A Life-Cycle Overlapping-Generations Model of the Small Open Economy: Mathematical Appendix

Ben J. Heijdra* University of Groningen

Ward E. Romp♯ University of Groningen

March 2006

A.1 Introduction

In this Mathematical Appendix the details of the more important derivations are presented.

A.2 Key model derivations

Since $\Phi'(s) = \phi(s)$, $0 \le \Phi(s) < 1$, and $\Phi(0) = 0$ we have that:

$$\begin{aligned} -m(s) &= \frac{d}{ds}\ln[1-\Phi(s)] \quad \Rightarrow \\ -\int_0^{\tau-v} m(s)\,ds &= \int_0^{\tau-v} d\ln[1-\Phi(s)] \\ &= \ln[1-\Phi(\tau-v)] - \ln[1-\Phi(0)] \\ &= \ln[1-\Phi(\tau-v)]. \end{aligned} \tag{A.1}$$

It follows from (A.1) that:

$$1-\Phi(\tau-v) = \exp\left\{-\int_0^{\tau-v} m(s)\,ds\right\} \equiv \exp\{-M(\tau-v)\}. \tag{A.2}$$

Obviously, it follows from the definition of $M(\tau-v)$ that:

$$M'(\tau-v) = m(\tau-v), \tag{A.3}$$

$$\phi(\tau-v) = M'(\tau-v)\exp\{-M(\tau-v)\}. \tag{A.4}$$

By expanding the lifetime utility function of a newborn we obtain:

$$\begin{aligned} \Lambda(v,v) &\equiv \int_v^t U[\bar{c}(v,\tau)]\,e^{-[\theta(\tau-v)+M(\tau-v)]}d\tau + \int_t^\infty U[\bar{c}(v,\tau)]\,e^{-[\theta(\tau-t)+M(\tau-v)-M(t-v)]}d\tau \\ &= \int_v^t U[\bar{c}(v,\tau)]\,e^{-[\theta(\tau-v)+M(\tau-v)]}d\tau + \Lambda(v,t), \end{aligned} \tag{A.5}$$

*Department of Economics, University of Groningen, P.O. Box 800, 9700 AV Groningen, The Netherlands. Phone: +31-50-363-7303, Fax: +31-50-363-7337, E-mail: b.j.heijdra@rug.nl.

♯Department of Economics, University of Groningen, P.O. Box 800, 9700 AV Groningen, The Netherlands. Phone: +31-50-363-7190, Fax: +31-50-363-7337, E-mail: w.e.romp@rug.nl.

where $\Lambda(v,t)$ is defined as:

$$\Lambda(v,t) \equiv e^{M(t-v)} \int_t^\infty U\left[\bar{c}(v,\tau)\right] e^{-[\theta(\tau-t)+M(\tau-v)]} d\tau. \tag{A.6}$$

From the perspective of period t ($> v$), the first term on the right-hand side of (A.5) reflects the stream of felicity due to past choices ("water under the bridge"), whilst the second term deals with current and future felicity. The term $\exp\{M(t-v)\}$ appearing in (A.6) is a constant which does not affect the optimal choices of the household. It must be recognized, however, in the section on welfare effects (see below).

The household chooses paths for $\bar{c}(v,\tau)$ and $\bar{a}(v,\tau)$ in order to maximize $\Lambda(v,t)$ subject to a transversality condition, the budget identity,

$$\dot{\bar{a}}(v,\tau) = [r + m(\tau-v)]\,\bar{a}(v,\tau) + \bar{w}(\tau) - \bar{z}(\tau) - \bar{c}(v,\tau), \tag{A.7}$$

and taking as given the initial stock of assets, $\bar{a}(v,t)$. The Hamiltonian is:

$$\begin{aligned}\mathcal{H} \;&\equiv\; U\left[\bar{c}(v,\tau)\right] e^{-[\theta(\tau-t)+M(\tau-v)]} \\ &\quad +\mu(v,\tau)\left[\,[r+m(\tau-v)]\,\bar{a}(v,\tau)+\bar{w}(\tau)-\bar{z}(\tau)-\bar{c}(v,\tau)\,\right],\end{aligned}$$

and the first-order necessary conditions are:

$$\frac{\partial\mathcal{H}}{\partial\bar{c}(v,\tau)} = \bar{c}(v,\tau)^{-1/\sigma} e^{-[\theta(\tau-t)+M(\tau-v)]} - \mu(v,\tau) = 0, \tag{A.8}$$

$$\dot{\mu}(v,\tau) = -\frac{\partial\mathcal{H}}{\partial\bar{a}(v,\tau)} = -[r+m(\tau-v)]\,\mu(v,\tau). \tag{A.9}$$

Using (A.8)-(A.9) we can derive the consumption Euler equation for this case:

$$\begin{aligned}\dot{\mu}(v,\tau)\,\bar{c}(v,\tau)^{1/\sigma} + \frac{1}{\sigma}\mu(v,\tau)\,\bar{c}(v,\tau)^{1/\sigma-1}\,\dot{\bar{c}}(v,\tau) &= -[\theta+m(\tau-v)]\,e^{-[\theta(\tau-t)+M(\tau-v)]} \\ \frac{\dot{\mu}(v,\tau)\,\bar{c}(v,\tau)^{1/\sigma}}{\mu(v,\tau)\,\bar{c}(v,\tau)^{1/\sigma}} + \frac{1}{\sigma}\frac{\mu(v,\tau)\,\bar{c}(v,\tau)^{1/\sigma-1}\,\dot{\bar{c}}(v,\tau)}{\mu(v,\tau)\,\bar{c}(v,\tau)^{1/\sigma}} &= -[\theta+m(\tau-v)] \\ -[r+m(\tau-v)] + \frac{1}{\sigma}\frac{\dot{\bar{c}}(v,\tau)}{\bar{c}(v,\tau)} &= -[\theta+m(\tau-v)] \\ \frac{\dot{\bar{c}}(v,\tau)}{\bar{c}(v,\tau)} &= \sigma\,[r-\theta]. \end{aligned} \tag{A.10}$$

This is **equation (8)** in the paper.

The life-time budget constraint is obtained by integrating (A.7) forward in time and imposing the initial and terminal conditions. The integrating factor is:

$$R(v,t,\tau) \equiv e^{-[r(\tau-t)+M(\tau-v)-M(t-v)]}, \tag{A.11}$$

$$\frac{dR(v,t,\tau)}{d\tau} = -[r+m(\tau-v)]\,R(v,t,\tau). \tag{A.12}$$

Using (A.7) we find:

$$\begin{aligned}&\left[\dot{\bar{a}}(v,\tau) - [r+m(\tau-v)]\,\bar{a}(v,\tau)\right] R(v,t,\tau) = [\bar{w}(\tau)-\bar{z}(\tau)-\bar{c}(v,\tau)]\,R(v,t,\tau) \\ &\frac{d}{d\tau}\bar{a}(v,\tau)\,R(v,t,\tau) = [\bar{w}(\tau)-\bar{z}(\tau)-\bar{c}(v,\tau)]\,R(v,t,\tau) \\ &\int_t^\infty d\bar{a}(v,\tau)\,R(v,t,\tau) = \int_t^\infty [\bar{w}(\tau)-\bar{z}(\tau)-\bar{c}(v,\tau)]\,R(v,t,\tau)\,d\tau \\ &\lim_{\tau\to\infty} R(v,t,\tau)\,\bar{a}(v,\tau) - \bar{a}(v,t) = \int_t^\infty [\bar{w}(\tau)-\bar{z}(\tau)-\bar{c}(v,\tau)]\,R(v,t,\tau)\,d\tau.\end{aligned} \tag{A.13}$$

The transversality condition for the household ensures that the first term on the left-hand side of (A.13) is zero:

$$\lim_{\tau\to\infty} R(v,t,\tau)\,\bar{a}(v,\tau)=0, \tag{A.14}$$

so that (A.13) reduces to:

$$\int_t^\infty \bar{c}(v,\tau)\,R(v,t,\tau)\,d\tau = \bar{a}(v,t)+\bar{h}(v,t), \tag{A.15}$$

$$\bar{h}(v,t) \equiv \int_t^\infty [\bar{w}(\tau)-\bar{z}(\tau)]\,R(v,t,\tau)\,d\tau, \tag{A.16}$$

where $h(v,t)$ is human wealth at time t of an household of vintage v. It features a generations index because the annuity rate of interest depends on it. **Equation (10)** in the paper is obtained by rewriting (A.16) somewhat.

To solve for consumption in the planning period we note that the Euler equation (A.10) can be solved to yield:

$$\bar{c}(v,\tau)=\bar{c}(v,t)\,e^{\sigma[r-\theta](\tau-t)}. \tag{A.17}$$

Using (A.11) and (A.17) we find that (A.15) can be rewritten as:

$$\begin{aligned}\bar{a}(v,t)+\bar{h}(v,t) &= \bar{c}(v,t)\int_t^\infty e^{\sigma[r-\theta](\tau-t)}e^{-[r(\tau-t)+M(\tau-v)-M(t-v)]}d\tau \\ &= \bar{c}(v,t)\,\Delta(v,t),\end{aligned} \tag{A.18}$$

where $\Delta(v,t)$ is defined as:

$$\begin{aligned}\Delta(v,t) &\equiv \int_t^\infty e^{\theta(t-\tau)-M(\tau-v)+M(t-v)}d\tau \\ &= e^{r^*(t-v)+M(t-v)}\int_t^\infty e^{-r^*(\tau-v)-M(\tau-v)}d\tau \\ &= e^{r^*u+M(u)}\int_u^\infty e^{-[r^*s+M(s)]}ds \equiv \Delta(u,r^*),\end{aligned} \tag{A.19}$$

where $r^*\equiv r-\sigma[r-\theta]$ and $u\equiv t-v$ is the household's age in the planning period. The general expression for $\Delta(u,\lambda)$ is found in **equation (11)** in the paper. Equation (A.18) is the same as **equation (9)** in the paper.

To derive the per capita expressions we first note that:

$$l(v,t)=be^{-[n(t-v)+M(t-v)]} \tag{A.20}$$

so that:

$$\frac{\dot{l}(v,t)}{l(v,t)}=-[n+m(t-v)]. \tag{A.21}$$

Furthermore, it follows from equation (A.16) that human capital evolves according to:

$$\dot{\bar{h}}(v,t)=[r+m(t-v)]\,\bar{h}(v,t)-\bar{w}(t)+\bar{z}(t). \tag{A.22}$$

Per capita consumption is defined as $c(t)\equiv\int_{-\infty}^t l(v,t)\bar{c}(v,t)dv$ so that:

$$\begin{aligned}\dot{c}(t) &= l(t,t)\,\bar{c}(t,t)+\int_{-\infty}^t l(v,t)\,\dot{\bar{c}}(v,t)\,dv+\int_{-\infty}^t \dot{l}(v,t)\,\bar{c}(v,t)\,dv \\ &= b\bar{c}(t,t)+\sigma[r-\theta]\,c(t)-\int_{-\infty}^t [n+m(t-v)]\,l(v,t)\,\bar{c}(v,t)\,dv,\end{aligned} \tag{A.23}$$

where we have used (A.10) and (A.21) to get to the second line. This is **equation (18)** in the paper.

Per capita financial wealth is defined as $a(t) \equiv \int_{-\infty}^{t} l(v,t)\bar{a}(v,t)dv$ so that:

$$\begin{aligned}
\dot{a}(t) &= l(t,t)\,\bar{a}(t,t) + \int_{-\infty}^{t} l(v,t)\,\dot{\bar{a}}(v,t)\,dv + \int_{-\infty}^{t} \dot{l}(v,t)\,\bar{a}(v,t)\,dv \\
&= \int_{-\infty}^{t} l(v,t)\,\big[\,[r+m(t-v)]\,\bar{a}(v,t) + \bar{w}(t) - \bar{z}(t) - \bar{c}(v,t)\,\big]dv \\
&\quad - \int_{-\infty}^{t} [n+m(t-v)]\,l(v,t)\,\bar{a}(v,t)\,dv \\
&= (r-n)\,a(t) + w(t) - z(t) - c(t), \qquad \text{(A.24)}
\end{aligned}$$

where $w(t) = \bar{w}(t)$, $z(t) = \bar{z}(t)$, and we have used equation (A.7) and noted the fact that newborns are born without financial assets ($a(t,t)=0$). This is **equation (19)** in the paper.

Finally, per capita human wealth is defined as $h(t) \equiv \int_{-\infty}^{t} l(v,t)\bar{h}(v,t)dv$ so that $\dot{h}(t)$ can be written as:

$$\begin{aligned}
\dot{h}(t) &= l(t,t)\,\bar{h}(t,t) + \int_{-\infty}^{t} l(v,t)\,\dot{\bar{h}}(v,t)\,dv + \int_{-\infty}^{t} \dot{l}(v,t)\,\bar{h}(v,t)\,dv \\
&= b\bar{h}(t,t) + \int_{-\infty}^{t} l(v,t)\,\big[\,[r+m(t-v)]\,\bar{h}(v,t) - \bar{w}(t) + \bar{z}(t)\,\big]dv \\
&\quad - \int_{-\infty}^{t} [n+m(t-v)]\,l(v,t)\,\bar{h}(v,t)\,dv \\
&= (r-n)\,h(t) + b\bar{h}(t,t) - w(t) + z(t), \qquad \text{(A.25)}
\end{aligned}$$

where we have used (A.22) to get to the second line. This is **equation (20)** in the paper.

A.3 Coherency condition

In the text we define the aggregate mortality rate as:

$$\begin{aligned}
\bar{m} &= \int_{-\infty}^{t} m(t-v)\,\frac{L(v,t)}{L(t)}dv \\
&= b\int_{-\infty}^{t} m(t-v)\,e^{-[n(t-v)+M(t-v)]}dv \\
&= b\int_{0}^{\infty} m(s)\,e^{-ns-M(s)}ds, \qquad \text{(A.26)}
\end{aligned}$$

where we assume that $n \equiv b - \bar{m} \neq 0$. To solve this integral we write:

$$\begin{aligned}
-\frac{d}{ds}\left[e^{-ns-M(s)}\right] &= ne^{-ns-M(s)} + m(s)e^{-ns-M(s)} \quad \Leftrightarrow \\
-\left. e^{-ns-M(s)}\right|_{s=0}^{\infty} &= n\int_{0}^{\infty} e^{-ns-M(s)}ds + \int_{0}^{\infty} m(s)\,e^{-ns-M(s)}ds. \qquad \text{(A.27)}
\end{aligned}$$

By using (A.26) to simplify the second term on the right-hand side of (A.27) we obtain:

$$\begin{aligned}
1 &= (b-\bar{m})\int_{0}^{\infty} e^{-ns-M(s)}ds + \frac{\bar{m}}{b} \quad \Leftrightarrow \\
\frac{1}{b} &= \int_{0}^{\infty} e^{-ns-M(s)}ds \equiv \Delta(0,n), \qquad \text{(A.28)}
\end{aligned}$$

where $\Delta(u,\lambda)$ is defined in the text. For a given birth rate b, equation (A.28) implicitly defines the coherent solution for n and thus for $\bar{m} \equiv b - n$. (Vice versa, for a given n, (A.28) determines the coherent solution for b.) Note that (A.28) coincides with **equation (16)** in the paper.

A.4 Financial assets

To derive the effect on per capita assets, start with the definition of $\hat{a}$ and use equations (24)–(26) in the paper:

$$\begin{aligned}\hat{a} &\equiv \int_0^\infty l(u)\hat{\bar{a}}(u)du \\ &= \int_0^\infty l(u)\Delta(u,r^*)\hat{\bar{c}}(u) - l(u)\hat{\bar{h}}(u)du \\ &= [\hat{w} - \hat{z}]\Delta(0,r)b\int_0^\infty e^{[r-n]u}\Omega(u,r,r^*)du, \end{aligned} \tag{A.29}$$

where $\Omega(u,r,r^*)$ is given by:

$$\Omega(u,r,r^*) \equiv \frac{\int_u^\infty e^{-r^*s-M(s)}ds}{\Delta(0,r^*)} - \frac{\int_u^\infty e^{-rs-M(s)}ds}{\Delta(0,r)}.$$

To determine the sign of the integral we need to determine the sign of $\Omega(u,r,r^*)$. It is easy to see that $\Omega(u,r,r^*) = 0$ and $\lim_{u\to\infty}\Omega(u,r,r^*) = 0$. Differentiation of $\Omega(u,r,r^*)$ with respect to u gives

$$\Omega'(u,r,r^*) = \frac{e^{-ru-M(u)}}{\Delta(0,r)} - \frac{e^{-r^*u-M(u)}}{\Delta(0,r^*)}, \tag{A.30}$$

which clearly has just one root and $\Omega'(0,r,r^*) > 0$ (for $r > r^*$). This in combination with continuity of $\Omega(u,r,r^*)$ shows that $\Omega(u,r,r^*) > 0$ for $u > 0$ (and $r > r^*$), so the whole integral in equation (A.29) must be positive. This determines the sign of **equation (33)** because:

$$\frac{d\hat{a}}{d\hat{z}} = \frac{1}{r-n}\left[1 + \frac{d\hat{c}}{d\hat{z}}\right] = -b\Delta(0,r)\int_0^\infty e^{[r-n]u}\Omega(u,r,r^*)du < 0. \tag{A.31}$$

Note that $\hat{c}$ can be written as:

$$\begin{aligned}\hat{c} &= \hat{\bar{c}}(0)\frac{\Delta(0,n^*)}{\Delta(0,n)} \\ &= [\hat{w} - \hat{z}]\frac{\Delta(0,r)}{\Delta(0,r^*)}\frac{\Delta(0,n^*)}{\Delta(0,n)},\end{aligned}$$

where $n^* \equiv n - \sigma[r - \theta]$. Hence, it follows that $\hat{a}$ can be written as:

$$\begin{aligned}(r-n)\hat{a} &= \hat{c} - [\hat{w} - \hat{z}] \\ &= [\hat{w} - \hat{z}]\left[1 - \frac{\Delta(0,r)}{\Delta(0,n)}\frac{\Delta(0,n^*)}{\Delta(0,r^*)}\right].\end{aligned} \tag{A.32}$$

Since $\hat{a} > 0$ it follows from (A.32) that, since $r > n$, we have:

$$\frac{\Delta(0,r)}{\Delta(0,n)}\frac{\Delta(0,n^*)}{\Delta(0,r^*)} < 1.$$

This result is very hard to prove directly.

A.5 Calibrating the distribution function

As was shown above (in equation (A.2)), the proportion of the population surviving up to age u is given by:

$$1 - \Phi(u) = e^{-M(u)}, \tag{A.33}$$

where $M(u) = \int_0^u m(s)ds$. To calibrate the distribution function of the lifetime we first have to choose a parametric family. We consider four different functional forms for the instantaneous mortality rate:

1. *Constant* mortality rate (used by Blanchard to allow for exact aggregation):

$$m(s) = \mu_0, \tag{A.34}$$
$$M(u) = \mu_0 u. \tag{A.35}$$

2. Mortality rate *linear in age*:

$$m(s) = \mu_0 + 2\mu_1^2 s, \tag{A.36}$$
$$M(u) = \mu_0 u + \mu_1^2 u^2 \tag{A.37}$$

3. Mortality rate *piecewise linear* in age:

$$m(s) = \mu_0 + 2\mu_1^2 \max(0, s - \bar{u}), \tag{A.38}$$
$$M(u) = \mu_0 u + \delta(u) \mu_1^2 (u - \bar{u})^2, \tag{A.39}$$

where $\delta(u) = 0$ for $0 \leq u < \bar{u}$ and $\delta(u) = 1$ for $u \geq \bar{u}$.

4. *Gompertz-Makeham* mortality rate:

$$m(s) = \mu_0 + \mu_1 e^{\mu_2 s}, \tag{A.40}$$
$$M(u) = \mu_0 u + (\mu_1/\mu_2)\left[e^{\mu_2 u} - 1\right]. \tag{A.41}$$

As is explained in the paper, we use actual demographic data for the cohort born in 1920 in the Netherlands. The estimates for the different models are presented in Table A.1. In that table, $b_0 = 0.0236$ in the implied birth rate for the G-M model (see **Footnote 17** in the paper). This birth rate is held constant for the other models, implying different population growth rates, $\hat{n}(b_0)$. The implied $\Delta(u, \theta)$ functions and steady-state values for $\hat{\bar{h}}(u)$, $\hat{\bar{c}}(u)$, and $\hat{\bar{a}}(u)$ are visualized in Figure A.1.

A.6 Expressions for $\Delta(u, \lambda)$

The $\Delta(u, \lambda)$ plays a crucial role in the model. For all models it can be expressed in terms of known functions.

1. Constant mortality. For the Blanchard case equations (A.34)-(A.35) are relevant. It follows that:

$$\begin{aligned}\Delta(u,\lambda) &\equiv e^{\lambda u+M(u)}\int_u^\infty e^{-[\lambda s+M(s)]}ds \\ &= e^{\lambda u+\mu_0 u}\int_u^\infty e^{-[\lambda s+\mu_0 s]}ds=\frac{1}{\lambda+\mu_0}.\end{aligned} \tag{A.42}$$

This result is mentioned in the paper in the first paragraph of **Subsection 3.2**.

2. Linear-in-age mortality. For this case, equations (A.36)-(A.37) are relevant. We find:

$$\begin{aligned}\Delta(u,\lambda) &= e^{[\lambda+\mu_0]u+\mu_1^2u^2}\int_u^\infty e^{-[\lambda+\mu_0]s-\mu_1^2s^2}ds \\ &= \frac{\sqrt{\pi}}{2\mu_1}\operatorname{erfcx}\left(\mu_1 u+\frac{\lambda+\mu_0}{2\mu_1}\right).\end{aligned} \tag{A.43}$$

3. Piecewise-linear mortality. For this model, equations (A.38)-(A.39) are relevant. For the first part ($u<\bar{u}$) we get for $\Delta(u,\lambda)$:

$$\begin{aligned}\Delta(u,\lambda) &= e^{[\lambda+\mu_0]u}\left[\int_u^{\bar{u}} e^{-[\lambda+\mu_0]s}ds+\int_{\bar{u}}^\infty e^{-[\lambda+\mu_0]s-\mu_1^2[s-\bar{u}]^2}ds\right] \\ &= \frac{1-e^{-[\lambda+\mu_0][\bar{u}-u]}}{\lambda+\mu_0}+e^{-[\lambda+\mu_0][\bar{u}-u]}e^{[\lambda+\mu_0-2\mu_1^2\bar{u}]\bar{u}+\mu_1^2\bar{u}^2}\int_{\bar{u}}^\infty e^{[\lambda+\mu_0-2a_1^2\bar{u}]s+\mu_1^2s^2}ds \\ &= \frac{1-e^{-[\lambda+\mu_0][\bar{u}-u]}}{\lambda+\mu_0}+e^{-[\lambda+\mu_0][\bar{u}-u]}\frac{\sqrt{\pi}}{2\mu_1}\operatorname{erfcx}\left(\frac{\lambda+\mu_0}{2\mu_1}\right) \\ &= \frac{1}{\lambda+\mu_0}+e^{-[\lambda+\mu_0][\bar{u}-u]}\Gamma_\lambda,\end{aligned} \tag{A.44}$$

where Γ_λ is defined as:

$$\Gamma_\lambda\equiv\frac{\sqrt{\pi}}{2\mu_1}\operatorname{erfcx}\left(\frac{\lambda+\mu_0}{2\mu_1}\right)-\frac{1}{\lambda+\mu_0}. \tag{A.45}$$

For the second part ($u\geq\bar{u}$) we get

$$\begin{aligned}\Delta(u,\lambda) &= e^{[\lambda+\mu_0]u+\mu_1^2[u-\bar{u}]^2}\int_u^\infty e^{-[\lambda+\mu_0]s-\mu_1^2[s-\bar{u}]^2}ds \\ &= e^{[\lambda+\mu_0]u+\mu_1^2[u-\bar{u}]^2-\mu_1^2\bar{u}^2}\int_u^\infty e^{-[\lambda+\mu_0-2\mu_1^2\bar{u}]s-\mu_1^2s^2}ds \\ &= e^{[\lambda+\mu_0-2\mu_1^2\bar{u}]u+\mu_1^2u^2}\int_u^\infty e^{-[\lambda+\mu_0-2\mu_1^2\bar{u}]s-\mu_1^2s^2}ds \\ &= \frac{\sqrt{\pi}}{2\mu_1}\operatorname{erfcx}\left(\mu_1[u-\bar{u}]+\frac{\lambda+\mu_0}{2\mu_1}\right).\end{aligned} \tag{A.46}$$

Concluding:

$$\Delta(u,\lambda)=\begin{cases}\frac{1}{\lambda+\mu_0}+e^{-[\lambda+\mu_0][\bar{u}-u]}\Gamma_\lambda & \text{for } 0<u<\bar{u} \\ \frac{\sqrt{\pi}}{2\mu_1}\operatorname{erfcx}\left(\mu_1[u-\bar{u}]+\frac{\lambda+\mu_0}{2\mu_1}\right) & \text{for } u\geq\bar{u}\end{cases} \tag{A.47}$$

Properties of the functions used are covered in the following Lemma.

Lemma A.1 *The error function* ($\mathrm{erf}\,(x)$)*, complementary error function* ($\mathrm{erfc}\,(x)$)*, and scaled complementary error function* ($\mathrm{erfcx}\,(x)$) *are defined as follows.*

$$\begin{aligned}
\mathrm{erf}(x) &= \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt, \\
\mathrm{erfc}\,(x) &\equiv \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt = 1-\mathrm{erf}(x), \\
\mathrm{erfcx}\,(x) &\equiv e^{x^2}\,\mathrm{erfc}(x).
\end{aligned}$$

For non-negative values of x, these functions have the following properties:

(i) $0 < \mathrm{erf}(x), \mathrm{erfc}(x), \mathrm{erfcx} < 1$ *for* $0 < x \ll \infty$.

(ii) $\mathrm{erf}(0) = 1-\mathrm{erfc}(0) = 1-\mathrm{erfcx}(0) = 0$.

(iii) $\lim_{x\to\infty}\mathrm{erf}(x) = 1$, $\lim_{x\to\infty}\mathrm{erfc}(x) = \lim_{x\to\infty}\mathrm{erfcx}(x) = 0$.

(iv) $\mathrm{erf}'(x) > 0$, $\mathrm{erfc}'(x) < 0$, $\mathrm{erfcx}'(x) < 0$.

(v) $\mathrm{erfcx}\,(x) \approx 1/\left(x\sqrt{\pi}\right)$ *for large x.*

4. Gompertz-Makeham mortality. For this model equations (A.40)-(A.41) are relevant. The demographic discount function for the G-M process can be written as:

$$\Delta(u,\lambda) = e^{[\lambda+\mu_0]u+\frac{\mu_1}{\mu_2}e^{\mu_2 u}}\int_u^\infty e^{-[\lambda+\mu_0]s-\frac{\mu_1}{\mu_2}e^{\mu_2 s}}ds. \tag{A.48}$$

We define $\beta\,(u) \equiv \frac{\mu_1}{\mu_2}e^{\mu_2 u}$ and $t = \beta\,(s)$ so that $dt = \beta'\,(s)\,ds = \mu_2\beta\,(s)\,ds$. We see that (A.48) can be rewritten as:

$$\Delta(u,\lambda) = e^{[\lambda+\mu_0]u+\beta(u)}\int_u^\infty e^{-\beta(s)}e^{-[\lambda+\mu_0]s}ds. \tag{A.49}$$

We can write:

$$e^{-\beta(s)} = e^{-t} \tag{A.50}$$

$$\begin{aligned}
e^{-[\lambda+\mu_0]s}ds &= \frac{e^{-[\lambda+\mu_0]s}}{\mu_2\beta\,(s)}\mu_2\beta\,(s)\,ds \\
&= \frac{e^{-[\lambda+\mu_0]s}}{\mu_2 t}dt \\
&= \frac{1}{\mu_2}\left(\frac{\mu_2}{\mu_1}\right)^\alpha t^{\alpha-1}dt,
\end{aligned} \tag{A.51}$$

where we have made use of the following fact to get to the final expression:

$$t = \frac{\mu_1}{\mu_2}e^{\mu_2 s} \quad\Leftrightarrow\quad \left(\frac{\mu_2 t}{\mu_1}\right)^\alpha = e^{\alpha\mu_2 s} = e^{-[\lambda+\mu_0]s}, \tag{A.52}$$

for $\alpha \equiv -\left(\lambda+\mu_0\right)/\mu_2$. Since $s \in [u,\infty)$ it follows that $t \in [\beta\,(u)\,,\infty)$. By using (A.50) and (A.51) in (A.49) we obtain:

$$\Delta(u,\lambda) = \frac{\mu_2^{\alpha-1}}{\mu_1^\alpha}e^{[\lambda+\mu_0]u+\beta(u)}\Gamma(\alpha,\beta(u)), \tag{A.53}$$

where $\Gamma(\alpha,\beta(u))$ is the upper tailed incomplete gamma function (defined in general terms as $\Gamma(\alpha,x) = \int_x^\infty t^{\alpha-1}e^{-t}dt$; see Kreyszig (1999, p. A55)). The incomplete gamma function is well documented (see every mathematical/statistical handbook) and, more importantly, most software packages have very fast routines to calculate it.

There is one slight complication: the incomplete gamma function is usually only defined for $\alpha \geq 0$, whereas we need to evaluate it for $\alpha < 0$. We can solve this problem by using the 'functional relation of the incomplete gamma function'. Indeed, by integrating the incomplete gamma function by parts we obtain the following recursion formula:

$$\Gamma(\alpha, x) = \frac{1}{\alpha} e^{-t} t^{\alpha} \Big|_{t=x}^{\infty} + \frac{1}{\alpha} \int_x^{\infty} t^{\alpha} e^{-t} dt = -\frac{1}{\alpha} e^{-x} x^{\alpha} + \frac{1}{\alpha} \Gamma(\alpha + 1, x). \tag{A.54}$$

Repeated application of eq. (A.54) gives for $k = 0, 1, 2, \ldots$:

$$\begin{aligned}\Gamma(\alpha, x) = &- e^{-x} x^{\alpha} \left[\frac{1}{\alpha} + \frac{1}{\alpha} \frac{1}{\alpha+1} x + \frac{1}{\alpha} \frac{1}{\alpha+1} \frac{1}{\alpha+2} x^2 + \cdots + \frac{1}{\alpha} \frac{1}{\alpha+1} \cdots \frac{1}{\alpha+k-1} x^{k-1} \right] \\ &+ \frac{1}{\alpha} \frac{1}{\alpha+1} \cdots \frac{1}{\alpha+k-1} \Gamma(\alpha+k, x).\end{aligned} \tag{A.55}$$

Hence, by choosing the smallest integer k such that $\alpha + k$ is non-negative, the value of $\Gamma(\alpha, x)$ can be computed in a standard fashion.

A.7 Macroeconomic shocks

All the shocks studied (or mentioned) in Section 4 of the paper can be expressed in terms of the following functions:

$$w(t) = \begin{cases} \hat{w} & \text{for } t < 0 \\ \hat{w} + dw_0 e^{-\xi t} & \text{for } t \geq 0 \end{cases}, \tag{A.56}$$

$$r(t) = \begin{cases} r & \text{for } t < 0 \\ r_N \equiv r + dr & \text{for } t \geq 0 \end{cases}, \tag{A.57}$$

$$g(t) = \begin{cases} 0 & \text{for } t < 0 \\ d\hat{g} & \text{for } t \geq 0 \end{cases}, \tag{A.58}$$

$$z(t) = \begin{cases} 0 & \text{for } t < 0 \\ -dz_0 e^{-\chi t} + d\hat{z}[1 - e^{-\chi t}] & \text{for } t \geq 0 \end{cases}. \tag{A.59}$$

Government consumption $d\hat{g}$ and the path of government debt are related to the other parameters according to:

$$d\hat{g} = \frac{\chi d\hat{z}}{r - n + \chi} - \frac{(r-n)\, dz_0}{r - n + \chi}, \tag{A.60}$$

$$d(t) = \frac{d\hat{g} + dz_0}{\chi} [1 - e^{-\chi t}]. \tag{A.61}$$

The time at which the shock occurs is normalized to zero, and the intertemporal substitution elasticity is equal to unity ($\sigma = 1$)

The three shocks explicitly studied in the text are:

- Unanticipated and permanent balanced-budget increase in government consumption: $g(t)$ set as in (A.58), $z(t)$ set according to (A.59) and (A.48) with $\chi \to \infty$, i.e. $d\hat{g} = d\hat{z}$. No debt financing occurs, i.e. $d(t) = 0$ for all $t \geq 0$.

- Ricardian equivalence experiment, temporary tax cut: $g(t) = 0$, $z(t)$ set according to (A.59) and (A.48) with $0 < \chi \ll \infty$, and the (stable) path of debt is set according to (A.53).

- Unanticipated and permanent increase in the world interest rate: $dr > 0$ for $t \geq 0$.

A fourth shock is only discussed here because its effects are very similar to those of the temporary tax cut:

- Temporary productivity shock: $g(t) = z(t) = d(t) = 0$, $w(t)$ set according to (A.56) with $0 < \xi \ll \infty$.

A.7.1 Post-Shock Profiles

The steady-state age profiles for the different variables before the shock occurs ($t < 0$) are defined for individual households in **equation (24)-(26)**. After the shock occurs ($t \geq 0$), the path for individual human wealth is:

$$\begin{aligned}\bar{h}(v,t) &= \hat{w}\Delta(t-v,r_N) + dw_0 e^{-\xi t}\Delta(t-v,r_N+\xi) - d\hat{z}\Delta(t-v,r_N) \\ &\quad +[dz_0 + d\hat{z}]e^{-\chi t}\Delta(t-v,r_N+\chi). \end{aligned} \tag{A.62}$$

For households who were born before the shock ($v < 0$), the age index at the time of the shock is $-v > 0$. For such households, the paths for individual consumption and asset holdings after the shock ($t \geq 0$) are given by:

$$\bar{c}_E(v,t) = \frac{\hat{\bar{a}}(-v) + \bar{h}(v,0)}{\Delta(-v,\theta)} e^{(r_N-\theta)t}, \tag{A.63}$$

$$\bar{a}_E(v,t) = \Delta(t-v,\theta)\bar{c}_E(v,t) - \bar{h}(v,t), \tag{A.64}$$

where the subscript "E" denotes *existing* households (at the time of the shock).

For households that are born after the shock ($v \geq 0$), the relevant age index at time t ($\geq v$) is defined as $t - v$. For such households the paths for consumption and asset holdings (at individual and cohort level) at time t (≥ 0) are given by:

$$\bar{c}_F(v,t) = \frac{\bar{h}(v,v)}{\Delta(0,\theta)} e^{(r_N-\theta)(t-v)}, \tag{A.65}$$

$$\bar{a}_F(v,t) = \Delta(t-v,\theta)\bar{c}_F(v,t) - \bar{h}(v,t), \tag{A.66}$$

where the subscript "F" denotes *future* households.

A.7.2 Welfare Effects

The welfare effects of the different shocks are illustrated in **Figure 6** in the text. For existing agents the change in welfare from the perspective of the shock period $t = 0$ is evaluated ($d\Lambda(v,0)$ for $v \leq 0$) whereas for future agents the welfare change from the perspective of their birth date is computed ($d\Lambda(v,v)$ for $v > 0$).

A.7.2.1 Existing generations

Equation (43) is derived as follows. The effect on welfare of existing agents at $t = 0$ can be written as a function of their age at that moment $(-v)$:

$$d\Lambda(v,0) = \int_0^\infty [\ln \bar{c}_E(v,\tau) - \ln \hat{\bar{c}}(v,\tau)] e^{-\theta\tau - M(\tau-v)+M(-v)} d\tau. \tag{A.67}$$

Consumption after the shock can be written in terms of pre-shock consumption:

$$\begin{aligned}
\bar{c}_E(v,\tau) &= \frac{\hat{\bar{a}}(-v) + \bar{h}(v,0)}{\Delta(-v,\theta)} e^{(r_N-\theta)\tau} \\
&= e^{(r_N-r)\tau} \left[\frac{\hat{\bar{a}}(-v) + \hat{\bar{h}}(-v)}{\Delta(-v,\theta)} e^{(r-\theta)\tau} + \frac{\bar{h}(v,0) - \hat{\bar{h}}(-v)}{\Delta(-v,\theta)} e^{(r-\theta)\tau} \right] \\
&= e^{(r_N-r)\tau} \left[\frac{\hat{\bar{a}}(-v) + \hat{\bar{h}}(-v)}{\Delta(-v,\theta)} + \frac{\bar{h}(v,0) - \hat{\bar{h}}(-v)}{\Delta(-v,\theta)} \right] e^{(r-\theta)\tau} \\
&= e^{(r_N-r)\tau} \left[1 + \frac{\bar{h}(v,0) - \hat{\bar{h}}(-v)}{\hat{\bar{a}}(-v) + \hat{\bar{h}}(-v)} \right] \hat{\bar{c}}(-v) e^{(r-\theta)\tau} \\
&= e^{(r_N-r)\tau} \left[\frac{\hat{\bar{a}}(-v) + \bar{h}(v,0)}{\hat{\bar{a}}(-v) + \hat{\bar{h}}(-v)} \right] \hat{\bar{c}}(v,\tau).
\end{aligned} \tag{A.68}$$

By taking logarithms of (A.68) and rewriting we obtain:

$$\ln \bar{c}_E(v,\tau) - \ln \hat{\bar{c}}(v,\tau) = (r_N - r)\,\tau + \ln \Gamma_E(v), \tag{A.69}$$

where $\Gamma_E(v)$ is defined in **equation (44)**. By substituting (A.69) into (A.67) and splitting the integral we get:

$$\begin{aligned}
d\Lambda(v,0) &= dr \int_0^\infty \tau e^{-\theta\tau - M(\tau-v)+M(-v)} d\tau + \left[e^{M(-v)} \int_0^\infty e^{-\theta\tau - M(\tau-v)} d\tau \right] \ln \Gamma_E(v) \\
&= dr \int_0^\infty \tau e^{-\theta\tau - M(\tau-v)+M(-v)} d\tau + \left[e^{-\theta v + M(-v)} \int_{-v}^\infty e^{-[\theta s + M(s)]} ds \right] \ln \Gamma_E(v) \\
&= dr \int_0^\infty \tau e^{-\theta\tau - M(\tau-v)+M(-v)} d\tau + \Delta(-v,\theta) \ln \Gamma_E(v).
\end{aligned} \tag{A.70}$$

Equation (A.70) coincides with **equation (43)** in the text.

A.7.2.2 Future generations

Equation (45) is derived as follows. For future households the welfare effect at birth is defined as:

$$d\Lambda(v,v) = \int_v^\infty [\ln \bar{c}_F(v,\tau) - \ln \hat{\bar{c}}(v,\tau)] e^{-\theta[\tau-v] - M(\tau-v)} d\tau. \tag{A.71}$$

Next we express the post-shock consumption path in terms of the pre-shock path as:

$$\begin{aligned}
\bar{c}_F(v,\tau) &= \frac{\bar{h}(v,v)}{\Delta(0,\theta)} e^{(r_N-\theta)(\tau-v)} \\
&= e^{(r_N-r)(\tau-v)} \frac{\hat{h}(v,v)}{\hat{\bar{h}}(0)} \frac{\hat{\bar{h}}(0)}{\Delta(0,\theta)} e^{(r-\theta)(\tau-v)} \\
&= e^{(r_N-r)(\tau-v)} \frac{\bar{h}(v,v)}{\hat{\bar{h}}(0)} \hat{\bar{c}}(v,\tau).
\end{aligned} \tag{A.72}$$

By taking logarithms of (A.72) and rewriting we obtain:

$$\ln \bar{c}_F(v,\tau) - \ln \hat{\bar{c}}(v,\tau) = (r_N - r)\,(\tau - v) + \ln \Gamma_F(v), \tag{A.73}$$

where $\Gamma_F(v)$ is defined in **equation (46)**. By substituting (A.73) into (A.71) and splitting the integral we get:

$$\begin{aligned} d\Lambda(v,v) &= dr \int_v^{\infty} (\tau - v)\, e^{-\theta(\tau-v) - M(\tau-v)} d\tau + \left[\int_v^{\infty} e^{-\theta(\tau-v)-M(\tau-v)} d\tau\right] \ln \Gamma_F(v) \\ &= dr \int_0^{\infty} s e^{-[\theta s + M(s)]} ds + \left[\int_0^{\infty} e^{-[\theta s + M(s)]} ds\right] \ln \Gamma_F(v) \\ &= dr \int_0^{\infty} s e^{-[\theta s + M(s)]} ds + \Delta(0,\theta) \ln \Gamma_F(v). \end{aligned} \tag{A.74}$$

Equation (A.74) coincides with **equation (45)** in the text.

Table A.1: Estimated Survival Functions (different models)

	$\hat{\mu}_0$	$\hat{\mu}_1$	$\hat{\mu}_2$	$\hat{\bar{u}}$	$\hat{\sigma}$	$\bar{m}$	$\hat{n}\left(b_0\right)$	$1-\widehat{\Phi\left(100\right)}$
1. *Constant* $M\left(u\right)=\mu_0 u$	0.0115 (14.29)	–	–	–	0.2213	0.0115	0.0129	31.77
2. *Linear* $M\left(u\right)=\mu_0 u+\mu_1^2 u^2$	-0.8824×10^{-2} (-8.47)	0.0174 (32.65)	–	–	0.0997	–	–	–
	–	0.0132 (42.24)	–	–	0.1312	0.0109	0.0134	17.57
3. *Piece-wise linear (PWL)* $M\left(u\right)=\mu_0 u+\delta\left(u\right)\mu_1^2\left(u-\bar{u}\right)^2$ $\delta\left(u\right)=\begin{cases}0 & \text{for } 0<u<\bar{u}\\ 1 & \text{for } u\geq\bar{u}\end{cases}$	0.3629×10^{-2} (32.15)	0.0441 (37.74)	–	54.84 (97.61)	0.0243	0.0101	0.0142	1.29
4. *Gompertz-Makeham (GM)* $M\left(u\right)=\mu_0 u+\left(\mu_1/\mu_2\right)\left[e^{\mu_2 u}-1\right]$	0.2437×10^{-2} (65.84)	0.5520×10^{-4} (20.51)	0.0964 (138.17)	–	0.0049	0.0102	0.0134	0.01

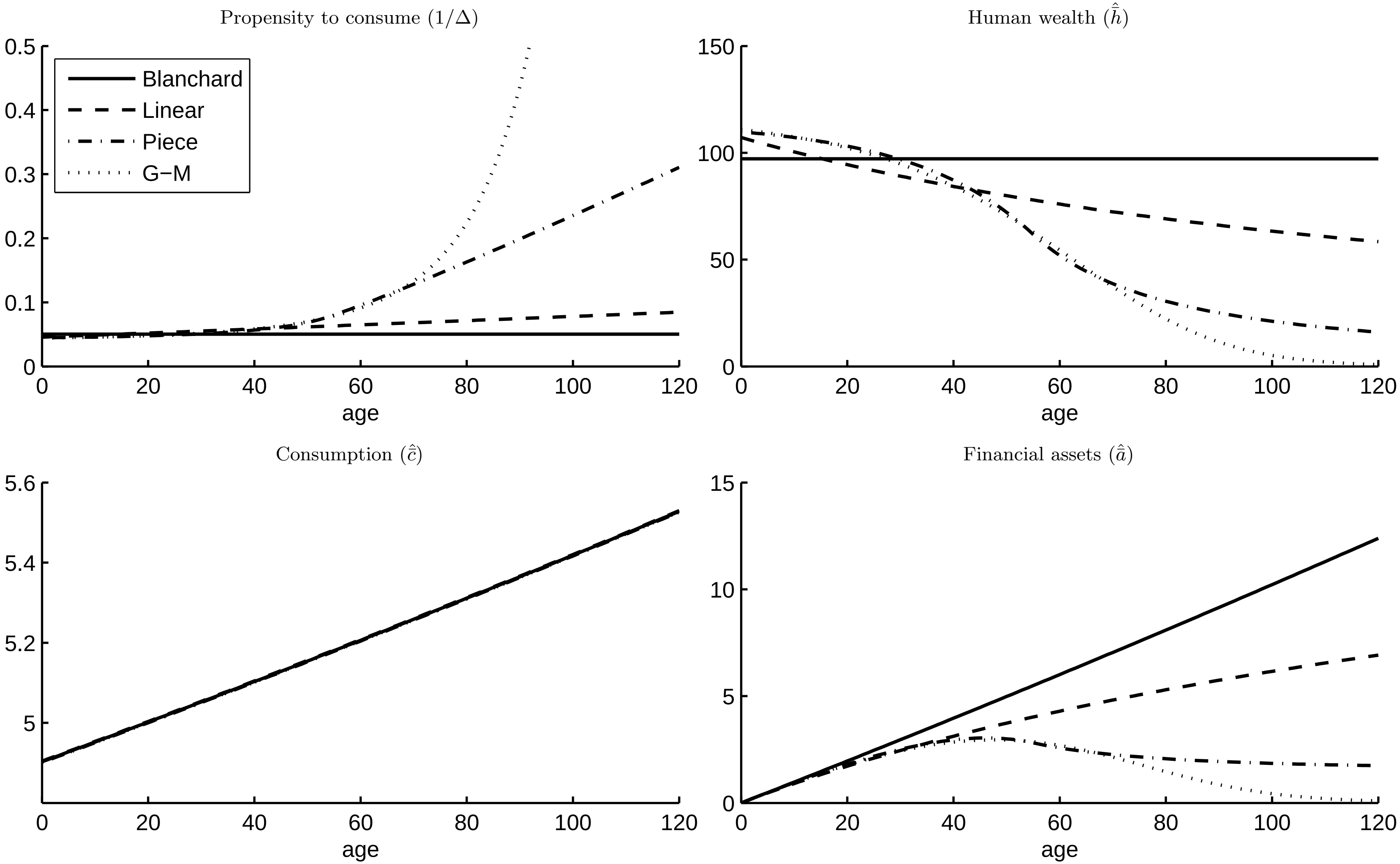

Fig. A.1. Steady-State Profiles for Individuals (different models)