The hiring subsidy *cum* firing tax in a search model of unemployment: Mathematical appendix

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A.1 Households

The current-value Hamiltonian for the household's optimization problem is (dropping the time index where it is obvious):

$$\mathcal{H}^H \equiv \log\left[C - \frac{(U+L)^{1+1/\sigma}}{1+1/\sigma}\right] + \eta_A\Big[rA + W(1-t_L)L + s_U U - T - C\Big] + \eta_L\left[fU - sL\right],$$

where A and L are the state variables, η_A and η_L are the corresponding co-state variables, and C and U are the control variables. (See Table A.1 for a list of all variables used).

The first-order conditions are the constraints,

$$\dot{L} \equiv fU - sL, \tag{A.1}$$

$$\dot{A} \equiv rA + W(1-t_L)L + s_U U - T - C, \tag{A.2}$$

the transversality conditions (see Chiang (1992, p. 252)),

$$0 = \lim_{\tau\to\infty} \eta_A(\tau)A(\tau)e^{\rho(t-\tau)}, \tag{A.3}$$

$$0 = \lim_{\tau\to\infty} \eta_L(\tau)L(\tau)e^{\rho(t-\tau)}, \tag{A.4}$$

and:

$$\frac{\partial\mathcal{H}^H}{\partial C} = 0: \ \frac{1}{X} = \eta_A, \tag{A.5}$$

$$\frac{\partial\mathcal{H}^H}{\partial U} = 0: \ \frac{(U+L)^{1/\sigma}}{X} = s_U\eta_A + f\eta_L, \tag{A.6}$$

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$$\dot{\eta}_A - \rho\eta_A = -\frac{\partial \mathcal{H}^H}{\partial A} = -r\eta_A, \tag{A.7}$$

$$\dot{\eta}_L - \rho\eta_L = -\frac{\partial \mathcal{H}^H}{\partial L} = \frac{(U+L)^{1/\sigma}}{X} - W(1-t_L)\eta_A + s\eta_L. \tag{A.8}$$

By combining (A.5) and (A.7) we obtain the Euler equation for full consumption:

$$\frac{\dot{X}}{X} = r - \rho. \tag{A.9}$$

This is **equation (5) in the paper**. In a small open economy there is only a well-defined steady state if $r = \rho$, so that $\dot{\eta}_A = \dot{X} = 0$ in that case. Combining (A.6) and (A.8) we obtain:

$$\dot{\eta}_L = (s + f + \rho)\eta_L - [W(1-t_L) - s_U]\,\eta_A. \tag{A.10}$$

A standard result from optimal control theory ensures that the co-state variables are the shadow prices of the corresponding stocks, i.e. for period t we have:

$$\eta_L(t) = \frac{\partial \Lambda(t)}{\partial L(t)}, \quad \eta_A(t) = \frac{\partial \Lambda(t)}{\partial A(t)}, \tag{A.11}$$

and for later periods $\eta_i(\tau)$ is the imputed value of stock $i \in \{A, L\}$ at time τ along the optimal path. A proof for these results is found in Léonard and Long (1992, pp. 152-153). Hence, since η_L is the utility value of the marginal job and η_A is the marginal utility of wealth, the *monetary value* of the marginal job is:

$$\lambda_L(\tau) \equiv \frac{\eta_L(\tau)}{\eta_A(\tau)}, \qquad \text{for } \tau \in [t, \infty). \tag{A.12}$$

Using (A.7) and (A.12) and noting that $\dot{\eta}_A = 0$, we can rewrite (A.10) as follows:

$$\dot{\lambda}_L = (s + f + r)\lambda_L - [W(1-t_L) - s_U]\,. \tag{A.13}$$

By integrating (A.13) we obtain two equivalent expressions for $\lambda_L(t)$:

$$\lambda_L(t) \equiv \int_t^\infty [W(\tau)(1-t_L) - s_U] \exp\left[-\int_t^\tau [s + f(\mu) + r]\, d\mu\right] d\tau \tag{A.14}$$

$$= \int_t^\infty [W(\tau)(1-t_L) - W_R(\tau)]\, e^{(s+r)(t-\tau)} d\tau, \tag{A.15}$$

where $W_R(\tau) \equiv s_U + f(\tau)\lambda_L(\tau)$ is the reservation wage. Equations (A.14)-(A.15) are reported in **equations (6)-(7) in the paper**.

Next we derive the closed-form solution for full consumption. First we note that the household lifetime budget constraint can be written as:

$$\int_t^\infty C(\tau)e^{r(t-\tau)}d\tau = A(t) - Z(t) \tag{A.16}$$

$$+ \int_t^\infty [W(\tau)(1-t_L)L(\tau) + s_U U(\tau)]\, e^{r(t-\tau)}d\tau,$$

where $Z(t)$ is the present value of lump-sum taxes:

$$Z(t) \equiv \int_t^\infty T(\tau)e^{r(t-\tau)}d\tau.$$

By combining (A.5)-(A.6) and (A.12) we can write labour market participation as:

$$U(\tau) + L(\tau) = W_R(\tau)^\sigma, \qquad W_R(\tau) \equiv s_U + f(\tau)\lambda_L(\tau), \tag{A.17}$$

where $W_R(\tau)$ is the reservation wage. This is **equation (9) in the paper**. Recall that:

$$X(\tau) \equiv C(\tau) - \frac{[U(\tau) + L(\tau)]^{1+1/\sigma}}{1 + 1/\sigma}. \tag{A.18}$$

By using (A.18) and (A.17), the present value of goods consumption can be written as:

$$\int_t^\infty C(\tau)e^{r(t-\tau)}d\tau = \int_t^\infty X(\tau)e^{r(t-\tau)}d\tau + \left(\frac{\sigma}{1+\sigma}\right)\int_t^\infty W_R(\tau)^{1+\sigma}e^{r(t-\tau)}d\tau. \tag{A.19}$$

The Euler equation for full consumption, (A.9), can be solved to yield:

$$X(\tau) = X(t)e^{(\rho-r)(t-\tau)}. \tag{A.20}$$

Combining (A.16), (A.19), and (A.20) yields:

$$X(t) = \rho\left[A(t) + \tilde{H}(t) - Z(t)\right], \tag{A.21}$$

where $\tilde{H}(t)$ is gross human wealth:

$$\tilde{H}(t) \equiv \int_t^\infty Y_F(\tau)e^{r(t-\tau)}d\tau, \tag{A.22}$$

and where $Y_F(\tau)$ is full income:

$$Y_F(\tau) \equiv W(\tau)(1-t_L)L(\tau) + s_U U(\tau) - \left(\frac{\sigma}{1+\sigma}\right)W_R(\tau)^{1+\sigma}. \tag{A.23}$$

Note that in the paper we work directly with after-tax human wealth ($H \equiv \tilde{H} - Z$). Equation (A.21) is **equation (8) in the paper**. Equations (A.22)-(A.23) are combined to yield **equation (10) in the paper**. We first state and prove the following proposition. This proposition yields **equations (11)-(12) in the paper**.

Proposition A.1 *Gross human wealth can be expressed in terms of the monetary shadow prices of work and unemployment.*

$$\tilde{H}(t) \equiv \lambda_L(t)L(t) + \lambda_U(t),$$

where $\lambda_L(t)$ is defined in (A.14) and $\lambda_U(t)$ is defined as:

$$\lambda_U(t) \equiv \left(\frac{1}{1+\sigma}\right)\int_t^\infty W_R(\tau)^{1+\sigma}e^{r(t-\tau)}d\tau.$$

Proof. From the definitions of λ_L and λ_U we find (ignoring the time index):

$$\frac{d}{d\tau}\left[e^{r(t-\tau)}(\lambda_L L + \lambda_U)\right] = e^{r(t-\tau)}\left[\dot{\lambda}_L L + \lambda_L\dot{L} - r(\lambda_L L + \lambda_U) + \dot{\lambda}_U\right]. \tag{a}$$

The term in square brackets on the right-hand side of (a) can be written as:

$$\begin{aligned}
[.] &= (s+f+r)\lambda_L L - [W(1-t_L)-s_U]\,L + \lambda_L\,[fU - sL] \\
&\quad - r\,[\lambda_L L + \lambda_U] + r\lambda_U - \frac{W_R^{1+\sigma}}{1+\sigma} \\
&= f\lambda_L(L+U) - [W(1-t_L)-s_U]\,L - \frac{W_R^{1+\sigma}}{1+\sigma} \\
&= (s_U + f\lambda_L)(L+U) - W(1-t_L)L - s_U U - \frac{W_R^{1+\sigma}}{1+\sigma} \\
&= -W(1-t_L)L - s_U U + \left(\frac{\sigma}{1+\sigma}\right) W_R^{1+\sigma} = -Y_F, \qquad \text{(b)}
\end{aligned}$$

where use is made of (A.17). By substituting (b) into (a) and integrating we obtain:

$$\lambda_L(t)L(t) + \lambda_U(t) = \int_t^\infty Y_F(\tau)e^{r(t-\tau)}d\tau \equiv \tilde{H}(t), \qquad \text{(c)}$$

where we have used the fact that:

$$\lim_{\tau\to\infty} e^{r(t-\tau)}\,[\lambda_L(\tau)L(\tau) + \lambda_U(\tau)] = 0. \qquad \text{(d)}$$

This completes the proof of the proposition. □

A.2 Firms

The current-value Hamiltonian for the optimization problem is (again dropping time indexes):

$$\mathcal{H}^P \equiv \omega_0 L - \gamma_V V - WL + (\mu_L + s_V)\,[qV - sL]\,, \qquad \text{(A.24)}$$

where L is the state variable, μ_L is the co-state variable, and V is the control variable. The first-order conditions can be written as:

$$\frac{\partial \mathcal{H}^P}{\partial V} = 0: \quad -\gamma_V + (\mu_L + s_V)q = 0, \qquad \text{(A.25)}$$

$$\dot{\mu}_L - r\mu_L = -\frac{\partial \mathcal{H}^P}{\partial L} = -[\omega_0 - W] + s(\mu_L + s_V), \qquad \text{(A.26)}$$

$$0 = \lim_{\tau\to\infty}\left[\mu_L(\tau)L(\tau)e^{r(t-\tau)}\right]. \qquad \text{(A.27)}$$

Equation (A.25) is **equation (16) in the paper**. The co-state variable $\mu_L(\tau)$ represents the pecuniary value of an additional job to the firm at time τ. By integrating (A.26) forward and imposing the terminal condition (A.27) we obtain:

$$\mu_L(t) = \int_t^\infty [\omega_0 - W(\tau) - ss_V]\,e^{(s+r)(t-\tau)}d\tau. \qquad \text{(A.28)}$$

This is **equation (18) in the paper**. Finally, we state and prove the following proposition (that gives rise to **equation (17) in the paper**).

Proposition A.2 *The value of the firm is equal to the shadow value of the number of occupied jobs.*

$$A_P(t) \equiv \mu_L(t)L(t)$$

Proof. First we note that:

$$
\begin{aligned}
-\frac{d}{d\tau}\left[e^{r(t-\tau)}\mu_L L\right] &= e^{r(t-\tau)}\left[r\mu_L L - \dot{\mu}_L L - \mu_L \dot{L}\right] \\
&= e^{r(t-\tau)}\left[\omega_0 L - \gamma_V V - WL + s_V\left(qV - sL\right)\right] \\
&= e^{r(t-\tau)}\left[\omega_0 L - \gamma_V V - WL + s_V \dot{L}\right], \qquad \text{(a)}
\end{aligned}
$$

where use has been made of (A.1), (A.25), and (A.26). Integrating both sides of (a) for $\tau \in [t, \infty)$ yields:

$$
\begin{aligned}
-\int_t^\infty d\mu_L(\tau)L(\tau)e^{r(t-\tau)} &= \int_t^\infty \left[\left(\omega_0 - W(\tau)\right)L(\tau) - \gamma_V V(\tau) + s_V \dot{L}(\tau)\right] e^{r(t-\tau)} d\tau \\
\mu_L(t)L(t) &= A_P(t), \qquad \text{(b)}
\end{aligned}
$$

where we have used (A.27) in the final step. This completes the proof of the proposition. □

A.3 Bargaining

This subsection is the only place where we need to introduce an index i to indicate that we are studying a particular pairing between an unemployed worker and a production firm. Following standard practice in the search literature, we assume that the wage, W_i, results from bargaining between the firm and the worker and is set in order to maximize the following expression:

$$
\mathcal{H}_i^W \equiv \left[\lambda_L^i(t)\right]^\zeta \left[\mu_L^i(t) + s_V\right]^{1-\zeta},
$$

where ζ and $1-\zeta$ are, respectively, the bargaining weights of the worker and the firm, and where $\lambda_L^i(t)$ and $\mu_L^i(t)$ are obtained from, respectively, (A.14) and (A.28) by substituting $W(\tau) = W^i(\tau)$. The first-order condition determining the **current** wage, $W^i(t)$, is:

$$
\frac{d\mathcal{H}_i^W}{dW^i(t)} = \left(\frac{\zeta \mathcal{H}_i^W}{\lambda_L^i(t)}\right)\left(\frac{\partial \lambda_L^i(t)}{\partial W^i(t)}\right) + \left(\frac{(1-\zeta)\mathcal{H}_i^W}{\mu_L^i(t) + s_V}\right)\left(\frac{\partial \mu_L^i(t)}{\partial W^i(t)}\right) = 0. \qquad \text{(A.29)}
$$

Since $\partial \lambda_L^i(t)/\partial W^i(t) = 1 - t_L$ and $\partial \mu_L^i(t)/\partial W^i(t) = -1$ we can simplify (A.29) to:

$$
\zeta(1-t_L)\left[\mu_L^i(t) + s_V\right] = (1-\zeta)\lambda_L^i(t). \qquad \text{(A.30)}
$$

By using (A.30) and its time derivative (and assuming $\dot{t}_L = \dot{s}_V = 0$, so that there are no anticipation effects) we get:

$$
\begin{aligned}
\zeta(1-t_L)\left[\dot{\mu}_L^i(t) - (s+r)(\mu_L^i(t) + s_V)\right] &= (1-\zeta)\left[\dot{\lambda}_L^i(t) - (s+r)\lambda_L^i(t)\right] \\
\zeta(1-t_L)\left[W^i(t) - rs_V - \omega_0\right] &= (1-\zeta)\left[s_U + f(t)\lambda_L^i(t) - W^i(t)(1-t_L)\right]. \qquad \text{(A.31)}
\end{aligned}
$$

By solving (A.31) for the wage rate we obtain:

$$
W^i(t) = \zeta\left[\omega_0 + rs_V\right] + (1-\zeta)\left[\frac{s_U + f(t)\lambda_L^i(t)}{1-t_L}\right]. \qquad \text{(A.32)}
$$

Since all production firms are identical they all choose the same wage. The index i can thus be dropped from (A.32). This yields **equation (21) in the paper**.

An alternative expression for the wage equation can be derived as follows. First we note that in the symmetric equilibrium (with all matches identical) we can use (A.17), (A.30), and (A.25) to write the tax-adjusted reservation wage as:

$$W_R = s_U + \left(\frac{\zeta(1-t_L)}{1-\zeta}\right)\gamma_V\theta, \tag{A.33}$$

where we have used the fact that $f/q = \theta$. Equation (A.33) is stated in **footnote 11 in the paper**. By substituting (A.33) into (A.32) (again noting symmetry) we obtain:

$$W = (1-\zeta)\left[\frac{s_U}{1-t_L}\right] + \zeta\left[\omega_0 + rs_V + \gamma_V\theta\right]. \tag{A.34}$$

This is **equation (22) in the paper**.

A.4 Model solution

A.4.1 Transitional dynamics

In order to investigate the dynamical properties of the model, we first derive the key expressions for the labour market. By using (A.25) (and its time derivative) in (A.26) (and noting that $\dot{s}_V = 0$) we obtain the following expression:

$$-\frac{\dot{q}}{q} = s + r - \left[\omega_0 + rs_V - W\right]\frac{q}{\gamma_V}. \tag{A.35}$$

The matching function implies that $q = \theta^{-\epsilon}$ so that $-\dot{q}/q = \epsilon\dot{\theta}/\theta$. Equation (A.35) can thus be rewritten as:

$$\epsilon\left(\frac{\dot{\theta}}{\theta}\right) = s + r - \left(\frac{\omega_0 + rs_V - W}{\gamma_V}\right)\theta^{-\epsilon}. \tag{A.36}$$

By using (A.34) we can eliminate the wage rate from (A.36) and obtain a nonlinear differential equation in θ:

$$\epsilon\left(\frac{\dot{\theta}}{\theta}\right) = s + r + \left[\left(\frac{1-\zeta}{\gamma_V}\right)\left(\frac{s_U}{1-t_L} - \omega_0 - rs_V\right) + \zeta\theta\right]\theta^{-\epsilon}. \tag{A.37}$$

This is the unnumbered **equation stated in the proof of Proposition 1 in the paper**. The key thing to note is that θ is the only endogenous variable appearing in (A.37). It follows from (A.37) that:

$$\epsilon\left(\frac{\partial\dot{\theta}}{\partial\theta}\right) = s + r + \theta^{1-\epsilon}\left[\zeta + (1-\epsilon)\left(\frac{W - (\omega_0 + rs_V)}{\theta\gamma_V}\right)\right]. \tag{A.38}$$

By evaluating (A.38) around the steady state we find:

$$\epsilon\left(\frac{\partial\dot{\theta}}{\partial\theta}\right) = \theta^{1-\epsilon}\left[\zeta + \epsilon\left(\frac{\omega_0 + rs_V - W}{\theta\gamma_V}\right)\right] > 0. \tag{A.39}$$

Hence, equation (A.37) is an unstable differential equation in θ and the only economically sensible solution is that θ is equal to its (constant) steady-state value. (Note that $\omega_0 + rs_V > W$ follows from the postulated existence of a steady state in (A.36)). We denote this steady-state value by θ^*.

Following any (unanticipated and permanent) shock to any of the exogenous variables in (A.37), θ immediately jumps to its new steady-state value. Since q and f depend only on θ, there is no transitional dynamics in these variables either. By (A.25), (A.30), (A.33), and (A.34) it follows that there is no transitional dynamics in μ_L, λ_L, W_R, and W either. We summarize the results obtained so far in the following proposition (**Proposition 1 in the paper**).

Proposition A.3 *In the absence of anticipation effects, there is no transitional dynamics in* θ*,* q*,* f*,* μ_L*,* λ_L*,* W_R*, and* W*. These variables jump to their new steady-state values following any unanticipated and permanent shock to any of the parameters* (s, r, ω_0, γ_V, ϵ) *or policy variables* (s_U, s_V, t_L)*.*

Proof. See text. □

Next we turn to the dynamics of the unemployment rate. The unemployment rate is defined in the usual fashion as the proportion of job seekers in the labour force:

$$u \equiv \frac{U}{U+L}. \tag{A.40}$$

By differentiating (A.40) with respect to time and noting that $U + L = W_R^{\sigma}$ we get:

$$\frac{\dot{u}}{u} = \frac{\dot{U}}{U} - \sigma\left(\frac{\dot{W}_R}{W_R}\right). \tag{A.41}$$

Since $1 - M = U + L$ we can write the unemployment flow as:

$$\dot{U} = -\dot{L} - \dot{M} = sL - fU - \dot{M}, \tag{A.42}$$

where use is made of (A.1). By using (A.42) and noting that $L/U = (1-u)/u$, $\dot{M}/(1-M) = -\sigma\dot{W}_R/W_R$, and $1 - M = U + L$ we obtain:

$$\frac{\dot{U}}{U} = s\left(\frac{1-u}{u}\right) - f + \frac{\sigma}{u}\left(\frac{\dot{W}_R}{W_R}\right). \tag{A.43}$$

By combining (A.41) and (A.43) we obtain:

$$\dot{u} = s - (s+f)u + (1-u)\sigma\left(\frac{\dot{W}_R}{W_R}\right). \tag{A.44}$$

It follows from Proposition A.3 that there is no transitional dynamics in W_R for unanticipated and permanent shocks in the parameters or policy variables, i.e. $\dot{W}_R = 0$. Equation (A.44) is a stable differential equation which determines a unique steady-state unemployment rate, u^*:

$$u^* = \frac{s}{s+f^*}, \tag{A.45}$$

where f^* is the steady-state job finding rate. By manipulating (A.40) and noting that $U+L = W_R^{\sigma}$ we can write the unemployment rate as follows:

$$u = \frac{(U+L)-L}{U+L} = \frac{W_R^{\sigma} - L}{W_R^{\sigma}} = 1 - W_R^{-\sigma}L. \tag{A.46}$$

Equation (A.46) clarifies the dynamical behaviour of the unemployment rate. If labour supply is exogenous (so that $\sigma = 0$) then $U + L = 1$ and $u = U = 1 - L$. Since the stock of employed workers (L) is predetermined, the same holds for the unemployment rate in that case. Following any (policy or parameter) shock, the dynamics of the unemployment rate mirrors that in the stock of employment. Matters are different if labour supply is endogenous (so that $\sigma > 0$). In that case, a non-zero impact effect on the unemployment rate is made possible by the change in labour force participation. Indeed, as is shown in (A.46), an increase in the reservation wage, W_R, leads to an increase (at impact) in the unemployment rate because households expand their labour force participation but cannot immediate locate a job due to the matching friction. Following its impact jump, the unemployment rate gradually moves to its steady-state equilibrium defined in (A.45).

Note that a useful expression for the differential equation for the stock of employment is obtained by using $U + L = W_R^\sigma$ in equation (A.1):

$$\dot{L} = f W_R^\sigma - (s + f)L. \tag{A.47}$$

Steady-state employment, L^*, and job-seeking activities, U^*, are thus equal to:

$$L^* = \left(\frac{f^*}{s + f^*}\right)(W_R^*)^\sigma, \qquad U^* = \left(\frac{s}{s + f^*}\right)(W_R^*)^\sigma, \tag{A.48}$$

These expressions are found in **equation (29) in the paper**.

A.4.2 Graphical apparatus

The steady-state effects on the labour market have been illustrated graphically in the text. The wage setting (WS) equation is defined in (A.34) and is upward sloping in (W, θ) space. An increase in s_U, t_L, s_V, ω_0, or γ_V leads to an upward shift in the WS curve. Since there is no transitional dynamics in μ_L (see Proposition A.3), we can use (A.25) and (A.26) to derive the following expression for the vacancy creation (VC) curve:

$$\frac{\omega_0 + r s_V - W}{s + r} = \frac{\gamma_V}{q(\theta)}. \tag{A.49}$$

This is **equation (26) in the paper**. Since $q'(\theta) < 0$ is follows from (A.49) that the VC curve is downward sloping. An increase in s_V or ω_0 and a decrease in γ_V all lead to an upward shift in the VC curve.

A.4.3 Steady-state effects of s_V

The effects of an unanticipated and permanent change in s_V can be computed as follows. First we differentiate the WS and VC curves (given in, respectively, (A.34) and (A.49)) to get:

$$\frac{r ds_V - dW}{s + r} = \frac{\epsilon \gamma_V}{q} \frac{d\theta}{\theta}, \tag{A.50}$$

$$dW = \zeta \left[r ds_V + \gamma_V d\theta \right], \tag{A.51}$$

where $\epsilon \equiv -\theta q'(\theta)/q(\theta)$. Solving (A.50)-(A.51) yields:

$$\frac{d\theta^*}{ds_V} = \frac{(1 - \zeta) r \theta^*}{\zeta \gamma_V \theta^* + \epsilon \left(\omega_0 + r s_V - W^*\right)} > 0, \tag{A.52}$$

$$\frac{dW^*}{ds_V} = \zeta r \left(\frac{\gamma_V \theta^* + \epsilon \left(\omega_0 + r s_V - W^*\right)}{\zeta \gamma_V \theta^* + \epsilon \left(\omega_0 + r s_V - W^*\right)} \right) > 0. \tag{A.53}$$

By rewriting (A.52)-(A.53) we obtain **equations (27)-(28) in the paper**. By using (A.45) we obtain the effect on the steady-state unemployment rate:

$$\frac{du^*}{ds_V} = -\frac{u^* f'(\theta^*)}{s + f^*} \left(\frac{d\theta^*}{ds_V}\right) < 0, \tag{A.54}$$

where $f'(\theta^*) > 0$. By using (A.33) we find the effect on the steady-state reservation wage:

$$\frac{dW_R^*}{ds_V} = \frac{\zeta\gamma_V(1-t_L)}{1-\zeta}\left(\frac{d\theta^*}{ds_V}\right) > 0, \tag{A.55}$$

so that it follows from (A.17) that labour force participation increases:

$$\frac{d(U+L)^*}{ds_V} = \sigma\left(W_R^*\right)^{\sigma-1}\left(\frac{dW_R^*}{ds_V}\right) > 0. \tag{A.56}$$

The long-run effect on the employment stock follows from the steady-state version of (A.46) (or, equivalently, from (A.48)):

$$\begin{aligned}\frac{dL^*}{ds_V} &= (1-u^*)\frac{d(U+L)^*}{ds_V} - (U+L)^*\frac{du^*}{ds_V} \\ &= \left(W_R^*\right)^{\sigma}\left[(1-u^*)\left(\frac{\sigma\zeta\gamma_V(1-t_L)}{(1-\zeta)W_R^*}\right) + u^*\left(\frac{f'(\theta^*)}{s+f^*}\right)\right]\left(\frac{d\theta^*}{ds_V}\right) > 0,\end{aligned} \tag{A.57}$$

where use is made of (A.54)-(A.56), and (A.17). The employment stock rises unambiguously because the unemployment rate falls and labour market participation rises. Since $U^* \equiv u^*(U+L)^*$ it follows that the long-run effect on the *level* of unemployment is:

$$\begin{aligned}\frac{dU^*}{ds_V} &= u^*\frac{d(U+L)^*}{ds_V} + (U+L)^*\frac{du^*}{ds_V} \\ &= u^*\left(W_R^*\right)^{\sigma}\left[\frac{\sigma\zeta\gamma_V(1-t_L)}{(1-\zeta)W_R^*} - \frac{f'(\theta^*)}{s+f^*}\right]\left(\frac{d\theta^*}{ds_V}\right) \lesseqgtr 0.\end{aligned} \tag{A.58}$$

The long-run effect on U is ambiguous because the unemployment *rate* falls but the labour market participation rises. If σ is small (large) the rate effect dominates (is dominated by) the participation effect so that the level of unemployment falls (rises). The long-run effect on vacancies is obtained by noting that $V^* \equiv \theta^* U^*$ so that:

$$\begin{aligned}\frac{dV^*}{ds_V} &= \theta^*\frac{dU^*}{ds_V} + U^*\frac{d\theta^*}{ds_V} \\ &= u^*\left(W_R^*\right)^{\sigma}\left[\frac{\sigma\theta^*\zeta\gamma_V(1-t_L)}{(1-\zeta)W_R^*} + \left(1 - \frac{f'(\theta^*)\theta^*}{s+f^*}\right)\right]\left(\frac{d\theta^*}{ds_V}\right) > 0,\end{aligned} \tag{A.59}$$

where the sign follows from the fact that the term in round brackets on the right-hand side is positive:

$$\begin{aligned}1 - \frac{f'(\theta^*)\theta^*}{s+f^*} &= 1 - \left(\frac{f^*}{s+f^*}\right)\left(\frac{f'(\theta^*)\theta^*}{f^*}\right) \\ &= 1 - \left(\frac{f^*}{s+f^*}\right)(1-\epsilon) > 0,\end{aligned}$$

since $1-\epsilon \equiv f'(\theta^*)\theta^*/f^*$ and $0 < \epsilon < 1$ (see above). Vacancies increase unambiguously. The increase is larger, the higher the labour supply elasticity. By using equation (A.25) we obtain the long-run result on μ_L:

$$\frac{d\mu_L^*}{ds_V} = (1-\zeta)\left(\frac{r}{s+r}\right)\left(\frac{\epsilon\left(\omega_0 + rs_V - W^*\right)}{\zeta\gamma_V\theta^* + \epsilon\left(\omega_0 + rs_V - W^*\right)}\right) - 1 < 0. \tag{A.60}$$

From (A.30) it follows that:

$$\begin{aligned}\frac{d\lambda_L^*}{ds_V} &= \frac{\zeta(1-t_L)}{1-\zeta}\left(\frac{d\mu_L^*}{ds_V} + 1\right) \\ &= \left(\frac{r\zeta(1-t_L)}{s+r}\right)\left(\frac{\epsilon\left(\omega_0 + rs_V - W^*\right)}{\zeta\gamma_V\theta^* + \epsilon\left(\omega_0 + rs_V - W^*\right)}\right) > 0.\end{aligned} \tag{A.61}$$

A.5 Welfare analysis

In this section we conduct the welfare analysis. In the first subsection we solve the first-best social planning programme. In the second subsection we study second-best issues.

A.5.1 The first-best optimum

The social planner maximizes the lifetime welfare of the representative agent which is given by:

$$\Lambda(t) \equiv \int_t^\infty \log\left[C(\tau) - \frac{[U(\tau)+L(\tau)]^{1+1/\sigma}}{1+1/\sigma}\right] e^{\rho(t-\tau)} d\tau. \tag{A.62}$$

The aggregate resource constraint facing the policy maker in the small open economy is given by the current account and the national solvency condition:

$$\begin{aligned} \dot{A}_F(t) &= rA_F(t) + \omega_0 L(t) - \gamma_V V(t) - C(t), && \text{(A.63)} \\ \lim_{\tau\to\infty} A_F(t)e^{r(t-\tau)} &= 0, && \text{(A.64)} \end{aligned}$$

where $A_F(t)$ is taken as given (the planner does not default on foreign debt if there is any). The differential equation for the stock of labour is given in (A.1). It can be rewritten (by noting that $fU = qV$) as:

$$\dot{L}(\tau) = q\left(\frac{V(\tau)}{U(\tau)}\right) V(\tau) - sL(\tau), \tag{A.65}$$

where $L(t)$ is taken as given. The current-value Hamiltonian for the social planning problem is:

$$\begin{aligned} \mathcal{H}^{SO} \equiv{}& \log\left[C(\tau) - \frac{[U(\tau)+L(\tau)]^{1+1/\sigma}}{1+1/\sigma}\right] + \eta_L\left[q\left(\frac{V}{U}\right)V - sL\right] \\ &+\eta_F\left[rA_F(t) + \omega_0 L(t) - \gamma_V V(t) - C(t)\right], && \text{(A.66)} \end{aligned}$$

where η_L and η_F are the co-state variables for, respectively, L and A_F. The control variables are C, U, and V. The state variables are L and A_F. The key first-order conditions characterizing the social optimum are:

$$\begin{aligned} &\frac{\partial \mathcal{H}^{SO}}{\partial C} = 0\text{:}\quad \frac{1}{X} = \eta_F, && \text{(A.67)} \\ &\frac{\partial \mathcal{H}^{SO}}{\partial U} = 0\text{:}\quad \frac{(U+L)^{1/\sigma}}{X} = -\eta_L \theta^2 q'(\theta), && \text{(A.68)} \\ &\frac{\partial \mathcal{H}^{SO}}{\partial V} = 0\text{:}\quad \eta_L\left[q(\theta) + \theta q'(\theta)\right] = \eta_F \gamma_V, && \text{(A.69)} \\ &\dot{\eta}_L - \rho\eta_L = -\frac{\partial \mathcal{H}^{SO}}{\partial L} = \frac{(U+L)^{1/\sigma}}{X} - \omega_0\eta_F + s\eta_L, && \text{(A.70)} \\ &\dot{\eta}_F - \rho\eta_F = -\frac{\partial \mathcal{H}^{P}}{\partial A_F} = -r\eta_F, && \text{(A.71)} \end{aligned}$$

where $\theta \equiv V/U$. In the small open economy we must have that $r = \rho$ so that it follows from (A.71) that $\dot{\eta}_F = 0$ and from (A.67) that $\dot{X} = 0$ (flat profile for full consumption). By combining (A.67) and (A.68) we obtain the following expression for the optimal search effort:

$$(U+L)^{1/\sigma} = \xi_L f(\theta)\epsilon, \qquad \xi_L \equiv \frac{\eta_L}{\eta_F}, \tag{A.72}$$

where we have used the fact that $\epsilon \equiv -\theta q'(\theta)/q(\theta)$ and $f(\theta) = \theta q(\theta)$. Using the definition of ξ_L, equations (A.69) and (A.70) can be rewritten as:

$$\begin{aligned} \frac{\gamma_V}{q(\theta)} &= \xi_L(1-\epsilon), && \text{(A.73)} \\ \dot{\xi}_L &= [s+\rho+f(\theta)\epsilon]\,\xi_L - \omega_0. && \text{(A.74)} \end{aligned}$$

By using (A.73) and its time derivative in (A.74) (and assuming that ϵ is constant) we obtain an unstable nonlinear differential equation in θ:

$$\epsilon\left(\frac{\dot{\theta}}{\theta}\right) = s+\rho+f(\theta)\epsilon - \frac{\omega_0(1-\epsilon)q(\theta)}{\gamma_V}. \tag{A.75}$$

As in the market solution, it is optimal to have a constant θ in the social optimum. It follows from (A.73) that ξ_L is also time-invariant.

In order to derive the efficiency property of the market solution we match the conditions for the first-best social optimum with the corresponding ones for the market solution. The latter are summarized by:

$$\begin{aligned} \dot{X} &= 0, && \text{(A.76)} \\ U+L &= [s_U + f(\theta)\lambda_L]^{\sigma}, && \text{(A.77)} \\ \frac{\gamma_V}{q(\theta)} &= \mu_L + s_V, && \text{(A.78)} \\ \mu_L + s_V &= \frac{(1-\zeta)\left[\omega_0 + \rho s_V - \left(\frac{s_U}{1-t_L}\right)\right]}{s+\rho+\zeta f(\theta)}. && \text{(A.79)} \end{aligned}$$

Equation (A.76) is obtained from (A.9) by setting $r = \rho$, (A.77) is the same as (A.17), and (A.78) is the same as (A.25). The derivation of (A.79) warrants some comment. It is obtained by substituting (A.32) (in symmetric equilibrium) into (A.26) (with $\dot{\mu}_L = 0$ imposed) and noting (A.30) (also in symmetric equilibrium).

Like in the social optimum, the market solution selects a flat time profile for full consumption. Equations (A.72) and (A.77) coincide if $\lambda_L = \epsilon\xi_L$ and $s_U = 0$. Equations (A.73) and (A.78) match up if $\xi_L(1-\epsilon) = \mu_L + s_V = (1-\zeta)\lambda_L/[\zeta(1-t_L)]$, or, in view of the earlier condition, if $\zeta = \epsilon$ and $t_L = 0$. Finally, the steady-state version of (A.74) matches with (A.79) if $\zeta = \epsilon$ and $s_V = 0$. We summarize the results in the following proposition. (**Proposition 2 in the paper**).

Proposition A.4 *The market solution is efficient if the following conditions hold:*

$$\zeta = \epsilon, \quad t_L = s_V = s_U = 0.$$

The first of these conditions is the so-called Hosios condition [after Hosios (1990)]. With a Cobb-Douglas matching function ϵ is a constant and the Hosios condition is a knife-edge condition.

A.5.2 The second-best optimum

In this section we study the optimum vacancy subsidy/ firing tax in the presence of pre-existing distortions. We call this the second-best optimal s_V. We work directly with the decentralized (market) equilibrium. In the market equilibrium there is no transition in X, θ, $f(\theta)$, $q(\theta)$, W, and

W_R. Hence, lifetime utility of the representative agent is equal to:

$$\begin{aligned}\Lambda(t) &\equiv \int_t^{\infty} \log X(\tau) e^{\rho(t-\tau)} d\tau \\ &= \frac{\log X(\tau)}{\rho} \quad \Rightarrow \\ \rho\Lambda(t) &= \log\rho + \log\left[A(t) + \tilde{H}(t) - Z(t)\right], \end{aligned} \tag{A.80}$$

where we have used (A.21) in the final step. Maximizing welfare is thus equivalent to maximizing total wealth, $\Omega(t) \equiv A(t) + \tilde{H}(t) - Z(t)$. Recall the following definitions:

$$\tilde{H}(t) \equiv \int_t^{\infty} \left[W(\tau)(1-t_L)L(\tau) + s_U U(\tau) - \left(\frac{\sigma}{1+\sigma}\right) W_R(\tau)^{1+\sigma}\right] e^{r(t-\tau)} d\tau, \tag{A.81}$$

$$A_P(t) = \int_t^{\infty} \left[Y(\tau) - \gamma_V V(\tau) - W(\tau)L(\tau) + s_V \dot{L}(\tau)\right] e^{r(t-\tau)} d\tau, \tag{A.82}$$

$$Z(t) = \int_t^{\infty} \left[s_U U(\tau) + s_V \dot{L}(\tau) - t_L W(\tau) L(\tau)\right] e^{r(t-\tau)} d\tau, \tag{A.83}$$

$$A(\tau) \equiv A_P(\tau) + A_F(\tau). \tag{A.84}$$

Total wealth can be rewritten by using (A.81)-(A.84):

$$\Omega(t) = A_F(t) + \int_t^{\infty} \left[\omega_0 L(\tau) - \gamma_V V(\tau) - \left(\frac{\sigma}{1+\sigma}\right) W_R^{1+\sigma}\right] e^{\rho(t-\tau)} d\tau, \tag{A.85}$$

where $A_F(t)$ is the pre-existing stock of net foreign assets. After some manipulation we can rewrite the term in square brackets on the right-hand side of (A.85) as:

$$[.] = (\omega_0 + \gamma_V\theta)L(\tau) - \gamma_V\theta W_R^{\sigma} - \left(\frac{\sigma}{1+\sigma}\right) W_R^{1+\sigma}. \tag{A.86}$$

By using (A.1) we can write $\dot{L} = fW_R^{\sigma} - (s+f)L$. Solving this differential equation yields:

$$L(\tau) = \left[1 - e^{(s+f)(t-\tau)}\right]\left(\frac{f}{s+f}\right) W_R^{\sigma} + e^{(s+f)(t-\tau)} L(t), \tag{A.87}$$

where $L(t)$ is the initial stock of employment. Equation (A.87) is the same as **equation (30) in the paper**. (In the text of the paper we add asterisks to make sure the reader is reminded of the fact that variables like f and W_R always attain their respective steady-state values). By using (A.86) and (A.87) we can rewrite (A.85) as:

$$\Omega(t) = A_F(t) + \left(\frac{1}{\rho}\right)\left[\left(\frac{\omega_0 + \gamma_V\theta}{s+f+\rho}\right)\left[fW_R^{\sigma} + \rho L(t)\right] - \gamma_V\theta W_R^{\sigma} - \left(\frac{\sigma}{1+\sigma}\right) W_R^{1+\sigma}\right]. \tag{A.88}$$

Recall (from (A.33)) that the reservation wage can be written as:

$$W_R = s_U + \frac{\zeta(1-t_L)}{1-\zeta}\gamma_V\theta. \tag{A.89}$$

Equations (A.88)-(A.89) express $\Omega(t)$ in terms of predetermined variables ($A_F(t)$ and $L(t)$), policy parameters (s_V, s_U, and t_L) and the labour market tightness variable (θ). The second-best optimal vacancy subsidy/ firing tax is computed by setting $d\Omega(t)/ds_V = 0$ and noting (A.52).

A.5.2.1 Exogenous labour supply

We first look at a special case of the model for which labour supply is exogenous (so that $\sigma = 0$). Equation (A.88) reduces to:

$$\Omega(t) = A_F(t) + \left(\frac{1}{\rho}\right)\left[\left(\frac{\omega_0 + \gamma_V \theta}{s+f+\rho}\right)[f + \rho L(t)] - \gamma_V \theta\right]. \tag{A.90}$$

This is the unnumbered equation stated in the **proof of Proposition 3 in the paper**. By differentiating (A.90) with respect to θ we obtain:

$$\begin{aligned}
\rho\left(\frac{d\Omega(t)}{d\theta}\right) &= \left(1 - \frac{f+\rho L(t)}{s+f+\rho}\right)\left[\left(\frac{\omega_0+\gamma_V\theta}{s+f+\rho}\right)\frac{df}{d\theta} - \gamma_V\right] \\
&= \left(\frac{sq}{s+f}\right)\left(\frac{1}{s+f+\rho}\right)\left[(\omega_0+\gamma_V\theta)(1-\epsilon) - \frac{\gamma_V}{q}(s+f+\rho)\right], \qquad \text{(A.91)}
\end{aligned}$$

where $1-\epsilon \equiv \theta f'(\theta)/f(\theta)$, $f(\theta) \equiv \theta q(\theta)$, and we have assumed that the economy is initially in the steady state (so that $L(t) = L^* = f/(s+f)$. The term in square brackets on the right-hand side of (A.91) can be simplified as follows:

$$\begin{aligned}
[.] &= \omega_0(1-\epsilon) - \left(\frac{\gamma_V}{q}\right)(s+\epsilon f+\rho) \\
&= \omega_0(1-\epsilon) - (1-\zeta)\left[\omega_0 + \rho s_V - \left(\frac{s_U}{1-t_L}\right)\right]\left(\frac{s+\epsilon f+\rho}{s+\zeta f+\rho}\right) \\
&= \omega_0(\zeta-\epsilon)\left(\frac{s+f+\rho}{s+\zeta f+\rho}\right) - (1-\zeta)\left[\rho s_V - \left(\frac{s_U}{1-t_L}\right)\right]\left(\frac{s+\epsilon f+\rho}{s+\zeta f+\rho}\right), \qquad \text{(A.92)}
\end{aligned}$$

where we have used (A.78)-(A.79) (to eliminate γ_V/q) in going from the first to the second line. By substituting (A.92) into (A.91) we obtain:

$$\begin{aligned}
\rho\left(\frac{d\Omega(t)}{d\theta}\right) &= \left(\frac{qU^*}{s+\zeta f+\rho}\right)\times \\
&\quad \left[(\zeta-\epsilon)\omega_0 - (1-\zeta)\left(\frac{s+\epsilon f+\rho}{s+f+\rho}\right)\left(\rho s_V - \left(\frac{s_U}{1-t_L}\right)\right)\right], \qquad \text{(A.93)}
\end{aligned}$$

where $U^* = s/(s+f)$. This is the **equation stated in Proposition 3 in the paper**. The second-best optimal vacancy subsidy / firing tax, $\hat{s}_V$, is such as to render the term in square brackets on the right-hand side of (A.93) equal to zero:

$$\rho\hat{s}_V = \left(\frac{s_U}{1-t_L}\right) + (\zeta-\epsilon)\left(\frac{\omega_0}{1-\zeta}\right)\left(\frac{s+f+\rho}{s+\epsilon f+\rho}\right). \tag{A.94}$$

This expression is interpreted thoroughly in the text. See **equation (32) in the text**.

A.5.2.2 Endogenous labour supply

With endogenous labour supply ($\sigma > 0$) we need to work with (A.88) and recognize the endogeneity of the reservation wage, W_R. By differentiating (A.88) with respect to θ and simplifying we obtain:

$$\begin{aligned}
\rho\left(\frac{d\Omega(t)}{d\theta}\right) &= \sigma\Gamma + \left(\frac{qU^*}{s+\zeta f+\rho}\right)\times \\
&\quad \left[\omega_0(\zeta-\epsilon) - (1-\zeta)\left(\frac{s+\epsilon f+\rho}{s+f+\rho}\right)\left(\rho s_V - \left(\frac{s_U}{1-t_L}\right)\right)\right], \qquad \text{(A.95)}
\end{aligned}$$

where Γ is defined as:

$$\Gamma \equiv W_R^{\sigma-1} \left(\frac{dW_R}{d\theta}\right) \left[f \left(\frac{\omega_0 + \gamma_V \theta}{s+f+\rho}\right) - \gamma_V \theta - W_R\right]. \tag{A.96}$$

The derivation of the second term on the right-hand side of (A.95) is exactly the same as for the case of exogenous labour supply, except that now $L^* = fW_R^\sigma/(s+f)$ and $U^* = sW_R^\sigma/(s+f)$ (see (A.48) above). The term involving Γ represents the welfare effect operating via the reservation wage.

By using (A.33), the term in square brackets on the right-hand side of (A.96), which we denote by Γ_1, can be rewritten as follows:

$$\begin{aligned} \Gamma_1 &\equiv f \left(\frac{\omega_0 + \gamma_V \theta}{s+f+\rho}\right) - \gamma_V \theta - W_R \\ &= f \left(\frac{\omega_0 + \gamma_V \theta}{s+f+\rho}\right) - s_U - \left(\frac{1-\zeta t_L}{1-\zeta}\right) \gamma_V \theta \\ &= f \left[\frac{\omega_0 + \gamma_V \theta}{s+f+\rho} - \left(\frac{1-\zeta t_L}{1-\zeta}\right) \frac{\gamma_V}{q}\right] - s_U, \end{aligned} \tag{A.97}$$

where we have used the fact that $f(\theta) = \theta q(\theta)$ in going from the second to the third line. We denote the term in square brackets on the right-hand side of (A.97) by Γ_2. We can rewrite Γ_2 can be written as follows:

$$\begin{aligned} \Gamma_2 &= \frac{\omega_0}{s+f+\rho} + \left(\frac{1}{1-\zeta}\right) \left[\zeta t_L - \frac{s+\zeta f+\rho}{s+f+\rho}\right] \frac{\gamma_V}{q} \\ &= \frac{\omega_0}{s+f+\rho} + \left[\zeta t_L - \frac{s+\zeta f+\rho}{s+f+\rho}\right] \left[\frac{\omega_0 + \rho s_V - \left(\frac{s_U}{1-t_L}\right)}{s+\zeta f+\rho}\right] \\ &= \left(\frac{1}{s+f+\rho}\right) \left[t_L \phi_0 \omega_0 - (1 - t_L \phi_0) \left(\rho s_V - \left(\frac{s_U}{1-t_L}\right)\right)\right], \end{aligned} \tag{A.98}$$

where we have used (A.78)-(A.79) to go from the first to the second line and where ϕ_0 is defined as:

$$0 < \phi_0 \equiv \zeta \left(\frac{s+f+\rho}{s+\zeta f+\rho}\right) < 1. \tag{A.99}$$

By using (A.97)-(A.98) in (A.96) we obtain the final expression for Γ:

$$\begin{aligned} \Gamma \equiv\ & W_R^{\sigma-1} \left(\frac{dW_R}{d\theta}\right) \times \\ & \left[\left(\frac{f}{s+f+\rho}\right) \left[t_L \phi_0 \omega_0 - (1 - t_L \phi_0) \left(\rho s_V - \left(\frac{s_U}{1-t_L}\right)\right)\right] - s_U\right]. \end{aligned} \tag{A.100}$$

In **footnote 13 of the paper** we argue that even if $\zeta = \epsilon$ and $s_V = s_U = 0$, a positive labour income tax rate implies that welfare increases with the introduction of a hiring subsidy *cum* firing tax. This result follows by using (A.95) and (A.100):

$$\rho \left(\frac{d\Omega(t)}{d\theta}\right) = \sigma t_L W_R^{\sigma-1} \left(\frac{dW_R}{d\theta}\right) \left(\frac{f \phi_0 \omega_0}{s+f+\rho}\right) > 0. \tag{A.101}$$

Macroeconomic variables:		*Prices:*	
C	consumption	λ_L	value of job to households
M	leisure	μ_L	value of job to firm
L	employment	W	gross wage rate
U	unemployment	W_R	reservation wage
V	vacancies	r	world interest rate
X	full consumption		
Y_F	full income		
Wealth components:		*Fiscal variables:*	
A	financial assets	s_U	unemployment subsidy
A_F	net foreign assets	s_V	subsidy on net firm hirings
A_P	shares in production firms	T	lump-sum taxes
$\tilde{H}$	gross human wealth	t_L	tax on labour income
		Z	present value of lump-sum taxes
Matching variables:		*Structural parameters*:	
f	job-finding rate households	ϵ	elasticity of the $q(\theta)$ function
q	worker-finding rate ($f = \theta q$)	ζ	bargaining power household
s	job-destruction rate	ρ	rate of time preference
θ	labour market tightness (V/U)	σ	intratemporal labour supply elasticity

Table A.1: Notation used

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