11) implies that $\lim_{\chi_I \to \infty} r_I^{GR} = -\delta < 0$ and $r_I^{GR} = 0 \Leftrightarrow \chi_I = 1/(1-\tau_I)$. It follows that, for $\tau_I = 0$, $r_I^{GR} \leq 0$ for all $\chi_I \geq 1$. Hence, $r_I > \alpha > r_I^{GR}$, and the steady state is dynamically efficient.

The phase diagram in (C_I, K) space is drawn in Figure 1. The steady-state equilibrium is at E_0, and the capital stock is $K^{YB} < K^{MGR} < K^{GR}$, where K^{MGR} is the capital stock associated with the modified golden rule, for which $r_I = \alpha$, and K^{YB} is the Yaari-Blanchard equilibrium capital stock. By loglinearising the model around the stationary point E_0 we obtain:

$$\begin{bmatrix} \dot{\tilde{K}}(t) \\ \dot{\tilde{C}}_I(t) \end{bmatrix} = \Delta \begin{bmatrix} \tilde{K}(t) \\ \tilde{C}_I(t) \end{bmatrix}, \Delta \equiv \begin{bmatrix} (\delta/\omega_I)(\omega_C - \phi) & -\delta\omega_C/\omega_I \\ -(r_I - \alpha) - \phi(r_I + \delta) & r_I - \alpha \end{bmatrix}, \tag{A.12}$$

with $\omega_I \equiv 1 - \omega_C = I/Y = \delta K/Y$, $\omega_C \equiv C_I/Y$, $\dot{\tilde{K}}(t) \equiv d\dot{K}(t)/K$, $\dot{\tilde{C}}_I(t) \equiv d\dot{C}_I(t)C_I$, $\tilde{K}(t) \equiv dK(t)/K$, $\tilde{C}_I(t) \equiv dC_I(t)/C_I$. For stability it is necessary and sufficient that the determinant of Δ be negative. After some manipulation we obtain:

$$|\Delta| \equiv -r^* h^* = -\left(\frac{\delta\phi}{\omega_I}\right)[r_I - \alpha + \omega_C(r_I + \delta)] < 0, \tag{A.13}$$

so that $\mid \Delta \mid < 0$ is guaranteed since we assume that $\phi > 0$. From the saddle point property of the equilibrium it follows that Δ has distinct eigenvalues $-h^* < 0 < r^*$ (recall that $\mid \Delta \mid = -r^* h^* < 0$).

The steady-state interest rate r_I does not depend on χ_I or χ_C. This result is important in some of the proofs below, and can be demonstrated as follows. By using the equilibrium equations (A.6)-(A.7) and assuming that $B_I(0) = 0$, we obtain an equation in Y/K:

$$Y - \delta K - \frac{\beta(\alpha+\beta)(1-\tau_I)K}{r_I - \alpha} = 0 \Leftrightarrow$$

$$(r_I - \alpha)\left(\frac{Y}{K} - \delta\right) - \beta(\alpha+\beta)(1-\tau_I) = 0 \Leftrightarrow$$

$$\left(\frac{1-\epsilon_L}{1-\tau_I}\right)\left(\frac{Y}{K}\right)^2 - \left(\frac{\delta(1-\epsilon_L)}{1-\tau_I} + \delta + \alpha\right)\left(\frac{Y}{K}\right) + \delta(\delta+\alpha) - \beta(\alpha+\beta)(1-\tau_I) = 0, \tag{A.14}$$

where we have used the fact that $r_I + \delta = (1-\epsilon_L)(Y/K)/(1-\tau_I)$ (see (A.3)) to get from the second to the third line. The larger root of the quadratic (A.14) is the equilibrium output-capital ratio:

$$\begin{aligned}\frac{Y}{K} &= \frac{1-\tau_I}{2(1-\epsilon_L)}\left[\frac{\delta(1-\epsilon_L)}{1-\tau_I} + \delta + \alpha + \right. \\ &\left.\left[\left(\frac{\delta(1-\epsilon_L)}{1-\tau_I} + \delta + \alpha\right)^2 - 4\left(\frac{1-\epsilon_L}{1-\tau_I}\right)[\delta(\delta+\alpha) - \beta(\alpha+\beta)(1-\tau_I)]\right]^{1/2}\right] \\ &= \frac{1-\tau_I}{2(1-\epsilon_L)}\left[\frac{\delta(1-\epsilon_L)}{1-\tau_I} + \delta + \alpha + \left[\left(\delta + \alpha - \frac{\delta(1-\epsilon_L)}{1-\tau_I}\right)^2 + 4(1-\epsilon_L)\beta(\alpha+\beta)\right]^{1/2}\right],\end{aligned}$$

which implies that

$$\frac{Y}{K} = \frac{(\delta+\alpha)(1-\tau_I)}{1-\epsilon_L} > 0, \qquad \text{for } \beta = 0.$$

Since $r_I + \delta = (1-\epsilon_L)(Y/K)/(1-\tau_I)$, the steady-state rate of interest is given by:

$$r_I + \delta = \frac{1}{2}\left[\frac{\delta(1-\epsilon_L)}{1-\tau_I} + \delta + \alpha + \left[\left(\delta + \alpha - \frac{\delta(1-\epsilon_L)}{1-\tau_I}\right)^2 + 4(1-\epsilon_L)\beta(\alpha+\beta)\right]^{1/2}\right]. \tag{A.15}$$

Since (A.15) does not contain χ_I or χ_C, the steady-state interest rate r_I does not depend on these parameters.

3 Proof of Theorem 2

The current value Hamiltonian is:

$$\mathrm{H} = \log C_I + (\chi_C - \chi_I)\log N + \lambda_K [Y - C_I - \delta K] - \lambda_Y \left[Y - N^{\chi_I - \lambda}\mathrm{F}(K,1) + N^{\chi_I} f\right].$$

Let $N \geq N^{MIN}$. Then the first-order conditions are:

$$\left(\frac{\partial \mathrm{H}}{\partial N} \leq 0 :\right) \; N^{-1}\left[\chi_C - \chi_I - \lambda_Y\left(\chi_I N^{\chi_I} f - (\chi_I - \lambda) N^{\chi_I - \lambda}\mathrm{F}(K,1)\right)\right] \leq 0,$$

$$N \geq N^{MIN}, \left(N - N^{MIN}\right)\frac{\partial H}{\partial N} = 0, \tag{A.16}$$

$$\left(\frac{\partial \mathrm{H}}{\partial Y} = 0 :\right) \; \lambda_K - \lambda_Y = 0, \tag{A.17}$$

$$\left(\frac{\partial \mathrm{H}}{\partial C_I} = 0 :\right) \; C_I^{-1} - \lambda_K = 0, \tag{A.18}$$

$$\left(\dot{\lambda}_K - \alpha\lambda_K = -\frac{\partial \mathrm{H}}{\partial K}\right) \; \dot{\lambda}_K = (\alpha + \delta)\lambda_K - \lambda_Y N^{\chi_I - \lambda}\mathrm{F}_K(K,1), \tag{A.19}$$

$$\left(\frac{\partial H}{\partial \lambda_Y} = 0 :\right) \; Y = N^{\chi_I - \lambda}\mathrm{F}(K,1) - N^{\chi_I} f, \tag{A.20}$$

$$\left(\dot{K} = \frac{\partial \mathrm{H}}{\partial \lambda_K} :\right) \; \dot{K} = Y - C_I - \delta K.$$

The condition (A.19) can be rewritten by using (A.17) and (A.18) as:

$$\frac{\dot{\lambda}_K}{\lambda_K} = \alpha + \delta - N^{\chi_I - \lambda}\mathrm{F}_K(K,1) \;\Rightarrow\; \frac{\dot{C}_I}{C_I} = N^{\chi_I - \lambda}\mathrm{F}_K(K,1) - (\alpha + \delta). \tag{A.21}$$

The expression on the left-hand side of (A.21) can be written as:

$$\frac{\dot{C}_I}{C_I} = r_I - \alpha, \; r_I \equiv N^{\chi_I - \lambda}\mathrm{F}_K(K,1) - \delta, \tag{A.22}$$

which coincides with equation (23) in the text. Substituting (A.17), (A.18), and (A.20) into (A.18) yields

$$\begin{aligned}
\left[(\chi_C - \chi_I) - C_I^{-1}\left(\lambda N_I^{\chi} f - (\chi_I - \lambda)Y\right)\right]\left(N - N^{MIN}\right) &= 0 \Rightarrow \\
\left[(\chi_C - \chi_I) - (Y/C_I)\left[\lambda(\hat{\eta} - 1) - (\chi_I - \lambda)\right]\right]\left(N - N^{MIN}\right) &= 0 \Rightarrow \\
\left[\omega_C \chi_C + (1 - \omega_C)\chi_I - \lambda\hat{\eta}\right]\left(N - N^{MIN}\right) &= 0,
\end{aligned} \tag{A.23}$$

where ω_C is the share of private consumption in national product and where $\hat{\eta}$ represents the extent of increasing returns to scale at firm level due to the existence of fixed costs:

$$\hat{\eta} \equiv \frac{f + \hat{\hat{Y}}}{\hat{\hat{Y}}.} \tag{A.24}$$

Equation (A.23) coincides with (24) in the text.

The optimal number of firms can be inferred from (A.23)-(A.24). For K given, production per firm can be written as:

$$\bar{Y} + f = N^{-\lambda}\mathrm{F}(K, 1), \tag{A.25}$$

where we have used (A.20). Equation (A.25) can be used to infer the following results:

$$\lim_{N \to 0} \hat{\eta} = 1, \ \frac{d\hat{\eta}}{dN} > 0. \tag{A.26}$$

If there are very few large firms, fixed costs play an insignificant role and $\hat{\eta}$ goes to unity. The more firms there are, the smaller are individual firms, and the more significant become fixed costs. Hence, $\hat{\eta}$ rises with N.

We now distinguish two cases. First, for $\max[\chi_I, \chi_C] < \lambda$, the expression in square brackets in (A.23) is negative for all $N \geq 0$, because $\omega_C\chi_C + (1-\omega_C)\chi_I < \lambda$ and $\lambda\hat{\eta} \geq \lambda$. This ensures that the optimal number of firms equals its lower bound N^{MIN}. The second case refers to the situation with $\min[\chi_I, \chi_C] > \lambda$. This implies that $\omega_C\chi_C + (1-\omega_C)\chi_I > \lambda$ and, in view of (A.26), that the term in square brackets in (A.23) is positive for $N \to 0$. For a sufficiently small N^{MIN} an interior maximum occurs as $\hat{\eta}$ rises with N.□

4 Proof of Theorem 3

Decentralisation proceeds as follows. First, by using the results in (A.22), the marginal productivity condition for capital in the social optimum can be written as:

$$r_I + \delta = N^{\chi_I - \lambda}\mathrm{F}_K(K, 1). \tag{A.27}$$

By rewriting (T1.6), we obtain the following expression for the marginal product of capital:

$$N^{\chi_I - \lambda}\mathrm{F}_K[K, 1] = \mu(1-\tau_I)\left(r_I + \delta + \frac{\dot{\tau}_I}{1-\tau_I}\right). \tag{A.28}$$

By comparing (A.27) to (A.28) we see that, provided the number of firms and the capital stock are at their optimal levels, the market yields the socially optimal investment rate of interest if $\mu[1-\hat{\tau}_I] = 1$. This implies that the socially optimal investment subsidy satisfies $\hat{\tau}_I = 1/\sigma_C$ or, equivalently, $\hat{\tau}_I/(1-\hat{\tau}_I) = \mu - 1$. The optimal ITC is thus directed solely at the removal of the product market imperfection, incorporated in the mark-up μ of price over marginal cost.

It remains to match the conditions determining the number of firms. The internal solution for N statisfies, by equation (A.23), the following equality:

$$\lambda\hat{\eta} = \omega_C\chi_C + (1-\omega_C)\chi_I.$$

In view of the definition of excess profit per firm (see (17) in the text), the socially optimal firm size implies:

$$\lambda\hat{\eta}(1+\bar{\pi}) = \mu \Leftrightarrow (1+\bar{\pi})[\omega_C\chi_C + (1-\omega_C)\chi_I] = \mu, \tag{A.29}$$

where $\bar{\pi}$ is the excess profit rate per firm in the symmetric equilibrium. By solving (A.29) for $\bar{\pi}$ we obtain:

$$\bar{\pi} = \frac{(\mu-1) - [\omega_C\chi_C + (1-\omega_C)\chi_I - 1]}{\omega_C\chi_C + (1-\omega_C)\chi_I}. \tag{A.30}$$

The three cases mentioned in Theorem 3 follow directly from (A.30). First, if $\chi_I = \chi_C = \mu$, then $\bar{\pi} = 0$. Hence, no lump-sum firm transfers or taxes are needed. Second, if $\min[\chi_I, \chi_C] > \mu$ then $\bar{\pi} < 0$. It is socially optimal to produce lots of varieties and subsidies to firms are required. Third, if $\max[\chi_I, \chi_C] < \mu$ then $\bar{\pi} > 0$ and it is socially optimal to choke off the number of varieties that are produced in the market by levying lump-sum taxes on individual firms.□

5 Allocation effects of the investment tax credit

The effect of an investment tax credit on the equilibrium relations (A.6) and (A.7) is:

$$\begin{aligned} \left.\frac{\partial C_I}{\partial \tau_I}\right|_{\dot{K}=0} &= 0, \\ \left.\frac{\partial C_I}{\partial \tau_I}\right|_{\dot{C}_I=0} &= -\frac{\left[C_I\left(\frac{\partial r_I}{\partial \tau_I}\right) + \beta(\alpha+\beta)K\right]}{r_I - \alpha} < 0. \end{aligned}$$

It follows that, since $\partial r_I/\partial \tau_I > 0$, the $\dot{C}_I = 0$ schedule shifts down and to the right. Expanding around the steady state, the response path to an investment tax credit satisfies:

$$\begin{bmatrix} \dot{\tilde{K}}(t) \\ \dot{\tilde{C}}_I(t) \end{bmatrix} = \Delta \begin{bmatrix} \tilde{K}(t) \\ \tilde{C}_I(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \gamma(t) \end{bmatrix}, \tag{A.31}$$

where Δ is given in (A.12) and $\gamma(t)$ is the forcing term:

$$\gamma(t) \equiv \gamma_I \tilde{\tau}_I(t) - \dot{\tilde{\tau}}_I(t) - \gamma_B \tilde{B}_I(t),$$

where $\gamma_I \equiv r_I + \delta + r_I - \alpha > 0$ and $\gamma_B \equiv (r_I - \alpha)/[\omega_K(1-\tau_I)] > 0$. Taking the Laplace transform of (A.31) yields:

$$\begin{bmatrix} \mathcal{L}\{\tilde{K}, s\} \\ \mathcal{L}\{\tilde{C}_I, s\} \end{bmatrix} = (sI - \Delta)^{-1} \begin{bmatrix} \tilde{K}(0) \\ \tilde{C}_I(0) + \mathcal{L}\{\gamma, s\} \end{bmatrix}, \tag{A.32}$$

where $\mathcal{L}\{x, s\}$ denotes the Laplace transform of x with weighting factor s. Since is a predetermined state variable, $\tilde{K}(0) = 0$. Because $\mathcal{L}\{\tilde{C}_I, r^*\}$ is bounded, we obtain (see Judd, 1982):

$$\tilde{C}_I(0) = -\mathcal{L}\{\gamma, r^*\}, \tag{A.33}$$

where $r^* > 0$ is the unstable characteristic root of Δ (see (A.12)). The general solution for $(\tilde{K}(t), \tilde{C}_I(t))$ is obtained by substituting the expression for $\tilde{C}_I(0)$ given in (A.33) into equation (A.32):

$$\begin{bmatrix} \mathcal{L}\{\tilde{K}, s\} \\ \mathcal{L}\{\tilde{C}_I, s\} \end{bmatrix} = (sI - \Delta)^{-1} \begin{bmatrix} 0 \\ \mathcal{L}\{\gamma, s\} - \mathcal{L}\{\gamma, r^*\} \end{bmatrix}. \tag{A.34}$$

We obtain for the long-run multipliers:

$$\lim_{t\to\infty} \begin{bmatrix} \tilde{K}(t) \\ \tilde{C}_I(t) \end{bmatrix} = \lim_{s\downarrow 0} \begin{bmatrix} s\mathcal{L}\{\tilde{K}, s\} \\ s\mathcal{L}\{\tilde{C}_I, s\} \end{bmatrix} = -\Delta^{-1} \begin{bmatrix} 0 \\ \gamma(\infty) \end{bmatrix} = \left(\frac{\gamma(\infty)}{r^* h^*}\right) \begin{bmatrix} -\delta_{12} \\ \delta_{11} \end{bmatrix}. \tag{A.35}$$

We postulate that the impulse in the ITC is a unit-step impulse:

$$\tilde{\tau}_I(t) = \tilde{\tau}_I(0) = \tilde{\tau}_I(\infty), t \geq 0, \tag{A.36}$$

and that the debt path is parameterised as:

$$\tilde{B}_I(t) = \tilde{B}_I(0)e^{-\xi_B t} + \tilde{B}_I(\infty)\left(1 - e^{-\xi_B t}\right), \xi_B > 0. \tag{A.37}$$

Taking the Laplace transform of (A.36)-(A.37), we can derive the following result:

$$\frac{\mathcal{L}\{\gamma, s\} - \mathcal{L}\{\gamma, r^*\}}{s - r^*} = -\frac{\gamma_I \tilde{\tau}_I(\infty)}{r^*}\frac{1}{s} + \gamma_B\left[\frac{\tilde{B}_I(\infty)}{r^*}\frac{1}{s} + \frac{\tilde{B}_I(0) - \tilde{B}_I(\infty)}{r^* + \xi_B}\frac{1}{s + \xi_B}\right].$$

Equation (A.34) can be rewritten as:

$$\left[\begin{array}{c} \mathcal{L}\{\tilde{K}, s\} \\ \mathcal{L}\{\tilde{C}_I, s\} \end{array}\right] = \frac{1}{(s + h^*)(s - r^*)}\left[\begin{array}{cc} s - \delta_{22} & \delta_{12} \\ \delta_{21} & s - \delta_{11} \end{array}\right]\left[\begin{array}{c} 0 \\ \mathcal{L}\{\gamma, s\} - \mathcal{L}\{\gamma, r^*\} \end{array}\right], \tag{A.38}$$

where δ_{ij} is the ij-th element of the matrix Δ given in (A.12). The first equation can be rewritten as:

$$\begin{aligned} \mathcal{L}\{\tilde{K}, s\} &= \frac{\delta_{12}}{s + h^*}\left[\frac{\mathcal{L}\{\gamma, s\} - \mathcal{L}\{\gamma, r^*\}}{s - r^*}\right] \qquad \text{(A.39)} \\ &= \delta_{12}\left[\frac{\gamma_B \tilde{B}_I(\infty) - \gamma_I \tilde{\tau}_I(\infty)}{r^* h^*}\frac{h^*}{s(s + h^*)} + \frac{\gamma_B\left[\tilde{B}_I(0) - \tilde{B}_I(\infty)\right]}{r^* + \xi_B}\frac{1}{(s + h^*)(s + \xi_B)}\right]. \end{aligned}$$

By inverting equation (A.39) we obtain the path for the capital stock:

$$\tilde{K}(t) = \tilde{K}(\infty)\mathrm{A}(h^*, t) + \frac{\delta_{12}\gamma_B\left[\tilde{B}_I(0) - \tilde{B}_I(\infty)\right]}{r^* + \xi_B}\mathrm{T}(h^*, \xi_B, t), \tag{A.40}$$

where $\tilde{K}(\infty)$ is given by (A.35), $\mathrm{A}(h^*, t)$ is an adjustment term and $\mathrm{T}(h^*, \xi_B, t)$ is a transition term. Equation (27) in the text is obtained by setting $\tilde{B}_I(0) = \tilde{B}_I(\infty) = 0$. The adjustment and transition terms have the following properties:

Lemma A.1 *Let $A(\alpha_1, t)$ be a single adjustment function of the form:*

$$\mathrm{A}(\alpha_1, t) \equiv 1 - e^{-\alpha_1 t},$$

with $\alpha_1 > 0$. Then $\mathrm{A}(\alpha_1, t)$ has the following properties: (i) (positive) $\mathrm{A}(\alpha_1, t) > 0$ $t \in (0, \infty)$, (ii) $\mathrm{A}(\alpha_1, t) = 0$ for $t = 0$ and $\mathrm{A}(\alpha_1, t) \to 1$ in the limit as $t \to \infty$, (iii) (increasing) $d\mathrm{A}(\alpha_1, t)/dt \geq 0$, (iv) (step function as limit) As $\alpha_1 \to \infty$, $\mathrm{A}(\alpha_1, t) \to u(t)$, where $u(t)$ is a unit step function.

PROOF: Properties (i) and (ii) follow by simple substitution. Property (iii) follows from the fact that $d\mathrm{A}(\alpha_1, 0)/dt = \alpha_1[1 - \mathrm{A}(\alpha_1, t)]$ plus properties (i)-(ii). Property (iv) follows by comparing the Laplace transforms of $\mathrm{A}(\alpha_1, t)$ and $u(t)$ and showing that they converge as $\alpha_1 \to \infty$. Since $\mathcal{L}\{u, s\} = 1/s$ and $\mathcal{L}\{\mathrm{A}(\alpha_1, t), s\} = 1/s - 1/(s + \alpha_1)$ this result follows. □

Lemma A.2 *Let $T(\alpha_1, \alpha_2, t)$ be a single transition function of the form:*

$$\mathrm{T}(\alpha_1, \alpha_2, t) \equiv \begin{cases} \frac{e^{-\alpha_2 t} - e^{-\alpha_1 t}}{\alpha_1 - \alpha_2} & \text{for } \alpha_1 \neq \alpha_2 \\ t e^{-\alpha_1 t} & \text{for } \alpha_1 = \alpha_2, \end{cases}$$

with $\alpha_1 > 0$ and $\alpha_2 > 0$. Then $\mathrm{T}(\alpha_1, \alpha_2, t)$ has the following properties: (i) (positive) $\mathrm{T}(\alpha_1, \alpha_2, t) > 0$ $t \in (0, \infty)$, (ii) $\mathrm{T}(\alpha_1, \alpha_2, t) = 0$ for $t = 0$ and in the limit as $t \to \infty$, (iii) (single-peaked) $d\mathrm{T}(\alpha_1, \alpha_2, t)/dt > 0$ for $t \in (0, \hat{t})$, $d\mathrm{T}(\alpha_1, \alpha_2, t)/dt < 0$ for $t \in (\hat{t}, \infty)$, $d\mathrm{T}(\alpha_1, \alpha_2, t)/dt = 0$ for $t = \hat{t}$ and in the limit as $t \to \infty$, and $d\mathrm{T}(\alpha_1, \alpha_2, 0)/dt = 1$, (iv) $\hat{t} \equiv \ln(\alpha_1/\alpha_2)/(\alpha_1 - \alpha_2)$ if $\alpha_1 \neq \alpha_2$ and $\hat{t} \equiv 1/\alpha_1$ if $\alpha_1 = \alpha_2$; (v) (point of inflexion) $d^2\mathrm{T}(\alpha_1, \alpha_2, t)/dt^2 = 0$ for $t^* = 2\hat{t}$

PROOF: Property (i) follows by examining the three possible cases. The result is obvious if $\alpha_1 = \alpha_2$. If $\alpha_1 < (>)\alpha_2$, then $\alpha_2 - \alpha_1 > (<)0$ and $e^{-\alpha_1 t} > (<)\ e^{-\alpha_2 t}$ for all $t \in (0, \infty)$, and $\mathrm{T}(\alpha_1, \alpha_2, 0) > 0$. Property (ii) follows by direct substitution. Property (iii) follows by examining $d\mathrm{T}(\alpha_1, \alpha_2, t)/dt$:

$$\frac{d\mathrm{T}(\alpha_1, \alpha_2, t)}{dt} \equiv \begin{cases} \frac{\alpha_1 e^{-\alpha_1 t} - \alpha_2 e^{-\alpha_2 t}}{\alpha_1 - \alpha_2} & \text{for } \alpha_1 \neq \alpha_2 \\ [1 - \alpha_1 t]\, e^{-\alpha_1 t} & \text{for } \alpha_1 = \alpha_2. \end{cases}$$

Property (iv) is obtained by examining $d\mathrm{T}^2(\alpha_1, \alpha_2, t)/dt^2$:

$$\frac{d^2\mathrm{T}(\alpha_1, \alpha_2, t)}{d^2 t} \equiv \begin{cases} \frac{\alpha_1^2 e^{-\alpha_1 t} - \alpha_2^2 e^{-\alpha_2 t}}{\alpha_2 - \alpha_1} & \text{for } \alpha_1 \neq \alpha_2 \\ -\alpha_1 [2 - \alpha_1 t]\, e^{-\alpha_1 t} & \text{for } \alpha_1 = \alpha_2. \end{cases}$$

Hence, $d^2\mathrm{T}(\alpha_1, \alpha_2, 0)/dt^2 = -(\alpha_1 + \alpha_2) < 0$, and $\lim_{t \to \infty} d^2\mathrm{T}(\alpha_1, \alpha_2, t)/dt^2 = 0$. The inflexion point is found by finding the value of $t = t^*$ where $d^2\mathrm{T}(\alpha_1, \alpha_2, t)/dt^2 = 0$. □

The second equation of (A.38) can be rewritten as:

$$\begin{aligned} \mathcal{L}\{\tilde{C}_I, s\} &= \frac{s - \delta_{11}}{s + h^*} \left[\frac{\mathcal{L}\{\gamma, s\} - \mathcal{L}\{\gamma, r^*\}}{s - r^*} \right] \\ &= \frac{\tilde{C}_I(0) + \mathcal{L}\{\gamma, s\}}{s + h^*} + \frac{(r^* - \delta_{11})}{s + h^*} \left[\frac{\mathcal{L}\{\gamma, s\} - \mathcal{L}\{\gamma, r^*\}}{s - r^*} \right] \\ &= \frac{\tilde{C}_I(0)}{s + h^*} + \frac{\gamma_I \tilde{\tau}_I(\infty) - \gamma_B \tilde{B}_I(\infty)}{s(s + h^*)} - \frac{\gamma_B \left[\tilde{B}_I(0) - \tilde{B}_I(\infty) \right]}{(s + h^*)(s + \xi_B)} \\ &\quad + (r^* - \delta_{11}) \left[\frac{\gamma_B \tilde{B}_I(\infty) - \gamma_I \tilde{\tau}_I(\infty)}{r^* h^*} \frac{h^*}{s(s + h^*)} \right. \\ &\quad \left. + \frac{\gamma_B \left[\tilde{B}_I(0) - \tilde{B}_I(\infty) \right]}{r^* + \xi_B} \frac{1}{(s + h^*)(s + \xi_B)} \right]. \end{aligned} \tag{A.41}$$

By inverting equation (A.41) we obtain the path for consumption:

$$\tilde{C}_I(t) = \tilde{C}_I(0) \left[1 - \mathrm{A}(h^*, t) \right] + \tilde{C}_I(\infty) \mathrm{A}(h^*, t) - \frac{(\delta_{11} + \xi_B)\gamma_B \left[\tilde{B}_I(0) - \tilde{B}_I(\infty) \right]}{r^* + \xi_B} \mathrm{T}(h^*, \xi_B, t),$$

where $\tilde{C}_I(0)$ and $\tilde{C}_I(\infty)$ are given by (A.33) and (A.35), respectively. Equation (28) in the text is obtained by setting $\tilde{B}_I(0) = \tilde{B}_I(\infty) = 0$.

6 Welfare analysis

The utility of generation v at time t can be written as:

$$\begin{aligned} U(v, t) &= \int_t^\infty \log C(v, \tau) e^{-(\alpha + \beta)(\tau - t)} d\tau \\ &= \int_t^\infty \left[\log C_I(v, \tau) + (\chi_C - \chi_I) \log N(\tau) \right] e^{(\alpha + \beta)(t - \tau)} d\tau. \end{aligned} \tag{A.42}$$

The Euler equation for a household of vintage v reads $\dot{C}_I(v,\tau)/C_I(v,\tau) = (r_I - \alpha)$, which implies the following path for $C_I(v,\tau)$:

$$C_I(v,\tau) = C_I(v,t) \exp\left[\int_t^\tau (r_I(\mu) - \alpha)\, d\mu\right],$$

so that (A.42) can be rewritten as:

$$\begin{aligned} U(v,t) &= \int_t^\infty \left[\log C_I(v,t) + (\chi_C - \chi_I) \log N(\tau) + \int_t^\tau [r_I(\mu) - \alpha]\, d\mu\right] e^{-(\alpha+\beta)(\tau-t)} d\tau, \\ &\equiv U_C(v,t) + U_D(t) + U_R(t), \end{aligned} \tag{A.43}$$

where

$$U_C(v,t) \equiv \log C_I(v,t) \int_t^\infty e^{(\alpha+\beta)(t-\tau)} d\tau = \frac{\log C_I(v,t)}{\alpha+\beta}, \tag{A.44}$$

$$U_D(t) \equiv (\chi_C - \chi_I) \int_t^\infty \log N(\tau) e^{(\alpha+\beta)(t-\tau)} d\tau, \tag{A.45}$$

$$U_R(t) \equiv \int_t^\infty \left[\int_t^\tau (r_I(\mu) - \alpha)\, d\mu\right] e^{(\alpha+\beta)(t-\tau)} d\tau. \tag{A.46}$$

Hence, equations (A.43)-(A.46) imply $dU(v,t) = dU_C(v,t) + dU_D(t) + dU_R(t)$, with

$$dU_C(v,t) = \frac{1}{\alpha+\beta} \frac{dC_I(v,t)}{C_I(v,t)}, \tag{A.47}$$

$$\begin{aligned} dU_D(t) &= (\chi_C - \chi_I) \int_t^\infty \frac{dN(\tau)}{N(\tau)} e^{(\alpha+\beta)(t-\tau)} d\tau \\ &\approx (\chi_C - \chi_I) \int_t^\infty \tilde{N}(\tau) e^{(\alpha+\beta)(t-\tau)} d\tau, \end{aligned} \tag{A.48}$$

$$dU_R(t) = \frac{r_I}{\alpha+\beta} \int_t^\infty \tilde{r}_I(\tau) e^{(\alpha+\beta)(t-\tau)} d\tau. \tag{A.49}$$

Note that (A.48)-(A.49) satisfy the following properties:

$$\mathcal{L}\{dU_D, s\} = (\chi_C - \chi_I) \left[\frac{\mathcal{L}\{\tilde{N}, \alpha+\beta\} - \mathcal{L}\{\tilde{N}, s\}}{s - (\alpha+\beta)}\right], \tag{A.50}$$

$$\mathcal{L}\{dU_R, s\} = \left(\frac{r_I}{\alpha+\beta}\right) \left[\frac{\mathcal{L}\{\tilde{r}_I, \alpha+\beta\} - \mathcal{L}\{\tilde{r}_I, s\}}{s - (\alpha+\beta)}\right]. \tag{A.51}$$

In order to calculate $dU_C(v,t)$, we must distinguish between existing and future generations.

6.1 Existing generations

Designating the time of the shock by $t_0 = 0$, existing generations have a non-positive generations index, i.e. $v \leq 0$. We already know that $C_I(v,0) = (\alpha+\beta)[A_I(v,0) + H_I(0)]$, where $H_I(0)$ is the initial steady-state level of human wealth, and $A_I(0,0) = 0$ (newborns have no financial assets). Also, $A_I(0) = (1 - \tau_I + s_K)K(0)$, where s_K is a once-off subsidy to capital owners. Hence, we can write:

$$\tilde{C}_I(v,0) \equiv \frac{dC_I(v,0)}{C_I(v,0)} = (1 - \alpha_{HS}) \tilde{A}_I(v,0) + \alpha_{HS} \tilde{H}_I(0), \ \alpha_{HS} \equiv \frac{H_I(0)}{A_I(v,0) + H_I(0)}. \tag{A.52}$$

Furthermore, we have that $A_I(v,0) = (1 - \tau_I + s_K)K(v,0)$, so that

$$\tilde{A}_I(v,0) \equiv \frac{dA_I(v,0)}{A_I(v,0)} = \frac{ds_K - d\tau_I(0)}{1 - \tau_I} = \frac{s_K}{1 - \tau_I} - \tilde{\tau}_I(0) = \tilde{A}_I(0), \ (v < 0), \tag{A.53}$$

where we have used the fact that $s_K = 0$ initially, so that $ds_K = s_K$, and $dK(v,0) = 0$. For aggregate consumption we know that:

$$\begin{aligned} C_I(0) &= (\alpha+\beta)\left[A_I(0) + H_I(0)\right] \Rightarrow \\ \tilde{C}_I(0) &= (1-\omega_H)\tilde{A}_I(0) + \omega_H \tilde{H}_I(0),\ \omega_H \equiv \frac{H_I(0)}{A_I(0)+H_I(0)}. \end{aligned} \quad \text{(A.54)}$$

From the steady-state consumption profile we know that:

$$\begin{aligned} C_I(v,0) &= C_I(v,v)e^{-(r_I-\alpha)v}, (v \leq 0), \Rightarrow \\ (\alpha+\beta)\left[A_I(v,0)+H_I(0)\right] &= (\alpha+\beta)\left[A_I(v,v)+H_I(0)\right]e^{-(r_I-\alpha)v} \Rightarrow \\ \alpha_{HS} &= e^{(r_I-\alpha)v}, \end{aligned}$$

where we have used $A_I(v,v) = 0$. Hence, using (A.53)-(A.54), we can write (A.52) as follows:

$$\tilde{C}_I(v,0) \equiv \tilde{A}_I(0) + \left(\frac{\alpha_{HS}}{\omega_H}\right)\left[\tilde{C}_I(0) - \tilde{A}_I(0)\right],\ (v \leq 0). \quad \text{(A.55)}$$

Using the initial value theorem of the Laplace transform on (A.50)-(A.51), we obtain:

$$dU_D(0) \equiv \lim_{s\to\infty} s\mathcal{L}\{dU_D, s\} = (\chi_C - \chi_I)\,\mathcal{L}\{\tilde{N}, \alpha+\beta\}, \quad \text{(A.56)}$$

$$dU_R(0) \equiv \lim_{s\to\infty} s\mathcal{L}\{dU_R, s\} = \left(\frac{r_I}{\alpha+\beta}\right)\mathcal{L}\{\tilde{r}_I, \alpha+\beta\}. \quad \text{(A.57)}$$

By using (A.47)-(A.49) and (A.55)-(A.57) we obtain the expression for the change in welfare of existing generations ($v \leq 0$):

$$\begin{aligned} (\alpha+\beta)dU(v,0) &= \tilde{A}_I(0) + \left(\frac{\alpha_{HS}}{\omega_H}\right)\left[\tilde{C}_I(0) - \tilde{A}_I(0)\right] \\ &\quad +(\alpha+\beta)(\chi_C-\chi_I)\mathcal{L}\{\tilde{N},\alpha+\beta\} + r_I\mathcal{L}\{\tilde{r}_I,\alpha+\beta\},. \end{aligned} \quad \text{(A.58)}$$

The infinite horizon model is obtained from this expression by setting $\beta = 0$, $r_I = \alpha$, and $\alpha_{HS} = \omega_H$.

6.2 Future generations

Future generations have a generations index greater than $t_0 = 0$, i.e. $v = t \geq 0$. Their welfare level is evaluated at birth, i.e. we compute $dU(v,v)$. Starting with the generation-specific part of utility, we know that $C_I(v,v) = (\alpha+\beta)H_I(v)$, which implies that:

$$\tilde{C}_I(v,v) \equiv \frac{dC_I(v,v)}{C_I(0,0)} = \frac{(\alpha+\beta)dH_I(v)}{(\alpha+\beta)H_I(0)} \Rightarrow \tilde{C}_I(v,v) = \tilde{H}_I(v), \quad \text{(A.59)}$$

where $C_I(0,0)$ is the initial steady-state level of consumption by a newborn. In the aggregate we have that $C_I(v) = (\alpha+\beta)[A_I(0)+H_I(0)] = (\alpha+\beta)[(1-\tau_I(v))K(v) + B_I(v) + H_I(v)]$, so that:

$$\tilde{C}_I(v) \equiv \frac{dC_I(v)}{C_I(0)} = \omega_H\tilde{H}_I(v) + (1-\omega_H)\left[\tilde{K}(v) - \tilde{\tau}_I(v)\right] + \left[\frac{1-\omega_H}{\omega_K(1-\tau_I)}\right]\tilde{B}_I(v), \quad \text{(A.60)}$$

where we have used the fact that the initial debt is zero ($B_I(0) = 0$) and $\tilde{B}_I(v) \equiv r_I dB_I(v)/Y$. Equation (A.60) can be used to write $\tilde{H}_I(v)$ in terms of aggregate variables:

$$\tilde{H}_I(v) = \frac{\tilde{C}_I(v) - (1-\omega_H)\left[\tilde{K}(v) + \tilde{B}_I(v)/\left[\omega_K(1-\tau_I)\right] - \tilde{\tau}_I(v)\right]}{\omega_H}. \quad \text{(A.61)}$$

Using (A.47)-(A.51), (A.59), and (A.61), we obtain the expression for the change in welfare of future generations ($v \geq 0$):

$$
\begin{aligned}
(\alpha+\beta)dU(v,v) &= \frac{\tilde{C}_I(v) - (1-\omega_H)\left[\tilde{K}(v) + \tilde{B}_I(v)/\left[\omega_K(1-\tau_I)\right] - \tilde{\tau}_I(v)\right]}{\omega_H} \\
&\quad + (\alpha+\beta)\left(\chi_C - \chi_I\right)\mathcal{L}^{-1}\left[\frac{\mathcal{L}\{\tilde{N},\alpha+\beta\} - \mathcal{L}\{\tilde{N},s\}}{s-(\alpha+\beta)}\right] \\
&\quad + r_I\mathcal{L}^{-1}\left[\frac{\mathcal{L}\{\tilde{r}_I,\alpha+\beta\} - \mathcal{L}\{\tilde{r}_I,s\}}{s-(\alpha+\beta)}\right],.
\end{aligned} \tag{A.62}
$$

Obviously, (A.58) and (A.62) coincide for $v = 0$. This can be easily demonstrated by noting that $\tilde{B}_I(0) \equiv s_K\omega_K$, and $\tilde{K}(0) \equiv 0$. By comparing (A.58) and (A.62) for $v = 0$, the difference amounts to:

$$
\tilde{A}_I(0) + \tilde{\tau}_I(0) - \frac{\tilde{B}_I(0)}{\omega_K(1-\tau_I)} = \frac{s_K}{1-\tau_I} - \frac{s_K\omega_K}{\omega_K(1-\tau_I)} = 0,
$$

where we have used (A.53).

7 Derivation of equations (40) and (A.68)

Equation (A.62) can be rewritten in the more useful form of (40) (without bond policy) and (A.68) below (with bond policy). Since (40) is a special case of (A.68), obtained by setting $\tilde{B}_I(0) = \tilde{B}_I(\infty) = 0$, we consider the derivation of the most general case. By using the path for the capital stock as given in (A.40) we can derive that:

$$
\begin{aligned}
\frac{\mathcal{L}\{\tilde{K},\alpha+\beta\} - \mathcal{L}\{\tilde{K},s\}}{s-(\alpha+\beta)} &= \left(\frac{1}{s}\right)\mathcal{L}\{\tilde{K},\alpha+\beta\} \\
&\quad + \left(\frac{h^*}{s(s+h^*)}\right)\left[\frac{\tilde{K}(\infty)}{\alpha+\beta+h^*} - \frac{\Omega_K\left[\tilde{B}_I(\infty) - \tilde{B}_I(0)\right]}{(\alpha+\beta+h^*)(\alpha+\beta+\xi_B)}\right] \\
&\quad + \left(\frac{1}{(s+h^*)(s+\xi_B)}\right)\left[\frac{\Omega_K\left[\tilde{B}_I(\infty) - \tilde{B}_I(0)\right]}{\alpha+\beta+\xi_B}\right],
\end{aligned} \tag{A.63}
$$

where Ω_K is defined as:

$$
\Omega_K \equiv -\frac{\gamma_B\delta_{12}}{r^* + \xi_B} = \frac{\delta\omega_C(r_I-\alpha)}{\omega_I\omega_K(1-\tau_I)(r^*+\xi_B)} > 0.
$$

Note that (A.63) contains the Laplace transforms of unity ($\equiv 1/s$), of an adjustment term $\mathrm{A}(h^*,t)$ ($\equiv h^*/[s(s+h^*)]$), and of a transition term $\mathrm{T}(h^*,\xi_B,t)$ ($\equiv 1/[(s+\xi_B)(s+h^*)]$). This makes the Laplace inversion of terms involving (A.63) very simple.

Since $\tilde{N}(t) = (1-\epsilon_L)\tilde{K}(t)$ and (with an instantaneous ITC introduction) $r_I\tilde{r}_I(t) = (r_I + \delta)[-\phi\tilde{K}(t) + \tilde{\tau}_I]$, the terms in square brackets involving $\tilde{N}$ that appears in (A.62) can be rewritten as:

$$\begin{aligned}\frac{\mathcal{L}\{\tilde{N},\alpha+\beta\}-\mathcal{L}\{\tilde{N},s\}}{s-(\alpha+\beta)} &= \left(\frac{1}{s}\right)\mathcal{L}\{\tilde{N},\alpha+\beta\} \\ &+\left(\frac{h^*}{s(s+h^*)}\right)\left[\frac{\tilde{N}(\infty)}{\alpha+\beta+h^*}-\frac{(1-\epsilon_L)\Omega_K\left[\tilde{B}_I(\infty)-\tilde{B}_I(0)\right]}{(\alpha+\beta+h^*)(\alpha+\beta+\xi_B)}\right] \\ &+\left(\frac{1}{(s+h^*)(s+\xi_B)}\right)\left[\frac{(1-\epsilon_L)\Omega_K\left[\tilde{B}_I(\infty)-\tilde{B}_I(0)\right]}{\alpha+\beta+\xi_B}\right],\end{aligned} \tag{A.64}$$

and, using $r_I\tilde{r}_I(0)=(r_I+\delta)\tilde{\tau}_I$,

$$\begin{aligned} r_I\left[\frac{\mathcal{L}\{\tilde{r}_I,\alpha+\beta\}-\mathcal{L}\{\tilde{r}_I,s\}}{s-(\alpha+\beta)}\right] &= \left(\frac{1}{s}\right)r_I\mathcal{L}\{\tilde{r}_I,\alpha+\beta\}+\left(\frac{h^*}{s(s+h^*)}\right) \\ &\times\left[\frac{r_I\left[\tilde{r}_I(\infty)-\tilde{r}_I(0)\right]}{\alpha+\beta+h^*}+\frac{\phi(r_I+\delta)\Omega_K\left[\tilde{B}_I(\infty)-\tilde{B}_I(0)\right]}{(\alpha+\beta+h^*)(\alpha+\beta+\xi_B)}\right] \\ &-\left(\frac{1}{(s+h^*)(s+\xi_B)}\right)\left[\frac{\phi(r_I+\delta)\Omega_K\left[\tilde{B}_I(\infty)-\tilde{B}_I(0)\right]}{\alpha+\beta+\xi_B}\right].\end{aligned} \tag{A.65}$$

Equation (A.62) can be used to derive the expressions for $dU(0,0)$:

$$\begin{aligned}(\alpha+\beta)dU(0,0) &= \frac{\tilde{C}_I(0)-(1-\omega_H)\left[\tilde{B}_I(0)/\left[\omega_K(1-\tau_I)\right]-\tilde{\tau}_I\right]}{\omega_H} \\ &+(\alpha+\beta)\left(\chi_C-\chi_I\right)\mathcal{L}\{\tilde{N},\alpha+\beta\}+r_I\mathcal{L}\{\tilde{r}_I,\alpha+\beta\},\end{aligned} \tag{A.66}$$

and $dU(\infty,\infty)$:

$$\begin{aligned}(\alpha+\beta)dU(\infty,\infty) &= \frac{\tilde{C}_I(\infty)-(1-\omega_H)\left[\tilde{K}(\infty)+\tilde{B}_I(\infty)/\left[\omega_K(1-\tau_I)\right]-\tilde{\tau}_I\right]}{\omega_H} \\ &+(\alpha+\beta)\left(\chi_C-\chi_I\right)\left[\frac{\tilde{N}(\infty)}{\alpha+\beta}\right]+r_I\left[\frac{\tilde{r}_I(\infty)}{\alpha+\beta}\right].\end{aligned} \tag{A.67}$$

By substituting the paths for consumption, the capital stock, and debt into (A.62), using the Laplace inverses of (A.63)-(A.65), and collecting terms, the following expression is obtained:

$$\begin{aligned}dU(t,t) &= dU(0,0)+\left[dU(\infty,\infty)-dU(0,0)\right]\mathrm{A}(h^*,t) \\ &+\Omega_B(\xi_B)\left[\tilde{B}_I(0)-\tilde{B}_I(\infty)\right]\mathrm{T}(h^*,\xi_B,t),\end{aligned} \tag{A.68}$$

where $\Omega_B(\xi_B)$ is:

$$\begin{aligned}(\alpha+\beta)\Omega_B(\xi_B) &\equiv \frac{\gamma_B\left[\delta_{11}+\xi_B+\delta_{12}(1-\omega_H)\right]}{\omega_H(r^*+\xi_B)}-\frac{(1-\omega_H)(\xi_B-h^*)}{\omega_H\omega_K(1-\tau_I)} \\ &+\frac{\Omega_K\left[(\alpha+\beta)(\chi_C-\chi_I)(1-\epsilon_L)-\phi(r_I+\delta)\right]}{\alpha+\beta+\xi_B}.\end{aligned} \tag{A.69}$$

In the absence of bond policy, $\tilde{B}_I(0)=\tilde{B}_I(\infty)=0$, the transition term $T(h^*,\xi_B,t)$ drops out of the various expressions, and (A.68) collapses to (40).

8 Proof of Theorem 4

Theorem 4 can be proved by making use of (A.58) and setting $\beta = 0$, $r_I = \alpha$, and $\alpha_{HS} = \omega_H$ and deleting the generations index:

$$\begin{aligned} \alpha dU(0) &= \tilde{C}_I(0) + \alpha \mathcal{L}\{\tilde{r}_I, \alpha\} + \alpha \left(\chi_C - \chi_I\right) \mathcal{L}\{\tilde{N}, \alpha\} \\ &= \alpha \mathcal{L}\{\tilde{C}_I, \alpha\} + \alpha \left(\chi_C - \chi_I\right) \mathcal{L}\{\tilde{N}, \alpha\}. \end{aligned} \tag{A.70}$$

The following results can be derived for the infinite horizon case:

$$\alpha \mathcal{L}\{\tilde{C}_I, \alpha\} = \left(\frac{\alpha + \delta}{\alpha + h^*}\right) \tilde{\tau}_I, \tag{A.71}$$

$$\alpha \mathcal{L}\{\tilde{N}, \alpha\} = \left(\frac{\delta \omega_C (1 - \epsilon_L)(\alpha + \delta)}{\omega_I r^* (\alpha + h^*)}\right) \tilde{\tau}_I. \tag{A.72}$$

By substituting (A.71)-(A.71) into (A.70) we obtain:

$$\begin{aligned} \alpha dU(0) &= \left(\frac{(\alpha + \delta)\tilde{\tau}_I}{r^*(\alpha + h^*)}\right) \left[\frac{\delta(1 - \phi)}{\omega_I} - (\alpha + \delta) + \left(\chi_C - \chi_I\right) \frac{\delta \omega_C (1 - \epsilon_L)}{\omega_I}\right] \\ &= \left(\frac{(\alpha + \delta)^2 \tilde{\tau}_I}{r^*(\alpha + h^*)}\right) \left[\chi_I \left(1 - \tau_I\right) - 1 + \left(\chi_C - \chi_I\right) \left(1 - \tau_I\right) \omega_C\right], \end{aligned} \tag{A.73}$$

where we have used the fact that under infinite horizons $\delta(1 - \phi)/\omega_I = \chi_I(\alpha + \delta)(1 - \tau_I)$ and $\delta(1\epsilon_L)/\omega_I = (\alpha + \delta)(1 - \tau_I)$ in the final step. Since, for $\tau_I = 0$, the welfare effects collapses to:

$$\left[\alpha dU(0)\right]_{\tau_I = 0} = \left(\frac{(\alpha + \delta)^2 \tilde{\tau}_I}{r^*(\alpha + h^*)}\right) \left[\omega_C \left(\chi_C - 1\right) + \left(1 - \omega_C\right) \left(\chi_I - 1\right)\right] \geq 0, \tag{A.74}$$

where we have used the fact that $\chi_I \geq 1$ and $\chi_C \geq 1$. Equation (A.74) shows that the optimal ITC, denoted by τ_I^*, is non-negative, and is strictly positive if either $\chi_I > 1$ and/or $\chi_C > 1$. By setting $\alpha dU(0) = 0$ in (A.73), the optimal ITC can be calculated as:

$$\tau_I^* = \frac{\chi_I + \omega_C \left(\chi_C - \chi_I\right) - 1}{\chi_I + \omega_C \left(\chi_C - \chi_I\right)} = \frac{(1 - \omega_C)(\chi_I - 1) + \omega_C(\chi_C - 1)}{(1 - \omega_C)\chi_I + \omega_C \chi_C}.$$

It can also be expressed in a slightly more conventional form:

$$\left(\frac{\tau_I^*}{1 - \tau_I^*}\right) = (1 - \omega_C)(\chi_I - 1) + \omega_C(\chi_C - 1).$$

This coincides with the expression found in Theorem 4. □

9 Proof of Theorem 5

In order to prove the various components of this theorem, it is useful to first state and prove the following Lemma.

Lemma A.3 *Let $\tau_I = 0$ initially and define $f(s) \equiv |sI - \Delta|$. The unstable characteristic root satisfies: (i) if $(1 - \epsilon_L) < (\sqrt{5} - 1)/(\sqrt{5} + 1)$, then $r^* > r_I + \delta + r_I - \alpha$; and (ii) $r^* > r_I + \beta$.*

PROOF: In order to prove (i) $r^* > (r_I + \delta + r_I - \alpha)$, it is sufficient to show that $f(s) \equiv |sI - \Delta| < 0$ for $s = (r_I + \delta + r_I - \alpha)$. We can rewrite (A.13) by using $\tau_I = 0$, $\omega_C = 1 - \omega_I$, and $\delta/\omega_I = (r_I + \delta)/(1 - \epsilon_L)$:

$$|\Delta| \equiv -r^* h^* = -\phi(r_I + \delta) \left[\frac{r_I + \delta + r_I - \alpha}{1 - \epsilon_L} - \delta\right] < 0. \tag{A.75}$$

Using this result, the characteristic polynomial, evaluated in $s = r_I + \delta + r_I - \alpha$, equals:

$$f(r_I + \delta + r_I - \alpha) = (r_I + \delta + r_I - \alpha)\left[r_I + \delta + r_I - \alpha - (1-\phi)\left(\frac{\delta}{\omega_I}\right) + \delta - r_I + \alpha\right] + |\Delta|$$

$$\begin{aligned}
&= (r_I + \delta)\left[(1-\chi_I)(r_I+\delta) + \delta - \phi\left(\frac{r_I+\delta}{1-\epsilon_L} - \delta\right)\right] \\
&\quad +(r_I - \alpha)\left[(1-\chi_I)(r_I+\delta) + \delta - \frac{\phi(r_I+\delta)}{1-\epsilon_L}\right] \\
&= (r_I + \delta)\left[\delta + \left(1 - \frac{1}{1-\epsilon_L}\right)(r_I+\delta) + \delta\phi\right] \\
&\quad +(r_I - \alpha)\left[\left(1 - \frac{1}{1-\epsilon_L}\right)(r_I+\delta) + \delta\right] \\
&= (r_I + \delta)\left[\left(1 - \frac{1}{1-\epsilon_L}\right)r_I + \left(2 - \frac{1}{1-\epsilon_L} + \phi\right)\delta\right] \\
&\quad +(r_I - \alpha)\left[\left(1 - \frac{1}{1-\epsilon_L}\right)(r_I+\delta) + \delta\right] \\
&= (r_I + \delta)\left[\left(1 - \frac{1}{1-\epsilon_L}\right)r_I + \left(\frac{(2-\chi_I)(1-\epsilon_L)^2 + (1-\epsilon_L)\epsilon_L - \epsilon_L^2}{(1-\epsilon_L)}\right)\delta\right] \\
&\quad +(r_I - \alpha)\left[\left(1 - \frac{1}{1-\epsilon_L}\right)(r_I+\delta) + \delta\right] \\
&= (r_I + \delta)\left[\left(1 - \frac{1}{1-\epsilon_L}\right)r_I + \left(\frac{(2-\chi_I)(1-\epsilon_L)^2 + (1-\epsilon_L)\epsilon_L - \epsilon_L^2}{(1-\epsilon_L)}\right)\delta\right] \\
&\quad +(r_I - \alpha)\left[\left(1 - \frac{1}{1-\epsilon_L}\right)(r_I+\delta) + \delta\right].
\end{aligned}$$

Since r_I is independent of χ_I, this expression takes its maximum for $\chi_I = 1$. Using the factorisation $(1-\epsilon_L)^2 + \epsilon_L(1-\epsilon_L) - \epsilon_L^2 = (1 - \frac{1}{2}(1-\sqrt{5})\epsilon_L)(1 - \frac{1}{2}(1+\sqrt{5})\epsilon_L)$, this gives:

$$\begin{aligned}
f(r_I + \delta + r_I - \alpha) &\leq (r_I + \delta)\left[\left(1 - \frac{1}{1-\epsilon_L}\right)r_I + \left(\frac{(1-\epsilon_L)^2 + \epsilon_L(1-\epsilon_L) - \epsilon_L^2}{(1-\epsilon_L)}\right)\delta\right] \\
&\quad +(r_I - \alpha)\left[\left(1 - \frac{1}{1-\epsilon_L}\right)(r_I+\delta) + \delta\right] \\
&< 0,
\end{aligned}$$

for $1 - \frac{1}{2}(1+\sqrt{5})\epsilon_L < 0 \Leftrightarrow 1 - \epsilon_L < (\sqrt{5}-1)/(\sqrt{5}+1) \approx 0.382$.

The result (ii) $r^* > r_I + \beta$ is proved in a similar fashion. The characteristic polynomial, evaluated in $s = r_I + \beta$, equals:

$$
\begin{aligned}
f(r_I+\beta) &= (r_I+\beta)\left[r_I+\beta+r_I-(1-\phi)\left(\frac{\delta}{\omega_I}\right)+\delta-(r_I-\alpha)\right]+|\Delta| \\
&= (r_I+\beta)\left[\alpha+\beta-\chi_I(r_I+\delta)+\delta\right]+\left[\chi_I(r_I+\delta)-\delta\right](r_I-\alpha) \\
&\quad -\frac{\delta\omega_C}{\omega_I}\left[\phi(r_I+\delta)+(r_I-\alpha)\right] \\
&= \left[\chi_I(r_I+\delta)-\delta\right]\left[r_I-\alpha-(r_I+\beta)\right]+r_I(\alpha+\beta)-\frac{\delta\phi\omega_C}{\omega_I}(r_I+\delta) \\
&= (\alpha+\beta)\left[r_I+\delta-\chi_I(r_I+\delta)\right]-\frac{\delta\phi\omega_C}{\omega_I}(r_I+\delta) \\
&= (\alpha+\beta)(r_I+\delta)\left[1-\chi_I-\frac{\delta\phi\omega_C}{(\alpha+\beta)\omega_I}\right]<0.
\end{aligned}
$$

This establishes that $r^* > r_I + \beta$. This completes the proof of Lemma A.3. □

Part (i) of Theorem 5 is proved as follows. In the absence of bond policy, $\tilde{A}_I(0) = -\tilde{\tau}_I$ and equations (33)-(36) in the text predicts for existing generations ($v \leq 0$) that:

$$
(\alpha+\beta)\frac{\partial\left[dU(v,0)\right]}{\partial v}=\left(\frac{(r-\alpha)e^{(r-\alpha)v}}{\omega_H}\right)\left[\tilde{C}_I(0)+\tilde{\tau}_I\right],
$$

which shows that members of younger generations are better off than older generations if and only if $\tilde{C}_I(0) + \tilde{\tau}_I > 0$. From equation (29) in the text we find that:

$$
\tilde{C}_I(0)+\tilde{\tau}_I=\left(\frac{r^*-(r_I+\delta+r_I-\alpha)}{r^*}\right)\tilde{\tau}_I>0,
$$

where we have used part (i) of Lemma A.3 to determine the sign.

Part (ii) of Theorem 5 can be proved as follows. It is possible to derive that:

$$
\tilde{N}(\infty)=(1-\epsilon_L)\tilde{K}(\infty)=-\frac{(1-\epsilon_L)\delta_{12}\gamma_I\tilde{\tau}_I}{r^*h^*}, \tag{A.76}
$$

$$
r_I\tilde{r}_I(\infty)=(r_I+\delta)\left[-\phi\tilde{K}(\infty)+\tilde{\tau}_I\right]=\frac{\delta\phi(r_I+\delta)(r_I-\alpha)\tilde{\tau}_I}{r^*h^*}, \tag{A.77}
$$

$$
\tilde{C}_I(\infty)-(1-\omega_H)\left[\tilde{K}(\infty)-\tilde{\tau}_I\right]=\frac{\gamma_I\left[\chi_I(r_I+\delta)-\delta-(\alpha+\beta)\right]\tilde{\tau}_I}{r^*h^*}-\frac{(\alpha+\beta)\tilde{\tau}_I}{\delta_{12}}. \tag{A.78}
$$

By substituting (A.76)-(A.78) into (A.67) (with $\tilde{B}_I(\infty) = 0$ imposed) we obtain:

$$
(\alpha+\beta)dU(\infty,\infty)=\frac{\Gamma_2\tilde{\tau}_I}{\omega_H r^*h^*},
$$

where Γ_2 is defined as:

$$
\begin{aligned}
\Gamma_2 &\equiv \gamma_I\left[\chi_I(r_I+\delta)-\delta-(\alpha+\beta)\right]-\frac{(\alpha+\beta)r^*h^*}{\delta_{12}} \\
&\quad -\delta_{12}\omega_H\gamma_I(\chi_C-\chi_I)(1-\epsilon_L)+\frac{\omega_H(r_I+\delta)}{\alpha+\beta}\delta\phi(r_I-\alpha).
\end{aligned}
$$

By noting that $-(\alpha+\beta)/\delta_{12} \equiv 1-\omega_H = (r_I-\alpha)/\beta$, $\delta_{12}\omega_H = \alpha+\beta+\delta-(r_I+\delta)/(1-\epsilon_L)$, $\phi \equiv 1-\chi_I(1-\epsilon_L)$, and using the definition of r^*h^* in (A.75), we can rewrite Γ_2 as:

$$\begin{aligned}
\Gamma_2 \quad \equiv \quad & \omega_H\left[\gamma_I\left(\frac{r_I+\delta}{1-\epsilon_L}-\delta\right)+\left(\chi_C-\frac{1}{1-\epsilon_L}\right)(r_I+\delta)\gamma_I+\left[1-\chi_C(1-\epsilon_L)\right]\delta(r_I+\delta)\right] \\
& +\omega_H\left[\left[1-\chi_I(1-\epsilon_L)\right]\frac{\delta(r_I-\alpha)(r_I+\delta)}{\alpha+\beta}\right] \qquad \text{(A.79)} \\
& -\omega_H(\chi_C-\chi_I)\left[\gamma_I\epsilon_L(r_I+\delta\epsilon_L)+\delta(1-\epsilon_L)(r_I-\alpha)\right]-\left[1-\chi_I(1-\epsilon_L)\right]\delta(r_I+\delta).
\end{aligned}$$

The partial derivatives of Γ_2 with respect to χ_I and χ_C can be calculated from (A.79):

$$\begin{aligned}
\frac{\partial\Gamma_2}{\partial\chi_I} &= \delta(1-\epsilon_L)\left[\omega_H(r_I-\alpha)\left(1-\frac{r_I+\delta}{\alpha+\beta}\right)+(r_I+\delta)\right]+\omega_H\gamma_I\epsilon_L(r_I+\delta\epsilon_L)>0, \\
\frac{\partial\Gamma_2}{\partial\chi_C} &= \omega_H\gamma_I(1-\epsilon_L)(r_I+\delta\epsilon_L)>0,
\end{aligned}$$

Hence, a lower bound for Γ_2 is reached for $\chi_I=\chi_C=1$, so that (using $\gamma_I \equiv r_I+\delta+r_I-\alpha$)

$$\begin{aligned}
\Gamma_2 \quad > \quad & \omega_H(r_I+\delta+r_I-\alpha)r_I+\delta\epsilon_L(r_I+\delta)(r_I-\alpha)\left[\frac{\omega_H}{\alpha+\beta}-\frac{1}{\beta}\right] \\
= \quad & \left(1-\frac{r_I-\alpha}{\beta}\right)(r_I+\delta+r_I-\alpha)r_I+\delta\epsilon_L(r_I+\delta)(r_I-\alpha)\left[\frac{1}{\alpha+\beta}-\frac{r_I-\alpha}{\beta(\alpha+\beta)}-\frac{1}{\beta}\right] \\
= \quad & \left(\frac{\alpha+\beta-r_I}{\beta}\right)(r_I+\delta+r_I-\alpha)r_I-\delta\epsilon_L(r_I+\delta)(r_I-\alpha)\left[\frac{r_I}{\beta(\alpha+\beta)}\right].
\end{aligned}$$

We must first provide an estimate for $\delta(r_I-\alpha)$. The model solution satisfies:

$$g(r_I) \equiv (r_I-\alpha)\left[\frac{r_I+\delta}{1-\epsilon_L}-\delta\right]-\beta(\alpha+\beta)=0. \qquad \text{(A.80)}$$

It follows that

$$\frac{\partial\delta(r_I-\alpha)}{\partial\delta}=r_I-\alpha+\delta\frac{\partial r_I}{\partial\delta}=(r_I-\alpha)\left[1-\frac{\delta\epsilon_L}{r_I+\delta\epsilon_L+r_I-\alpha}\right]=\frac{(r_I-\alpha)(2r_I-\alpha)}{r_I+\delta\epsilon_L+r_I-\alpha}>0.$$

Furthermore, from (A.80) we see that $\lim_{\delta\to\infty}\delta(r_I-\alpha)=\beta(\alpha+\beta)(1-\epsilon_L)/\epsilon_L$. Hence

$$\delta(r_I-\alpha)<\frac{\beta(\alpha+\beta)(1-\epsilon_L)}{\epsilon_L},$$

and

$$(\alpha+\beta-r_I)(r_I+\delta+r_I-\alpha)r_I-\delta\epsilon_L(r_I+\delta)(r_I-\alpha)\frac{r_I}{\alpha+\beta}$$

$$>r_I\left[(\alpha+\beta-r_I)(r_I+\delta+r_I-\alpha)-(1-\epsilon_L)\beta(r_I+\delta)\right]$$

$$=r_I\left[(\alpha+\beta\epsilon_L-r_I)(r_I+\delta)+(\alpha+\beta-r_I)(r_I-\alpha)\right].$$

We have to show that $r_I<\alpha+\beta\epsilon_L \Leftrightarrow \omega_H>1-\epsilon_L$. To this end, substitute $r_I<\alpha+\beta\epsilon_L$ into (A.80). This yields

$$g(\alpha+\beta\epsilon_L)=\frac{\alpha\beta\left[2\epsilon_L-1\right]+\beta^2\left[\epsilon_L^2+\epsilon_L-1\right]}{1-\epsilon_L}-\beta\delta\epsilon_L$$

$$\geq \frac{\alpha\beta\left[2\epsilon_L - 1\right] - \beta^2\left[1 - \frac{1}{2}\left(1+\sqrt{5}\right)\epsilon_L\right]\left[1 - \frac{1}{2}\left(1-\sqrt{5}\right)\epsilon_L\right]}{1-\epsilon_L} > 0 \Leftarrow 1 - \epsilon_L < \frac{\sqrt{5}-1}{\sqrt{5}+1}.$$

Furthermore, for $r_I = \alpha$, (A.80) shows that $g(\alpha) < 0$ for $\beta > 0$. This proves that the larger root of (A.80) must be between α and $\alpha + \beta\epsilon_L$. This implies that $\Gamma_2 > 0$ and completes the proof of part (ii) of Theorem 5. □

10 Debt policy

The simulations reported in Table 2 in the text are performed as follows. It is assumed that the policy maker is able at time $t = 0$ to choose a path of debt that is parameterized as in (A.37). At time $t = 0$, the policy maker makes a discrete adjustment to its debt position, e.g. by providing a subsidy to capital owners (in which case $\tilde{B}_I(0) > 0$). After that, the government flow budget restriction (equation (T1.3)) and (A.36) together imply a path for lump-sum taxes, $\tilde{T}_I(t)$, which affects the welfare of present and future generations:

$$\tilde{T}_I(t) = \tilde{B}_I(\infty) + \left(\frac{r_I + \xi_B}{r_I}\right)\left(\tilde{B}_I(0) - \tilde{B}_I(\infty)\right)\left[1 - \mathrm{A}(\xi_B, t)\right] + (1-\omega_C)\tilde{\tau}_I.$$

With a positive birth rate Ricardian equivalence does not hold and debt has real effects. Equation (A.35) implies that the long-run results on the capital stock and consumption are equal to:

$$\begin{aligned}
\tilde{K}(\infty) &= \frac{\omega_C\left[(r_I + \delta + r_I - \alpha)\tilde{\tau}_I - \left[(r_I - \alpha)/[\omega_K(1-\tau_I)]\right]\tilde{B}_I(\infty)\right]}{\phi\left[r_I - \alpha + \omega_C(r_I + \delta)\right]}, \\
\tilde{C}_I(\infty) &= \frac{(\omega_C - \phi)\left[(r_I + \delta + r_I - \alpha)\tilde{\tau}_I - \left[(r_I - \alpha)/[\omega_K(1-\tau_I)]\right]\tilde{B}_I(\infty)\right]}{\phi\left[r_I - \alpha + \omega_C(r_I + \delta)\right]}.
\end{aligned}$$

The transition path taking the economy from the old to the new equilibrium is given by (A.37) and (A.38) which can be written as:

$$\begin{aligned}
\tilde{K}(t) &= \mathrm{A}(h^*, t)\tilde{K}(\infty) - \left(\frac{\delta\omega_C(r_I - \alpha)}{\omega_I\omega_K(1-\tau_I)(r^* + \xi_B)}\right)\left[\tilde{B}_I(0) - \tilde{B}_I(\infty)\right]\mathrm{T}(h^*, \xi_B, t), \\
\tilde{C}_I(t) &= \tilde{C}_I(0)[1 - \mathrm{A}(h^*, t)] + \mathrm{A}(h^*, t)\tilde{C}_I(\infty) \\
&\quad - \left(\frac{(r_I - \alpha)\left[\xi_B + \delta(\omega_C - \phi)/\omega_I\right]}{\omega_K(1-\tau_I)(r^* + \xi_B)}\right)\left[\tilde{B}_I(0) - \tilde{B}_I(\infty)\right]\mathrm{T}(h^*, \xi_B, t),
\end{aligned}$$

where $T(h^*, \xi_B, t)$ is a single transition term (see Lemma A.2), and the initial jump in consumption is equal to (see (A.33)):

$$\begin{aligned}
\tilde{C}_I(0) &= -\left(\frac{r_I + \delta + r_I - \alpha}{r^*}\right)\tilde{\tau}_I \\
&\quad + \left(\frac{r_I - \alpha}{\omega_K(1-\tau_I)r^*}\right)\left[\left(\frac{r^*}{r^* + \xi_B}\right)\tilde{B}_I(0) + \left(\frac{\xi_B}{r^* + \xi_B}\right)\tilde{B}_I(\infty)\right]. \qquad \text{(A.81)}
\end{aligned}$$

The welfare effects on current and future generations are also affected by the presence of debt policy. The welfare path of current generations is still given by equation (33) in the text, but the change in the value of the capital stock and the Laplace transforms for the rate of interest and the number of firms are all affected by the parameters of the debt path:

$$\mathcal{L}\{\tilde{r}_I, \alpha+\beta\} \equiv \left(\frac{r_I+\delta}{r_I(\alpha+\beta)}\right) \times \left[\tilde{\tau}_I - \frac{\phi h^*}{\alpha+\beta+h^*}\tilde{K}(\infty) + \frac{\phi\delta\omega_C(r_I-\alpha)(\alpha+\beta)[\tilde{B}_I(0)-\tilde{B}_I(\infty)]}{\omega_I\omega_K(1-\tau_I)(r^*+\xi_B)(\alpha+\beta+h^*)(\alpha+\beta+\xi_B)}\right], \tag{A.82}$$

$$\mathcal{L}\{\tilde{N}, \alpha+\beta\} \equiv \left(\frac{1-\epsilon_L}{\alpha+\beta}\right) \times \left[\frac{h^*}{\alpha+\beta+h^*}\tilde{K}(\infty) - \frac{\delta\omega_C(r_I-\alpha)(\alpha+\beta)[\tilde{B}_I(0)-\tilde{B}_I(\infty)]}{\omega_I\omega_K(1-\tau_I)(r^*+\xi_B)(\alpha+\beta+h^*)(\alpha+\beta+\xi_B)}\right], \tag{A.83}$$

$$\tilde{A}_I(0) = \frac{s_K}{1-\tau_I} - \tilde{\tau}_I = \frac{\tilde{B}_I(0)}{\omega_K(1-\tau_I)} - \tilde{\tau}_I, \tag{A.84}$$

where s_K is a once-off subsidy to capital owners aimed at compensating them for the capital loss they suffer as a result of the introduction of the ITC.

For future generations, the (change of) utility $dU(v,v)$ can be written as in (A.85)-(A.86). With the aid of equations (33), (A.68)-(A.69), and (A.81-(A.84), it is possible to show that the policy maker can devise several Pareto-improving policies.

Under the egalitarian policy the policy maker must select the right path for debt (or, equivalently, for lump-sum taxes), in order to spread the efficiency gains (or losses) equally over all existing and future generations. The policy maker has three instruments, $\tilde{B}_I(0)$ (which is regulated by the once-off subsidy s_K), $\tilde{B}_I(\infty)$, and ξ_B, with which to distribute the common (endogenously determined) utility gain, dU, to all generations (or loss, if $dU < 0$). First, very old generations must be given a subsidy such that they gain in net terms:

$$\begin{aligned} dU &= dU(-\infty, 0) \Leftrightarrow \\ dU &= \frac{s_K}{1-\tau_I} - \tilde{\tau}_I + r_I\mathcal{L}\{\tilde{r}_I, \alpha+\beta\} + (\alpha+\beta)(\chi_C-\chi_I)\mathcal{L}\{\tilde{N}, \alpha+\beta\}, \end{aligned}$$

where we have used (33) (with $v \to -\infty$), and (A.84). Second, all existing generations must be equally well-off. In terms of (33), this implies that the generation-specific term must be neutralised:

$$dU(v,0) = dU \Leftrightarrow \left[\tilde{C}_I(0) - \tilde{A}_I(0)\right] = 0. \tag{A.85}$$

The policy maker has the instruments to influence the initial jump in consumption. In view of (A.81), an appropriate combination of ξ_B, $\tilde{B}_I(0)$, and $\tilde{B}_I(\infty)$ can be used to ensure that (A.85) holds. Third, the generations born in the new steady state must also gain dU:

$$\begin{aligned} (\alpha+\beta)dU(\infty,\infty) &= \left(\frac{\tilde{C}_I(\infty) - (1-\omega_H)\left[\tilde{K}(\infty) + [1/[\omega_K(1-\tau_I)]\,\tilde{B}_I(\infty) - \tilde{\tau}_I\right]}{\omega_H}\right) \\ &\quad + r_I\left(\frac{\tilde{r}_I(\infty)}{\alpha+\beta}\right) + (\chi_C-\chi_I)\,\tilde{N}(\infty) \\ &= (\alpha+\beta)dU. \end{aligned} \tag{A.86}$$

In Table 3 we have reported such egalitarian policies for different values of τ_I and β.

References

Judd, K. L. (1982). An alternative to steady-state comparisons in perfect foresight models. *Economics Letters*, 10:55–59.