Environmental abatement and intergenerational distribution: Mathematical appendix

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1 Model solution and stability

1.1 Construction of Figure 1 and proof of Proposition 1

We first construct the phase diagram. Dropping time indices, the steady-state of the model in Table 1 can be written as:

$$C = \gamma_0 K^{1-\epsilon_L} - G, \qquad (\dot{K}=0 \text{ line})$$
$$C = \frac{\lambda(\rho+\lambda)\,[K+B]}{r(K)-\rho}, \qquad (\dot{C}=0 \text{ line})$$

where we write $r(K) \equiv \gamma_0(1-\epsilon_L)K^{-\epsilon_L}$, from which the following results can be derived:

$$r(K) \geq 0, \ \lim_{K\to 0} r(K) = \infty, \ \lim_{K\to\infty} r(K) = 0,$$
$$r_K(K) \equiv -\epsilon_L \left(\frac{r(K)}{K}\right) \leq 0, \ \lim_{K\to 0} r_K(K) = \infty, \ \lim_{K\to\infty} r_K(K) = 0.$$

The $\dot{K}=0$ line is concave towards the origin (as drawn in Figure 1):

$$\left(\frac{dC}{dK}\right)_{\dot{K}=0} = r(K), \ \lim_{K\to 0}\left(\frac{dC}{dK}\right)_{\dot{K}=0} = \infty,$$
$$\lim_{K\to\infty}\left(\frac{dC}{dK}\right)_{\dot{K}=0} = 0, \ \left(\frac{d^2C}{dK^2}\right)_{\dot{K}=0} = r_K(K) < 0.$$

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The slope of the $\dot{C}=0$ line can be written as:

$$\left(\frac{dC}{dK}\right)_{\dot{C}=0} = \frac{\lambda(\rho+\lambda)\left[r(K)-\rho-r_K(K)\left[K+B\right]\right]}{\left[r(K)-\rho\right]^2}.$$

The Keynes-Ramsey capital stock, K^{KR}, is implicitly defined by $r(K^{KR})=\rho$. Hence, the $\dot{C}=0$ line has a vertical asymptote at $K=K^{KR}$, is horizontal at $K=0$, and is upward sloping for $0<K<K^{KR}$:

$$\left(\frac{dC}{dK}\right)_{\dot{C}=0} \geq 0, \; \lim_{K\to 0}\left(\frac{dC}{dK}\right)_{\dot{C}=0} = 0, \; \lim_{K\to K^{KR}}\left(\frac{dC}{dK}\right)_{\dot{C}=0} = \infty,$$

$$\begin{aligned}\left(\frac{d^2C}{dK^2}\right)_{\dot{C}=0} &= \frac{-\lambda(\lambda+\rho)(r(K)-\rho)}{\left[r(K)-\rho\right]^4}\Big[2r_K(K)(r(K)-\rho) \\ &+ \left[(r(K)-\rho)r_{KK}(K)-2\left(r_K(K)\right)^2\right](K+B)\Big] > 0\end{aligned}$$

where $r_{KK}(K)\equiv\epsilon_L(1+\epsilon_L)r(K)/K^2>0$, and we have used the fact that $(r(K)-\rho)r_{KK}(K)-2r_K^2(K)<0$. Hence, the $\dot{C}=0$ line is convex towards the origin as drawn in Figure 1. There is a unique non-trivial equilibrium at point E_0 which lies to the left of the Keynes-Ramsey point A. (The equilibrium at point C is unstable and can thus be dismissed from further consideration.)

A very simple expression for the equilibrium output-capital ratio can be computed by equating the $\dot{C}=0$ and $\dot{K}=0$ lines:

$$\begin{aligned}&\left[r(K)-\rho\right]\left[Y-G\right]=\lambda(\rho+\lambda)\left[K+B\right]\Leftrightarrow \\ &\left[(1-\epsilon_L)\left(\frac{Y}{K}\right)-\rho\right]\left[1-\frac{G}{Y}\right]=\lambda(\rho+\lambda)\left[\frac{K}{Y}+\frac{B}{Y}\right]\Rightarrow \\ &\omega_C(1-\epsilon_L)\left(\frac{Y}{K}\right)^2-\left[\rho\omega_C+\lambda(\rho+\lambda)\omega_B\right]\left(\frac{Y}{K}\right)-\lambda(\rho+\lambda)=0,\end{aligned}$$

where $\omega_B\equiv B/Y$ and $\omega_C\equiv C/Y$. Taking the positive root of this quadratic equation yields the expression for the output-capital ratio associated with the Blanchard-Yaari equilibrium.

$$\left(\frac{Y}{K}\right)^{BY} = \frac{\rho\omega_C+\lambda(\rho+\lambda)\omega_B+\left[\left[\rho\omega_C+\lambda(\rho+\lambda)\omega_B\right]^2+4\lambda(\rho+\lambda)\omega_C(1-\epsilon_L)\right]^{1/2}}{2\omega_C(1-\epsilon_L)}.$$

Obviously, since $r\equiv(1-\epsilon_L)(Y/K)$, we also have the expression for the equilibrium interest rate:

$$r^{BY} = \frac{\rho\omega_C+\lambda(\rho+\lambda)\omega_B+\left[\left[\rho\omega_C+\lambda(\rho+\lambda)\omega_B\right]^2+4\lambda(\rho+\lambda)\omega_C(1-\epsilon_L)\right]^{1/2}}{2\omega_C}.$$

Throughout the paper (and in the remainder of this appendix) we assume that the initial debt is zero ($\omega_B=B=0$), so that r^{BY} is simplified to:

$$r^{BY} = \frac{\rho\omega_C+\left[\rho^2\omega_C^2+4\lambda(\rho+\lambda)\omega_C(1-\epsilon_L)\right]^{1/2}}{2\omega_C}, \text{ (for } B=0). \quad \text{(A.1)}$$

Since $r^{BY}=\rho+\lambda(1-\omega_H)$, where $\omega_H\equiv H/(H+K)$, it follows that $\rho<r^{BY}<\rho+\lambda$ because $0<\omega_H<1$. This last inequality can be deduced from the assumption $\omega_G\equiv 1-\omega_C<\epsilon_L$ plus the fact that $(r^{BY}+\lambda)(H/K)^{BY}\equiv(W^N/K)^{BY}=[\epsilon_L-\omega_G](Y/K)^{BY}$. This implies that

$(H/K)^{BY} > 0$ and hence $H^{BY} > 0$ ($W^N \equiv W - T = W - G$). In the steady state we have: $\omega_C(1-\omega_H) \equiv (\rho+\lambda)(K/Y)^{BY}$ and $(K/Y)^{BY} \equiv (1-\epsilon_L)/r^{BY}$. It is thus straightforward to show that $\rho < r^{BY} < \rho + \lambda$ implies the following inequality for $1 - \omega_H$:

$$1 - \epsilon_L < \omega_C(1-\omega_H) < (1-\epsilon_L)(1+\lambda/\rho).$$

Equation (A.1) can be used to derive the following results for the steady-state interest rate (holding the share of government spending, $1-\omega_C$, constant).

$$0 < \frac{dr^{BY}}{d\rho} = \frac{\omega_C r^{BY} + \lambda(1-\epsilon_L)}{\omega_C\left[2r^{BY}-\rho\right]} = \frac{r^{BY} + \lambda(1-\epsilon_L)/\omega_C}{r^{BY} + \lambda(1-\omega_H)} < 1, \tag{A.2}$$

$$\frac{dr^{BY}}{d\lambda} = \frac{(1-\epsilon_L)(\rho+2\lambda)}{\omega_C\left[2r^{BY}-\rho\right]} > 1 - \omega_H > 0, \tag{A.3}$$

$$\frac{dr^{BY}}{d\epsilon_L} = -\frac{\lambda(\rho+\lambda)}{\omega_C\left[2r^{BY}-\rho\right]} < 0, \tag{A.4}$$

$$\frac{dr^{BY}}{d\omega_C} = -\frac{dr^{BY}}{d\omega_G} = -\frac{\lambda(\rho+\lambda)(1-\epsilon_L)}{\omega_C^2\left[2r^{BY}-\rho\right]} < 0, \tag{A.5}$$

where we have also used the fact that $(\lambda+\rho)(K/Y)^{BY} \equiv \omega_C(1-\omega_H) > (1-\epsilon_L)$ because $r^{BY} < \lambda + \rho$. The proof for $dr^{BY}/d\lambda$ runs as follows:

$$\begin{aligned}
\frac{dr^{BY}}{d\lambda} - (1-\omega_H) &= \frac{(1-\epsilon_L)(\rho+2\lambda)}{\omega_C\left[2r^{BY}-\rho\right]} - (1-\omega_H) \\
&= \left(\frac{1}{\omega_C}\right)\left(\frac{K}{Y}\right)^{BY}\left[\left(\frac{\rho+2\lambda}{r^{BY}+\lambda(1-\omega_H)}\right)(1-\epsilon_L)\left(\frac{Y}{K}\right)^{BY} - (\rho+\lambda)\right] \\
&= \left(\frac{1}{\omega_C}\right)\left(\frac{K}{Y}\right)^{BY}\left[\frac{(\rho+2\lambda)r^{BY} - (\rho+\lambda)\left[r^{BY}+\lambda(1-\omega_H)\right]}{r^{BY}+\lambda(1-\omega_H)}\right] \\
&= \left(\frac{1}{\omega_C}\right)\left(\frac{K}{Y}\right)^{BY}\left(\frac{\lambda\rho\omega_H}{r^{BY}+\lambda(1-\omega_H)}\right) > 0.
\end{aligned}$$

Since $1-\omega_H \equiv (r^{BY}-\rho)/\lambda$, it is also straightforward to derive the results for the steady-state share of human wealth:

$$\frac{d\omega_H}{d\lambda} = \frac{1}{\lambda}\left[1-\omega_H - \frac{dr^{BY}}{d\lambda}\right] < 0, \quad \frac{d\omega_H}{d\rho} = \frac{1}{\lambda}\left[1 - \frac{dr^{BY}}{d\rho}\right] > 0,$$

$$\frac{d\omega_H}{d\epsilon_L} = -\frac{1}{\lambda}\frac{dr^{BY}}{d\epsilon_L} > 0, \quad \frac{d\omega_H}{d\omega_C} = -\frac{1}{\lambda}\frac{dr^{BY}}{d\omega_C} > 0.$$

The linearized model given in Table 2 can be written in terms of a single matrix equation:

$$\begin{aligned}
\begin{bmatrix} \dot{\tilde{C}}(t) \\ \dot{\tilde{K}}(t) \end{bmatrix} &= \begin{bmatrix} r-\rho & -(r\epsilon_L + r - \rho) \\ -\frac{r\omega_C}{1-\epsilon_L} & r \end{bmatrix}\begin{bmatrix} \tilde{C}(t) \\ \tilde{K}(t) \end{bmatrix} \\
&+ \begin{bmatrix} 0 & -\frac{r-\rho}{1-\epsilon_L} \\ -\frac{r}{1-\epsilon_L} & 0 \end{bmatrix}\begin{bmatrix} \tilde{G}(t) \\ \tilde{B}(t) \end{bmatrix},
\end{aligned} \tag{A.6}$$

where $r \equiv r^{BY}$, $\tilde{C}(t) \equiv dC(t)/C$, $\tilde{K}(t) \equiv dK(t)/K$, $\dot{\tilde{C}}(t) \equiv \dot{C}(t)/C$, $\dot{\tilde{K}}(t) \equiv \dot{K}(t)/K$, $\tilde{B}(t) \equiv rdB(t)/Y$, and $\tilde{G}(t) \equiv dG(t)/Y$ (and we have used $d\dot{C}(t) = \dot{C}(t)$ and $d\dot{K}(t) = \dot{K}(t)$). The 2 by 2 matrix of coefficients for the endogenous variables is denoted by Δ (with typical element δ_{ij}) and the 2 by 2 matrix of coefficients for the exogenous variables is represented by Γ (with non-zero elements $\gamma_K \equiv -r/(1-\epsilon_L)$, $\gamma_C \equiv -(r-\rho)/(1-\epsilon_L)$).

We let r^* and $-h^*$ stand for the characteristic roots of Δ, and we wish to show that the equilibrium E_0 in Figure 1 is saddle-point stable, i.e. that $r^* > 0$ and $h^* > 0$. The characteristic roots are equal to:

$$
\begin{aligned}
r^* &= \frac{\mathrm{tr}(\Delta)}{2}\left[1+\left(1-\frac{4|\Delta|}{\mathrm{tr}(\Delta)^2}\right)^{1/2}\right] > \mathrm{tr}(\Delta), \\
h^* &= -\frac{\mathrm{tr}(\Delta)}{2}\left[1-\left(1-\frac{4|\Delta|}{\mathrm{tr}(\Delta)^2}\right)^{1/2}\right].
\end{aligned}
$$

The adjustment speed of the economy is represented by h^*. The proof of saddle-point stability proceeds as follows. Recall that $|\Delta| = -r^*h^*$ and $\mathrm{tr}(\Delta) = r^* - h^*$. The determinant of Δ can be written as follows:

$$
|\Delta| \equiv -r^*h^* = -\frac{r\left[r\epsilon_L\omega_C + (\epsilon_L - \omega_G)(r-\rho)\right]}{1-\epsilon_L} < 0. \tag{A.7}
$$

We assume in the text that the labour income share exceeds the share of government spending, i.e., $\epsilon_L > \omega_G$, so that the after-tax share of labour is positive. Hence, the roots alternate in sign, and $r^* > 0$ and $h^* > 0$. This completes the proof of Proposition 1. □

1.2 Proof of Proposition 2

In order to prove the inequalities concerning the characteristic roots, we define $\mathrm{f}(s) \equiv |sI - \Delta|$. Obviously, since the system is saddle-point stable, $\mathrm{f}(s)$ is a quadratic function with roots $s_1 = -h^* < 0$ and $s_2 = r^* > 0$, and $\mathrm{f}(0) = |\Delta| < 0$. To prove the inequality for h^* $(> \rho + \lambda - r)$, all we need to show is that $\mathrm{f}(\bar{s}) < 0$ for $\bar{s} \equiv r - (\rho + \lambda)$. By simple substitutions we obtain:

$$
\begin{aligned}
\mathrm{f}(\bar{s}) &= \lambda(\rho+\lambda) - \frac{r\omega_C\left[r-\rho+r\epsilon_L\right]}{1-\epsilon_L} \\
&= \lambda(\rho+\lambda) - \left(\frac{\rho+\lambda}{1-\omega_H}\right)\left[\lambda(1-\omega_H)+r\epsilon_L\right] = -\frac{\epsilon_L\omega_C r^2}{1-\epsilon_L} < 0,
\end{aligned}
$$

where we have used the fact that $r - \rho = \lambda(1-\omega_H)$ and $r\omega_C/(1-\epsilon_L) = (\rho+\lambda)/(1-\omega_H)$. Since $r^* = h^* + \mathrm{tr}(\Delta)$, the inequality for r^* follows directly from the trace condition:

$$
r^* = h^* + r - \rho + r > (\rho + \lambda - r) + r - \rho + r = \lambda + r.
$$

This concludes the proof of Proposition 2. □

Of course, since $r^* = h^* + \mathrm{tr}(\Delta)$, the unstable root exceeds the rate of interest, i.e., $r^* > r$ and *a fortiori* $r^* > r - \rho$.

1.3 Model solution

We use the Laplace transform techniques pioneered by Judd (1982, 1985, 1987). By taking the Laplace transform of (A.6) and using

$$
\mathcal{L}\{\dot{\tilde{C}}, s\} = s\mathcal{L}\{\tilde{C}, s\} - \tilde{C}(0) \text{ and } \mathcal{L}\{\dot{\tilde{K}}, s\} = s\mathcal{L}\{\tilde{K}, s\},
$$

we obtain the following expression:

$$
(s\mathrm{I} - \Delta)\begin{bmatrix} \mathcal{L}\{\tilde{C}, s\} \\ \mathcal{L}\{\tilde{K}, s\} \end{bmatrix} = \begin{bmatrix} \tilde{C}(0) + \gamma_C\mathcal{L}\{\tilde{B}, s\} \\ \gamma_K\mathcal{L}\{\tilde{G}, s\} \end{bmatrix}. \tag{A.8}
$$

Define $A(s) \equiv sI - \Delta$, so that $|A(s)| \equiv (s - r^*)(s + h^*)$. By pre-multiplying (A.8) by $\text{adj}(A(r^*))$, we arrive at the initial condition for the jump in consumption:

$$\begin{aligned} \text{adj}[\text{A}(r^*)]\text{A}(r^*) \begin{bmatrix} \mathcal{L}\{\tilde{C}, r^*\} \\ \mathcal{L}\{\tilde{K}, r^*\} \end{bmatrix} &= \\ \begin{bmatrix} r^* - \delta_{22} & \delta_{12} \\ \delta_{21} & r^* - \delta_{11} \end{bmatrix} \begin{bmatrix} \tilde{C}(0) + \gamma_C \mathcal{L}\{\tilde{B}, r^*\} \\ \gamma_K \mathcal{L}\{\tilde{G}, r^*\} \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned} \tag{A.9}$$

Since the characteristic roots of Δ are distinct, $\text{rank}(\text{adj}(A(r^*))) = 1$ and there is exactly one independent equation determining the jump in consumption, $\tilde{C}(0)$. Hence, either row of (A.9) may be used to find $\tilde{C}(0)$:

$$(r^* - \delta_{22}) \left[\tilde{C}(0) + \gamma_C \mathcal{L}\{\tilde{B}, r^*\} \right] + \delta_{12} \gamma_K \mathcal{L}\{\tilde{G}, r^*\} = 0, \tag{A.10}$$

$$\delta_{21} \left[\tilde{C}(0) + \gamma_C \mathcal{L}\{\tilde{B}, r^*\} \right] + (r^* - \delta_{11}) \gamma_K \mathcal{L}\{\tilde{G}, r^*\} = 0. \tag{A.11}$$

Using either (A.10) or (A.11) to eliminate $\tilde{C}(0)$ from (A.8), we arrive at the general perfect foresight solution of the model in terms of Laplace transforms. Consider the first row of (A.8) in combination with (A.10). After some simplification it can be written as follows:

$$\begin{aligned} (s + h^*)\mathcal{L}\{\tilde{C}, s\} &= \tilde{C}(0) + \gamma_C \mathcal{L}\{\tilde{B}, s\} + \delta_{12}\gamma_K \left[\frac{\mathcal{L}\{\tilde{G}, s\} - \mathcal{L}\{\tilde{G}, r^*\}}{s - r^*} \right] \\ &\quad + (r^* - \delta_{22})\gamma_C \left[\frac{\mathcal{L}\{\tilde{B}, s\} - \mathcal{L}\{\tilde{B}, r^*\}}{s - r^*} \right]. \end{aligned} \tag{A.12}$$

The second row of (A.8) can be combined with (A.11), after which the following expression is obtained:

$$\begin{aligned} (s + h^*)\mathcal{L}\{\tilde{K}, s\} &= \gamma_K \mathcal{L}\{\tilde{G}, s\} + (r^* - \delta_{11})\gamma_K \left[\frac{\mathcal{L}\{\tilde{G}, s\} - \mathcal{L}\{\tilde{G}, r^*\}}{s - r^*} \right] \\ &\quad + \delta_{21}\gamma_C \left[\frac{\mathcal{L}\{\tilde{B}, s\} - \mathcal{L}\{\tilde{B}, r^*\}}{s - r^*} \right]. \end{aligned} \tag{A.13}$$

The *long-run* effects of shocks in abatement spending ($\tilde{G}$) and debt ($\tilde{B}(\infty)$) are obtained from (A.12) and (A.13) by applying the final-value theorem (Spiegel, 1965, p. 20).

$$\tilde{C}(\infty) \equiv \lim_{s \downarrow 0} s\mathcal{L}\{\tilde{C}, s\} = \frac{-\delta_{12}\gamma_K \tilde{G} + \delta_{22}\gamma_C \tilde{B}(\infty)}{r^* h^*}, \tag{A.14}$$

$$\tilde{K}(\infty) \equiv \lim_{s \downarrow 0} s\mathcal{L}\{\tilde{K}, s\} = \frac{\delta_{11}\gamma_K \tilde{G} - \delta_{21}\gamma_C \tilde{B}(\infty)}{r^* h^*}. \tag{A.15}$$

By making the appropriate substitutions for the γ_K, γ_C and the δ_{ij}-terms from (A.6), we find the expressions in section 3.1 of the text by setting $\tilde{B}(\infty) = 0$.

The initial responses of the time rates of change of the capital stock and consumption are obtained by applying the initial-value theorem (Spiegel, 1965, p. 20).

$$\begin{aligned} \dot{\tilde{K}}(0) &\equiv \lim_{s \to \infty} s\mathcal{L}\{\dot{\tilde{K}}, s\} = \lim_{s \to \infty} s^2 \mathcal{L}\{\tilde{K}, s\} \\ &= \gamma_K \left[\tilde{G}(0) - (r^* - \delta_{11})\mathcal{L}\{\tilde{G}, r^*\} \right] - \delta_{21}\gamma_C \mathcal{L}\{\tilde{B}, r^*\} = \gamma_K \tilde{G}(0) + \delta_{21}\tilde{C}(0), \end{aligned} \tag{A.16}$$

$$\begin{aligned}\dot{\tilde{C}}(0) &\equiv \lim_{s\to\infty} s\mathcal{L}\{\dot{\tilde{C}}, s\} = \lim_{s\to\infty} s\left[s\mathcal{L}\{\tilde{C}, s\} - \tilde{C}(0)\right] \\ &= \gamma_C\left[\tilde{B}(0) - \delta_{11}\mathcal{L}\{\tilde{B}, r^*\}\right] - \left(\frac{\gamma_K(r^*-\delta_{11})\delta_{11}}{\delta_{21}}\right)\mathcal{L}\{\tilde{G}, r^*\} \\ &= \delta_{11}\tilde{C}(0) + \gamma_C\tilde{B}(0).\end{aligned}$$

By making the appropriate substitutions, we find the expressions in section 3 of the text.

By taking the Laplace transform of equation (T2.4) in Table 2, and imposing the fact that the quality of the environment is a predetermined variable (so that $\tilde{E}(0) = 0$), we obtain the following expression:

$$\mathcal{L}\{\tilde{E}, s\} = \frac{-\alpha_K\alpha_E\mathcal{L}\{\tilde{K}, s\} + \alpha_G\alpha_E\mathcal{L}\{\tilde{G}, s\}}{s+\alpha_E}. \tag{A.17}$$

Hence, the Laplace transform of environmental quality is linked directly to the Laplace transforms for the capital stock and public abatement. It follows immediately from (T2.4) that:

$$\dot{\tilde{E}}(0) = -\alpha_K\alpha_E\tilde{K}(0) - \alpha_E\tilde{E}(0) + \alpha_G\alpha_E\tilde{G}(0) = \alpha_G\alpha_E\tilde{G}(0),$$

(since $\tilde{K}(0) = \tilde{E}(0) = 0$) and:

$$\begin{aligned}\ddot{\tilde{E}}(0) &= -\alpha_K\alpha_E\dot{\tilde{K}}(0) - \alpha_E\dot{\tilde{E}}(0) + \alpha_G\alpha_E\dot{\tilde{G}}(0) \\ &= -\alpha_K\alpha_E\dot{\tilde{K}}(0) - \alpha_E^2\alpha_G\tilde{G}(0) + \alpha_G\alpha_E\dot{\tilde{G}}(0).\end{aligned} \tag{A.18}$$

In order to calculate the transition paths for capital and consumption, the intertemporal paths of $\tilde{G}(t)$ and $\tilde{B}(t)$ must be specified. The cases discussed in the text are based on the following parameterizations:

$$\tilde{G}(t) = \left(1 - e^{-\xi_G t}\right)\tilde{G}, \quad \tilde{B}(t) = b_0 + \sum_{i=1}^{2}\left(1 - e^{-\xi_i t}\right)b_i, \tag{A.19}$$

with $\xi_G > 0$ and $\xi_i > 0$. In that case, it is possible to derive the following expressions:

$$\frac{\mathcal{L}\{\tilde{G}, s\} - \mathcal{L}\{\tilde{G}, r^*\}}{s - r^*} = \left[\frac{1}{(r^*+\xi_G)(s+\xi_G)} - \frac{1}{r^* s}\right]\tilde{G}, \tag{A.20}$$

$$\frac{\mathcal{L}\{\tilde{B}, s\} - \mathcal{L}\{\tilde{B}, r^*\}}{s - r^*} = -\frac{b_0}{r^* s} + \sum_{i=1}^{2}\left[\frac{1}{(r^*+\xi_i)(s+\xi_i)} - \frac{1}{r^* s}\right]b_i. \tag{A.21}$$

It is also useful to recognize that:

$$\frac{1}{(s+h^*)(s+\xi_i)} = \frac{1}{\xi_i - h^*}\left[\frac{1}{s+h^*} - \frac{1}{s+\xi_i}\right], \quad \frac{1}{(s+h^*)s} = \frac{1}{h^*}\left[\frac{1}{s} - \frac{1}{s+h^*}\right], \tag{A.22}$$

for $i = 1, 2, G$. Using (A.19)-(A.22) in (A.13) and recognizing (A.15), we obtain the transition path for the capital stock by inverting the Laplace transforms:

$$\begin{aligned}\tilde{K}(t) &= \mathrm{A}(h^*, t)\tilde{K}(\infty) - \gamma_K\left(\frac{\delta_{11}+\xi_G}{r^*+\xi_G}\right)\tilde{G}\mathrm{T}(\xi_G, h^*, t) \\ &\quad + \delta_{21}\gamma_C\sum_{i=1}^{2}\left(\frac{b_i}{r^*+\xi_i}\right)\mathrm{T}(\xi_i, h^*, t).\end{aligned} \tag{A.23}$$

Equation (A.23) contains a single adjustment term and three (temporary) single transition terms, about which the following properties can be established, respectively.

Lemma A.1 *Let $A(\alpha_1, t)$ be a single adjustment function of the form:*

$$\mathrm{A}(\alpha_1, t) \equiv 1 - e^{-\alpha_1 t},$$

with $\alpha_1 > 0$. Then $\mathrm{A}(\alpha_1, t)$ has the following properties: (i) (positive) $\mathrm{A}(\alpha_1, t) > 0$ $t \in (0, \infty)$, (ii) $\mathrm{A}(\alpha_1, t) = 0$ for $t = 0$ and $\mathrm{A}(\alpha_1, t) \to 1$ in the limit as $t \to \infty$, (iii) (increasing) $d\mathrm{A}(\alpha_1, t)/dt \geq 0$, (iv) (step function as limit) As $\alpha_1 \to \infty$, $\mathrm{A}(\alpha_1, t) \to u(t)$, where $u(t)$ is a unit step function.

PROOF: Properties (i) and (ii) follow by simple substitution. Property (iii) follows from the fact that $d\mathrm{A}(\alpha_1, 0)/dt = \alpha_1[1 - \mathrm{A}(\alpha_1, t)]$ plus properties (i)-(ii). Property (iv) follows by comparing the Laplace transforms of $\mathrm{A}(\alpha_1, t)$ and $u(t)$ and showing that they converge as $\alpha_1 \to \infty$. Since $\mathcal{L}\{u, s\} = 1/s$ and $\mathcal{L}\{\mathrm{A}(\alpha_1, t), s\} = 1/s - 1/(s + \alpha_1)$ this result follows. □

Lemma A.2 *Let $T(\alpha_1, \alpha_2, t)$ be a single transition function of the form:*

$$\mathrm{T}(\alpha_1, \alpha_2, t) \equiv \begin{cases} \frac{e^{-\alpha_2 t} - e^{-\alpha_1 t}}{\alpha_1 - \alpha_2} & \text{for } \alpha_1 \neq \alpha_2 \\ t e^{-\alpha_1 t} & \text{for } \alpha_1 = \alpha_2, \end{cases}$$

with $\alpha_1 > 0$ and $\alpha_2 > 0$. Then $\mathrm{T}(\alpha_1, \alpha_2, t)$ has the following properties: (i) (positive) $\mathrm{T}(\alpha_1, \alpha_2, t) > 0$ $t \in (0, \infty)$, (ii) $\mathrm{T}(\alpha_1, \alpha_2, t) = 0$ for $t = 0$ and in the limit as $t \to \infty$, (iii) (single-peaked) $d\mathrm{T}(\alpha_1, \alpha_2, t)/dt > 0$ for $t \in (0, \hat{t})$, $d\mathrm{T}(\alpha_1, \alpha_2, t)/dt < 0$ for $t \in (\hat{t}, \infty)$, $d\mathrm{T}(\alpha_1, \alpha_2, t)/dt = 0$ for $t = \hat{t}$ and in the limit as $t \to \infty$, and $d\mathrm{T}(\alpha_1, \alpha_2, 0)/dt = 1$, (iv) $\hat{t} \equiv \ln(\alpha_1/\alpha_2)/(\alpha_1 - \alpha_2)$ if $\alpha_1 \neq \alpha_2$ and $\hat{t} \equiv 1/\alpha_1$ if $\alpha_1 = \alpha_2$; (v) (point of inflexion) $d^2\mathrm{T}(\alpha_1, \alpha_2, t)/dt^2 = 0$ for $t^* = 2\hat{t}$

PROOF: Property (i) follows by examining the three possible cases. The result is obvious if $\alpha_1 = \alpha_2$. If $\alpha_1 < (>) \alpha_2$, then $\alpha_2 - \alpha_1 > (<) 0$ and $e^{-\alpha_1 t} > (<)\ e^{-\alpha_2 t}$ for all $t \in (0, \infty)$, and $\mathrm{T}(\alpha_1, \alpha_2, 0) > 0$. Property (ii) follows by direct substitution. Property (iii) follows by examining $d\mathrm{T}(\alpha_1, \alpha_2, t)/dt$:

$$\frac{d\mathrm{T}(\alpha_1, \alpha_2, t)}{dt} \equiv \begin{cases} \frac{\alpha_1 e^{-\alpha_1 t} - \alpha_2 e^{-\alpha_2 t}}{\alpha_1 - \alpha_2} & \text{for } \alpha_1 \neq \alpha_2 \\ [1 - \alpha_1 t]\, e^{-\alpha_1 t} & \text{for } \alpha_1 = \alpha_2. \end{cases}$$

Property (iv) is obtained by examining $d\mathrm{T}^2(\alpha_1, \alpha_2, t)/dt^2$:

$$\frac{d^2\mathrm{T}(\alpha_1, \alpha_2, t)}{d^2 t} \equiv \begin{cases} \frac{\alpha_1^2 e^{-\alpha_1 t} - \alpha_2^2 e^{-\alpha_2 t}}{\alpha_2 - \alpha_1} & \text{for } \alpha_1 \neq \alpha_2 \\ -\alpha_1\, [2 - \alpha_1 t]\, e^{-\alpha_1 t} & \text{for } \alpha_1 = \alpha_2. \end{cases}$$

Hence, $d^2\mathrm{T}(\alpha_1, \alpha_2, 0)/dt^2 = -(\alpha_1 + \alpha_2) < 0$, and $\lim_{t \to \infty} d^2\mathrm{T}(\alpha_1, \alpha_2, t)/dt^2 = 0$. The inflexion point is found by finding the value of $t = t^*$ where $d^2\mathrm{T}(\alpha_1, \alpha_2, t)/dt^2 = 0$. □

By using (A.19)-(A.22) in (A.12) and noting (A.14), we find the transition path for consumption by inverting the resulting Laplace transforms:

$$\begin{aligned} \tilde{C}(t) &= \tilde{C}(0)\, (1 - \mathrm{A}(h^*, t)) + \tilde{C}(\infty)\mathrm{A}(h^*, t) + \left(\frac{\delta_{12}\gamma_K}{r^* + \xi_G} \right) \tilde{G}\mathrm{T}(\xi_G, h^*, t) \\ &\quad -\gamma_C \sum_{i=1}^{2} \left(\frac{\delta_{22} + \xi_i}{r^* + \xi_i} \right) b_i \mathrm{T}(\xi_i, h^*, t), \end{aligned} \tag{A.24}$$

where the jump in consumption that occurs at impact can be calculated by using (A.11) in combination with (A.19):

$$\tilde{C}(0) = -\left(\frac{\gamma_K (r^* - \delta_{11})\xi_G}{r^* \delta_{21}(r^* + \xi_G)} \right) \tilde{G} - \gamma_C \left[\frac{b_0}{r^*} + \sum_{i=1}^{2} \left(\frac{b_i \xi_i}{r^*(r^* + \xi_i)} \right) \right]. \tag{A.25}$$

By setting $b_0 = b_1 = b_2 = 0$ and substituting δ_{ij} and γ_K, the equations in the text are obtained.

By substituting (A.13), (A.19)-(A.22) into (A.17) and inverting the Laplace transform, we derive the path for the environment:

$$\begin{aligned}\tilde{E}(t) \quad \equiv \quad & -\alpha_K \tilde{K}(\infty)\mathrm{A}(\alpha_E, h^*, t) + \alpha_G \tilde{G}\mathrm{A}(\alpha_E, \xi_G, t) \\ & + \left(\frac{\alpha_K \gamma_K (\delta_{11} + \xi_G)}{r^* + \xi_G}\right) \tilde{G}\mathrm{T}(\alpha_E, \xi_G, h^*, t) \\ & -\alpha_K \gamma_C \delta_{21} \sum_{i=1}^{2} \left(\frac{b_i}{r^* + \xi_i}\right) \mathrm{T}(\alpha_E, \xi_i, h^*, t).\end{aligned} \tag{A.26}$$

Note that $\tilde{E}(\infty)$ is defined as:

$$\tilde{E}(\infty) = -\alpha_K \tilde{K}(\infty) + \alpha_G \tilde{G}.$$

The $A(\alpha_E, \alpha_i, t)$ and $T(\alpha_E, \alpha_1, \alpha_2, t)$ terms in (A.26) are, respectively, multiple adjustment and multiple transition terms. The forms and properties of these terms are covered in Lemma A.3 to A.5 below.

Lemma A.3 *Let $A(\alpha_E, \alpha_i, t)$ be a multiple adjustment function of the form:*

$$\mathrm{A}(\alpha_E, \alpha_i, t) \equiv \begin{cases} 1 - \left(\frac{\alpha_i e^{-\alpha_E t} - \alpha_E e^{-\alpha_i t}}{\alpha_i - \alpha_E}\right) & \text{for } \alpha_i \neq \alpha_E \\ 1 - (1 + \alpha_E t) e^{-\alpha_E t} & \text{for } \alpha_i = \alpha_E, \end{cases}$$

with $\alpha_E > 0$ and $\alpha_i > 0$. Then $\mathrm{A}(\alpha_E, \alpha_i, t)$ has the following properties: (i) (increasing over time) $d\mathrm{A}(\alpha_E, \alpha_i, t)/dt > 0 \ \forall t \in (0, \infty)$, $d\mathrm{A}(\alpha_E, \alpha_i, t)/dt = 0$ (for $t = 0$ and in the limit as $t \to \infty$), (ii) (between 0 and 1) 0< $\mathrm{A}(\alpha_E, \alpha_i, t) < 1 \ \forall t \in (0, \infty)$ and $\mathrm{A}(\alpha_E, \alpha_i, 0) = 1 - \lim_{t \to \infty} \mathrm{A}(\alpha_E, \alpha_i, t) = 0$, (iii) (inflexion point) $d^2\mathrm{A}(\alpha_E, \alpha_i, t)/dt^2 = 0$ for $t = \hat{t} \equiv \ln(\alpha_E/\alpha_i)/(\alpha_E - \alpha_i)$ if $\alpha_E \neq \alpha_i$ and $\hat{t} \equiv 1/\alpha_E$ if $\alpha_E = \alpha_i$, (iv) As $\alpha_E \to \infty$, $\mathrm{A}(\alpha_E, \alpha_i, t) \to \mathrm{A}(\alpha_i, t)$.

Proof: Property (i): the derivative of $\mathrm{A}(\alpha_E, \alpha_i, t)$ with respect to time is itself proportional to a single transition term with properties covered in Lemma A.2:

$$\frac{d\mathrm{A}(\alpha_E, \alpha_i, t)}{dt} \equiv \alpha_E \alpha_i \mathrm{T}(\alpha_E, \alpha_i, t) \geq 0,$$

where for $t \in (0, \infty)$ the inequality is strict. Hence, $\mathrm{A}(\alpha_E, \alpha_i, t)$ itself is increasing over time. Property (ii) follows from the fact that $\mathrm{A}(\alpha_E, \alpha_i, 0) = 0$ and $\lim_{t \to \infty} \mathrm{A}(\alpha_E, \alpha_i, t) = 1$ plus the fact that $d\mathrm{A}(\alpha_E, \alpha_i, 0)/dt \geq 0$. Property (iii) follows from Lemma A.2 and:

$$\frac{\mathrm{d}^2\mathrm{A}(\alpha_E, \alpha_i, t)}{\mathrm{d}^2 t} = \alpha_E \alpha_i \left(\frac{\mathrm{dT}(\alpha_E, \alpha_i, t)}{dt}\right).$$

Finally, property (iv) follows trivially by letting $\alpha_E \to \infty$ in the definition of $\mathrm{A}(\alpha_E, \alpha_i, t)$.□

Lemma A.4 *Let $\mathrm{f}(t)$ be a function with the following Laplace transform $\mathrm{F}(s)$:*

$$\mathrm{F}(s) \equiv \frac{\alpha_E}{(s + \alpha_E)(s + \alpha_1)(s + \alpha_2)},$$

with 0< $\alpha_E, \alpha_1, \alpha_2 < \infty$. Then $\mathrm{f}(t) \geq 0 \forall t \in [0, \infty)$.

PROOF: Use the convolution property of the Laplace transform (Spiegel, 1965, p. 45). Define $\mathrm{G}(s) \equiv \alpha_E/(s+\alpha_E)$ and $\mathrm{H}(s) \equiv 1/(s+\alpha_1)(s+\alpha_2)$, so that $\mathrm{F}(s) = \mathrm{G}(s)\mathrm{H}(s)$. The inverse Laplace transforms of $\mathrm{G}(s)$ and $\mathrm{H}(s)$ are:

$$\mathrm{g}(t) \equiv \alpha_E e^{-\alpha_E t}, \quad \mathrm{h}(t) \equiv \mathrm{T}(\alpha_1, \alpha_2, t).$$

Then the convolution property states that $\mathrm{f}(t)$ is equal to:

$$\mathrm{f}(t) \equiv \mathcal{L}^{-1}\{\mathrm{G}(s), \mathrm{H}(s)\} = \int_0^t \mathrm{g}(\tau)\mathrm{h}(t-\tau)d\tau.$$

Since $\mathrm{g}(\tau) \geq 0$ and $\mathrm{h}(\tau) \geq 0$ $\forall\tau$ and $\alpha_E < \infty$, $\mathrm{f}(t)$ must be non-negative since it represents the discounted integral of a non-negative (single transition) function. □

Lemma A.5 *Let* $\mathrm{T}(\alpha_E, \alpha_1, \alpha_2, t)$ *be a multiple transition function of the form:*

$$\mathrm{T}(\alpha_E, \alpha_1, \alpha_2, t) \equiv \begin{cases} \left(\frac{\alpha_E}{\alpha_E - \alpha_1}\right)[\mathrm{T}(\alpha_1, \alpha_2, t) - \mathrm{T}(\alpha_E, \alpha_2, t)] & \text{for } \alpha_1 \neq \alpha_E \\ \alpha_E t^2 e^{-\alpha_E t}/2 & \text{for } \alpha_1 = \alpha_2 = \alpha_E, \end{cases}$$

with $\alpha_E > 0$, $\alpha_1 > 0$, *and* $\alpha_2 > 0$. *Then* $\mathrm{T}(\alpha_E, \alpha_1, \alpha_2, t)$ *has the following properties: (i) (positive)* $\mathrm{T}(\alpha_E, \alpha_1, \alpha_2, t) > 0$, *(ii)* $\mathrm{T}(\alpha_E, \alpha_1, \alpha_2, t) = 0$ *for* $t = 0$ *and in the limit as* $t \to \infty$, *(iii) (single-peaked)* $d\mathrm{T}(\alpha_E, \alpha_1, \alpha_2, t)/dt > 0$ *for* $0 < t < \bar{t}$ *and* $d\mathrm{T}(\alpha_E, \alpha_1, \alpha_2, t)/dt < 0$ *for* $t > \bar{t}$, $d\mathrm{T}(\alpha_E, \alpha_1, \alpha_2, t)/dt = 0$ *(for* $t = 0$, $t = \bar{t}$, *and* $t \to \infty$*), (iv) As* $\alpha_E \to \infty$, $\mathrm{T}(\alpha_E, \alpha_1, \alpha_2, t) \to \mathrm{T}(\alpha_1, \alpha_2, t)$.

PROOF: Property (i) follows applying Lemma A.4, because the Laplace transform of $\mathrm{T}(\alpha_E, \alpha_1, \alpha_2, t)$ has the required form. Property (ii) follows trivially by substitution. Property (iii) can be proved by examining the following differential equation in $\mathrm{T}(\alpha_E, \alpha_1, \alpha_2, t)$:

$$\frac{d\mathrm{T}(\alpha_E, \alpha_1, \alpha_2, t)}{dt} = \alpha_E \left[\mathrm{T}(\alpha_1, \alpha_2, t) - \mathrm{T}(\alpha_E, \alpha_1, \alpha_2, t)\right].$$

The properties of $\mathrm{T}(\alpha_1, \alpha_2, t)$ are covered in Lemma A.2. $\mathrm{T}(\alpha_1, \alpha_2, t)$ has a single maximum at $\hat{t} \equiv \ln(\alpha_1/\alpha_2)/(\alpha_1 - \alpha_2)$. $\mathrm{T}(\alpha_E, \alpha_1, \alpha_2, t)$ itself has a maximum in $\bar{t}$ where $\mathrm{T}(\alpha_1, \alpha_2, \bar{t}) = \mathrm{T}(\alpha_E, \alpha_1, \alpha_2, \bar{t})$ and $\ddot{\mathrm{T}}(\alpha_E, \alpha_1, \alpha_2, \bar{t}) = \dot{\mathrm{T}}(\alpha_1, \alpha_2, \bar{t}) < 0$. The maximum of $\mathrm{T}(\alpha_E, \alpha_1, \alpha_2, \bar{t})$ occurs later in time than that of $\mathrm{T}(\alpha_1, \alpha_2, \bar{t})$, or $\bar{t} > \hat{t}$ as $\dot{\mathrm{T}}(\alpha_1, \alpha_2, \bar{t}) < 0$.

That the maximum is unique can be proven by contradiction. If there were another maximum, there would also be a minimum for which $\mathrm{T}(\alpha_1, \alpha_2, t_{MIN}) = \alpha_E \mathrm{T}(\alpha_E, \alpha_1, \alpha_2, t_{MIN})$ (since $\dot{\mathrm{T}}(\alpha_E, \alpha_1, \alpha_2, t_{MIN}) = 0$) and $\ddot{\mathrm{T}}(\alpha_E, \alpha_1, \alpha_2, \bar{t}) = \dot{\mathrm{T}}(\alpha_1, \alpha_2, t_{MIN}) > 0$. This is impossible as $t_{MIN} > \hat{t}$ and $\ddot{\mathrm{T}}(\alpha_E, \alpha_1, \alpha_2, \bar{t}) < 0$. Finally, property (iv) follows trivially by letting $\alpha_E \to \infty$ in the definition of $\mathrm{T}(\alpha_E, \alpha_1, \alpha_2, t)$. □

2 Proof of Propositions 3 and 4

In order to characterize the paths of consumption and the capital stock, we derive expressions for the paths of the time rate of change of these variables. In sections 3-5 of the paper no bond policy is used. Hence, in terms of (A.19) we impose $\tilde{B}(t) = b_0 = b_1 = b_2 = 0$. Using (A.13) and (A.19)-(A.22), we can write the time path for the rate of change of the capital stock (i.e., investment) as follows:

$$\mathcal{L}\{\dot{\tilde{K}}, s\} \equiv s\mathcal{L}\{\tilde{K}, s\} = \left(\frac{-\gamma_K \xi_G}{r^* + \xi_G}\right)\left[\frac{r^* - \delta_{11}}{r^*} - \frac{\delta_{11} + \xi_G}{s + \xi_G}\right]\left[\frac{1}{s + h^*}\right]\tilde{G}. \tag{A.27}$$

Similarly, using (A.12) and (A.19)-(A.22), we can write the time path for the rate of change of consumption as follows:

$$\mathcal{L}\{\dot{\tilde{C}}, s\} \equiv s\mathcal{L}\{\tilde{C}, s\} - \tilde{C}(0) = \left(\frac{-\delta_{12}\gamma_K \xi_G}{r^*(r^* + \xi_G)}\right) \left[\frac{\delta_{11}}{h^* + \delta_{11}} + \frac{r^*}{s + \xi_G}\right] \left[\frac{1}{s + h^*}\right] \tilde{G}. \tag{A.28}$$

The statements regarding global monotonicity contained in Proposition 3 can now be proved with the aid of (A.27) and (A.28). Global monotonicity exists provided the rate of change in the variable does not change sign along the adjustment path. We first state and prove Lemma A.6.

Lemma A.6 *Let* $\mathrm{f}(t)$ *be a function with the following Laplace transform* $\mathrm{F}(s)$*:*

$$\mathrm{F}(s) \equiv \frac{A_1}{(s + \alpha_1)} + \frac{A_2}{(s + \alpha_1)(s + \alpha_2)},$$

with $\alpha_1 > 0$*,* $\alpha_2 > 0$ *and* $A_1 > 0$*. Then the sign of* $\mathrm{f}(t)$ *is as follows: (i)* $A_2 \geq 0 \Rightarrow \mathrm{f}(t) \geq 0$*, (ii)* $A_2 < 0$ *and* $(\alpha_2 - \alpha_1)(A_1/A_2) + 1 < 0 \Rightarrow \mathrm{f}(t) \geq 0\ \forall t$*, (iii)* $A_2 < 0$ *and* $(\alpha_2 - \alpha_1)(A_1/A_2) + 1 > 0$ $\Rightarrow \mathrm{f}(t) \geq 0\ \forall t \in [0, \bar{t}]$*, and* $\mathrm{f}(t) \leq 0\ \forall t \in [\bar{t}, \infty)$*, where* $\bar{t} = (\alpha_1 - \alpha_2)^{-1} \ln[1 + (\alpha_2 - \alpha_1)A_1/A_2]$ *if* $\alpha_1 \neq \alpha_2$ *and* $\bar{t} = -A_1/A_2$ *if* $\alpha_1 = \alpha_2$*.*

PROOF: The inverse of $\mathrm{F}(s)$ is:

$$\mathrm{f}(t) \equiv A_1 e^{-\alpha_1 t} + A_2 \mathrm{T}(\alpha_1, \alpha_2, t),$$

where both exponential terms are non-negative. Part (i) follows trivially if $A_2 \geq 0$. Parts (ii) and (iii) are derived by finding the conditions under which $\mathrm{f}(t)$ cuts the t-axis. Solving $\mathrm{f}(\bar{t}) = 0$, yields the solution $\bar{t} = (\alpha_1 - \alpha_2)^{-1} \ln[1 + (\alpha_2 - \alpha_1)A_1/A_2]$ if $\alpha_1 \neq \alpha_2$ and $\bar{t} = -A_1/A_2$ if $\alpha_1 = \alpha_2$. Hence, $\bar{t} < \infty$ exists iff $(\alpha_2 - \alpha_1)(A_1/A_2) + 1 > 0$. □

Armed with this Lemma, Proposition 3 can be proved. Proposition 3(i) is proved by showing that the terms in square brackets in front of $\tilde{G}$ appearing in (A.27) when inverted imply a function of time that changes sign for $t \in [0, \infty)$. Lemma A.6 implies that this property can be ascertained by looking at the Laplace transform directly. In terms of Lemma A.6, the relevant $\mathrm{F}(s)$ function has $\alpha_1 = h^*$, $\alpha_2 = \xi_G$, $A_1 = (r^* - \delta_{11})/r^* > 0$, and $A_2 = -(\delta_{11} + \xi_G) < 0$. For this configuration it is straightforward to show that Lemma A.6(iii) holds: adjustment of the capital stock must be non-monotonic. If $\xi_G \to \infty$, then $\mathrm{F}(s) \to -\delta_{11}/(r^*(s + h^*))$, which implies monotonic adjustment. Proposition 3(ii) can be verified by obtaining the relevant $\mathrm{F}(s)$ function from (A.28): $\alpha_1 = h^*$, $\alpha_2 = \xi_G$, $A_1 = \delta_{11}/(h^* + \delta_{11}) > 0$, and $A_2 = r^* > 0$. Hence, Lemma A.6(i) holds: adjustment of consumption must be monotonic. □

The proof of Proposition 4 proceeds as follows. Part (i) states that it is possible for the environment to deteriorate initially. By using (A.16) in (A.18) we obtain the condition for the environment to deteriorate initially (in the absence of bond policy):

$$\begin{aligned} \ddot{\tilde{E}}(0) &= -\alpha_K \alpha_E \dot{\tilde{K}}(0) - \alpha_E^2 \alpha_G \tilde{G}(0) + \alpha_G \alpha_E \dot{\tilde{G}}(0) \\ &= \alpha_K \alpha_E \gamma_K (r^* - \delta_{11}) \mathcal{L}\{\tilde{G}, r^*\} + \alpha_G \alpha_E \xi_G \tilde{G} < 0, \end{aligned}$$

since $\tilde{G}(0) = 0$ for $0 < \xi_G < \infty$. In view of (A.19), we know that $\mathcal{L}\{\tilde{G}, r^*\} = \xi_G/(r^*(r^* + \xi_G))$. By substituting this in the expression and simplifying, we obtain the inequality mentioned in the text and Proposition 4:

$$\left(\frac{\alpha_K}{\alpha_G(1 - \epsilon_L)}\right) \left(\frac{r^* - \delta_{11}}{r^*}\right) \left(\frac{r}{r^* + \xi_G}\right) > 1.$$

The results in part (i) follow from this inequality. A low birth rate (λ small) ensures that $\delta_{11} \equiv r - \rho \equiv \lambda(1-\omega_H)$ is close to zero so that $(r^* - \delta_{11})/r^*$ is close to unity. A slow introduction of the policy (ξ_G small) ensures that the term $r/(r^* + \xi_G)$ does not vanish.

To prove part (ii) of Proposition 4, we must show that the entire path of $\tilde{E}(t)$ is monotonic if $\xi_G \to \infty$. We have already demonstrated that the environment improves in the long-run. If $\xi_G \to \infty$, and in the absence of bond policy, all transition terms from equation (A.26) vanish and the environment improves gradually according to the multiple adjustment term $A(\alpha_E, h^*, t)$, the properties of which are covered in Lemma A.3. □

3 Crowding out of the capital stock

It is possible to prove a number of results regarding the degree to which the capital stock is crowded out by abatement. The results are shown as follows. The decline in the capital stock is given by:

$$\Omega \equiv \frac{r-\rho}{r\epsilon_L\omega_C + (\epsilon_L - \omega_G)(r-\rho)}.$$

In calculating the effects on Ω of ρ, λ, ϵ_L, and ω_G, it must be taken into account that the steady-state interest rate r depends on these parameters (see (A.2)-(A.5)). The comparative static effects are:

$$\begin{aligned}
\frac{d\Omega}{d\lambda} &= \left(\frac{\rho\epsilon_L\omega_C}{\left[r\epsilon_L\omega_C + (\epsilon_L - \omega_G)(r-\rho)\right]^2}\right)\frac{dr}{d\lambda} > 0, \\
\frac{d\Omega}{d\rho} &= \frac{-\epsilon_L\omega_C\left[r - \rho(dr/d\rho)\right]}{\left[r\epsilon_L\omega_C + (\epsilon_L - \omega_G)(r-\rho)\right]^2} < 0, \\
\frac{d\Omega}{d\omega_G} &= \frac{\rho\epsilon_L\omega_C(dr/d\omega_G) + (r-\rho)\left[r-\rho+r\epsilon_L\right]}{\left[r\epsilon_L\omega_C + (\epsilon_L - \omega_G)(r-\rho)\right]^2} > 0, \\
\frac{d\Omega}{d\epsilon_L} &= \frac{\rho\epsilon_L\omega_C(dr/d\epsilon_L) - (r-\rho)\left[r-\rho+r\omega_C\right]}{\left[r\epsilon_L\omega_C + (\epsilon_L - \omega_G)(r-\rho)\right]^2} < 0.
\end{aligned}$$

Capital crowding out increases if the birth rate rises, the rate of time preference falls, the initial share of government spending rises, or the initial share of labour income falls.

4 Intergenerational welfare analysis

The welfare implications of the different environmental policies can be derived in the manner suggested by Bovenberg (1993, 1994). The optimum utility level of generation v at time t is denoted by $U(v,t)$. It is obtained by substituting the optimum values for $C(v,\tau)$ (where τ runs from t to ∞) plus the policy-induced path for $E(\tau)$ into the utility functional:[1] $U(v,t) = U_{NE}(v,t) + \gamma_E U_E(t)$,

[1] We have modified the utility functional used in Bovenberg and Heijdra (1998) (viz. equation (1)) somewhat by making it logarithmic in $E(\tau)$ also (rather than linear):

$$U(v,t) \equiv \int_t^\infty \left[\log C(v,\tau) + \gamma_E \log E(\tau)\right] \exp\left[(\rho+\lambda)(t-\tau)\right] d\tau.$$

Since we work with the log-linearized model this change does not affect anything substantial but it simplifies the notation by not having to distinguish the initial level of environmental quality (denoted by E_0 in Bovenberg and Heijdra (1998)).

where $U_{NE}(v,t)$ is the private component of welfare, and $U_E(t)$ is the environmental component:

$$U_{NE}(v,t) \equiv \int_t^{\infty} \log C(v,\tau) \exp[(\rho+\lambda)(t-\tau)]d\tau, \tag{A.29}$$

$$U_E(t) \equiv \int_t^{\infty} \log E(\tau) \exp[(\rho+\lambda)(t-\tau)]d\tau. \tag{A.30}$$

Turn to the private component of welfare first. Using the Euler equation for the household, we can relate $C(v,\tau)$ to $C(v,t)$:

$$C(v,\tau) = C(v,t) \exp\left[\int_t^{\tau} (r(\mu)-\rho)d\mu\right], \quad \tau \geq t.$$

Substitution of this result in (A.29) yields:

$$U_{NE}(v,t) = \frac{\log C(v,t)}{\rho+\lambda} + \Delta(t), \tag{A.31}$$

$$\begin{aligned} \Delta(t) &\equiv \int_t^{\infty} \left[\int_t^{\tau} [r(\mu)-\rho]d\mu\right] \exp[(\rho+\lambda)(t-\tau)]d\tau \\ &= \int_t^{\infty} \left(\frac{r(\mu)-\rho}{\rho+\lambda}\right) \exp[(t-\mu)(\rho+\lambda)]d\mu. \end{aligned} \tag{A.32}$$

Linearising the expressions in (A.31) and (A.32), we find:

$$dU_{NE}(v,t) = \frac{\tilde{C}(v,t)}{\rho+\lambda} + d\Delta(t), \tag{A.33}$$

$$d\Delta(t) = \left(\frac{r}{\rho+\lambda}\right) \int_t^{\infty} \tilde{r}(\mu) e^{-(\mu-t)(\rho+\lambda)} d\mu. \tag{A.34}$$

In order to perform the welfare analysis, we must distinguish between 'existing' agents that are already alive at the time of announcement of the unanticipated shock, and 'future' generations that are not yet alive at that time. The time of the announcement is denoted by $t_0 = 0$. Hence, agents with a generations index smaller than or equal to 0 ($v \leq t_0 = 0$) are alive at the time of the shock. Those with an index larger than 0 are born later ($v = t > t_0 = 0$).

4.1 Existing Generations ($v \leq 0$)

For existing generations, the change in private utility is $dU_{NE}(v,0) = \tilde{C}(v,0)/(\rho+\lambda) + d\Delta(0)$. Using (A.34), and applying the initial-value theorem (Spiegel, 1965, p. 20), we find $d\Delta(0) = (r/(\rho+\lambda))\mathcal{L}\{\tilde{r}, \rho+\lambda\}$. The jump in consumption is a weighted average of the jump in human wealth and the change in financial wealth (although the capital stock is predetermined, a once-off tax on capital owners of τ_K (per unit of capital) implies $\tilde{A}(v,0) = \tilde{A}(0) = -\tau_K$ for $v < 0$). Hence, $\tilde{C}(v,0) = \alpha_{HS}\tilde{H}(0) + (1-\alpha_{HS})\tilde{A}(0)$, where $\tilde{H}(v,0) = \tilde{H}(0)$, and α_{HS} denotes the share of human wealth in total wealth of an agent belonging to generation v to be determined below. The human wealth term $\tilde{H}(0)$ can be eliminated by using the solution for the initial jump in aggregate consumption, $\tilde{C}(0) = (1-\omega_H)\tilde{A}(0) + \omega_H\tilde{H}(0)$, where $1-\omega_H \equiv K/(K+H) = (r-\rho)/\lambda$. This implies that $\tilde{C}(v,0)$ can be linked to $\tilde{C}(0)$ and $\tilde{A}(0)$ ($\equiv -\tau_K$):

$$\tilde{C}(v,0) \equiv -(1-\alpha_{HS})\tau_K + \alpha_{HS}\left[\frac{\tilde{C}(0) + (1-\omega_H)\tau_K}{\omega_H}\right],$$

where the revenue of the once-off capital tax ($\tau_K K(0)$) gives rise to the (negative) jump in debt at time $t = 0$: $\tilde{B}(0) = -\tau_K(1-\epsilon_L)$, where $rK/Y \equiv 1-\epsilon_L$. The human wealth share α_{HS} is determined by using the steady-state information for the optimal steady-state consumption profile of existing generations, i.e., $C(v,0) = C(v,v)\exp(-(r-\rho)v)$ for $v < 0$. Since all generations are born without financial wealth ($A(v,v) = 0$), human wealth is the same for all agents ($H(v,t) = H$), and consumption is proportional to total wealth ($C(v,t) = (\rho+\lambda)(A(v,t)+H)$), the following expression for α_{HS} is obtained:

$$\alpha_{HS} \equiv \frac{H}{A(v,0)+H} = e^{(r-\rho)v}.$$

This expression is intuitively clear. Old agents (i.e., with very negative v) have had a long time to accumulate financial assets and hence feature a relatively low share of human wealth in total wealth. A very young agent ($v \approx 0$) owns little or no financial wealth and hence exhibits a human wealth share of unity.

After combining all the information, we obtain the following relation:

$$(\rho+\lambda)dU_{NE}(v,0) = -\tau_K + \left(\frac{e^{(r-\rho)v}}{\omega_H}\right)\left[\tilde{C}(0)+\tau_K\right] + r\mathcal{L}\{\tilde{r},\rho+\lambda\}. \tag{A.35}$$

4.2 Future Generations ($v = t > 0$)

Future generations are born without financial wealth. Hence their consumption at birth is given by $C(t,t) = (\rho+\lambda)H(t)$, so that $\tilde{C}(t,t) = \tilde{H}(t)$. Human wealth can be eliminated by using the aggregate relation $\tilde{C}(t) = \omega_H\tilde{H}(t) + (1-\omega_H)\tilde{K}(t) + [(1-\omega_H)/(1-\epsilon_L)]\tilde{B}(t)$. This enables us to write $\tilde{C}(t,t)$ in terms of $\tilde{C}(t)$, $\tilde{K}(t)$ and $\tilde{B}(t)$:

$$\tilde{C}(t,t) = \left[\frac{\tilde{C}(t) - (1-\omega_H)\tilde{K}(t) - [(1-\omega_H)/(1-\epsilon_L)]\,\tilde{B}(t)}{\omega_H}\right],\ t > 0. \tag{A.36}$$

The following Lemma can be used to calculate the Laplace transform of $d\Delta(t)$.

Lemma A.7 *Let* $\mathrm{X}(t)$ *be a function defined as follows:*

$$\mathrm{X}(t) \equiv \int_t^\infty \mathrm{z}(\mu)e^{-(\rho+\lambda)(\mu-t)}d\mu.$$

Then $\mathcal{L}\{\mathrm{X},s\}$ *is given by:*

$$\mathcal{L}\{\mathrm{X},s\} = \left[\frac{\mathcal{L}\{\mathrm{z},\rho+\lambda\} - \mathcal{L}\{\mathrm{z},s\}}{s-(\rho+\lambda)}\right].$$

PROOF: $\mathrm{X}(t)$ satisfies the differential equation:

$$\dot{\mathrm{X}}(t) = -\mathrm{z}(t) + (\rho+\lambda)\mathrm{X}(t),\quad \mathrm{X}(0) = \mathcal{L}\{\mathrm{z},\rho+\lambda\}.$$

The Laplace transform of the differential equation amounts to:

$$s\mathcal{L}\{\mathrm{X},s\} - \mathrm{X}(0) = -\mathcal{L}\{\mathrm{z},s\} + (\rho+\lambda)\mathcal{L}\{\mathrm{X},s\}.$$

By substituting the initial condition $\mathrm{X}(0) = \mathcal{L}\{\mathrm{z},\rho+\lambda\}$ and gathering terms, we obtain the required result. □

The Laplace transform of $d\Delta(t)$ can be obtained by applying Lemma A.7 to (A.34):

$$\mathcal{L}\{d\Delta,s\} = \left(\frac{r}{\rho+\lambda}\right)\left[\frac{\mathcal{L}\{\tilde{r},\rho+\lambda\} - \mathcal{L}\{\tilde{r},s\}}{s-(\rho+\lambda)}\right]. \tag{A.37}$$

By combining (A.36) with (A.33), taking Laplace transforms, and using (A.37), we arrive at the following expression for the Laplace transform of private utility of future generations:

$$
\begin{aligned}
(\rho+\lambda)\mathcal{L}\{dU_{NE},s\} &= \left(\frac{\mathcal{L}\{\tilde{C},s\}-(1-\omega_H)\mathcal{L}\{\tilde{K},s\}-[(1-\omega_H)/(1-\epsilon_L)]\,\mathcal{L}\{\tilde{B},s\}}{\omega_H}\right) \\
&\quad +r\left(\frac{\mathcal{L}\{\tilde{r},\rho+\lambda\}-\mathcal{L}\{\tilde{r},s\}}{s-(\rho+\lambda)}\right). \qquad \text{(A.38)}
\end{aligned}
$$

By substituting the Laplace transforms for $\tilde{C}(t)$, $\tilde{K}(t)$, $\tilde{B}(t)$, and $\tilde{r}(t)$ $[\equiv -\epsilon_L\tilde{K}(t)]$, and inverting, the entire path for $dU_{NE}(t,t)$ is obtained (where t acts as the index for future generations, i.e., $t>0$).

In order to derive the results in sections 5-7 of the paper, the path for $dU_{NE}(t,t)$ is written in terms of $dU_{NE}(0,0)$, $dU_{NE}(\infty,\infty)$, and adjustment and transition terms like $\mathrm{A}(h^*,t)$ and $\mathrm{T}(\xi_i,h^*,t)$. This can be proved as follows. The crucial results that must be used are:

$$
\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{\mathcal{L}\{\tilde{K},\rho+\lambda\}-\mathcal{L}\{\tilde{K},s\}}{s-(\rho+\lambda)}\right\} &= \mathcal{L}\{\tilde{K},\rho+\lambda\} \\
&\quad -\left(\frac{\gamma_K(\delta_{11}+\xi_G)}{(r^*+\xi_G)(\rho+\lambda+\xi_G)}\right)\tilde{G}\mathrm{T}(\xi_G,h^*,t) \\
&\quad +\frac{\mathrm{A}(h^*,t)}{\rho+\lambda+h^*}\left[\tilde{K}(\infty)+\frac{\gamma_K(\delta_{11}+\xi_G)}{(\rho+\lambda+\xi_G)(r^*+\xi_G)}\tilde{G}-\delta_{21}\gamma_C\sum_{i=1}^{2}\left(\frac{b_i}{(r^*+\xi_i)(\rho+\lambda+\xi_i)}\right)\right] \\
&\quad +\delta_{21}\gamma_C\sum_{i=1}^{2}\left(\frac{b_i}{(r^*+\xi_i)(\rho+\lambda+\xi_i)}\right)\mathrm{T}(\xi_i,h^*,t),
\end{aligned}
$$

and:

$$
\begin{aligned}
&\tilde{C}(t)-(1-\omega_H)\tilde{K}(t)-[(1-\omega_H)/(1-\epsilon_L)]\,\tilde{B}(t) \\
&= \tilde{C}(0)+\left[\tilde{C}(\infty)-\tilde{C}(0)-(1-\omega_H)\tilde{K}(\infty)\right]\mathrm{A}(h^*,t) \\
&\quad -\left[\frac{1-\omega_H}{1-\epsilon_L}\right]\left[\tilde{B}(0)\left[1-\mathrm{A}(h^*,t)\right]+\tilde{B}(\infty)\mathrm{A}(h^*,t)+\sum_{i=1}^{2}(\xi_i-h^*)\mathrm{T}(\xi_i,h^*,t)b_i\right] \\
&\quad +\gamma_K\left(\frac{\delta_{12}+(1-\omega_H)(\delta_{11}+\xi_G)}{r^*+\xi_G}\right)\tilde{G}\mathrm{T}(\xi_G,h^*,t) \\
&\quad -\gamma_C\sum_{i=1}^{2}\left(\frac{\xi_i+\delta_{22}+\delta_{21}(1-\omega_H)}{r^*+\xi_i}\right)b_i\mathrm{T}(\xi_i,h^*,t).
\end{aligned}
$$

By substituting these results into the inverted equation (A.38), and gathering constant terms, and terms involving $\mathrm{A}(h^*,t)$ and $\mathrm{T}(\xi_i,h^*,t)$, $dU_{NE}(t,t)$ can be written as follows:

$$
\begin{aligned}
dU_{NE}(t,t) &= dU_{NE}(0,0)+\left[dU_{NE}(\infty,\infty)-dU_{NE}(0,0)\right]\mathrm{A}(h^*,t) \\
&\quad +\Omega_{NE}(\xi_G)\tilde{G}\mathrm{T}(\xi_G,h^*,t)-\sum_{i=1}^{2}\Lambda_{NE}(\xi_i)b_i\mathrm{T}(\xi_i,h^*,t), \qquad \text{(A.39)}
\end{aligned}
$$

where we have used the definitions of $dU_{NE}(0,0)$, $dU_{NE}(\infty,\infty)$, $\Omega_{NE}(\xi_G)$ and $\Lambda_{NE}(\xi_i)$:

$$
(\rho+\lambda)dU_{NE}(0,0)=\frac{\tilde{C}(0)-[(1-\omega_H)/(1-\epsilon_L)]\,\tilde{B}(0)}{\omega_H}-r\epsilon_L\mathcal{L}\{\tilde{K},\rho+\lambda\},
$$

$$
\begin{aligned}
(\rho+\lambda)dU_{NE}(\infty,\infty) &= \left[\frac{\tilde{C}(\infty)-(1-\omega_H)\tilde{K}(\infty)-[(1-\omega_H)/(1-\epsilon_L)]\,\tilde{B}(\infty)}{\omega_H}\right] \\
&\quad -\left(\frac{r\epsilon_L}{\rho+\lambda}\right)\tilde{K}(\infty),
\end{aligned}
$$

$$
\begin{aligned}
(\rho+\lambda)\Lambda_{NE}(\xi_i) &\equiv \left[\frac{(1-\omega_H)(\xi_i-h^*)}{\omega_H(1-\epsilon_L)}+\frac{\gamma_C r\delta_{21}\epsilon_L}{(r^*+\xi_i)(\rho+\lambda+\xi_i)}\right. \\
&\quad \left.+\frac{\gamma_C(\xi_i+\delta_{22}+\delta_{21}(1-\omega_H))}{\omega_H(r^*+\xi_i)}\right],
\end{aligned}
$$

$$
(\rho+\lambda)\Omega_{NE}(\xi_G)\equiv\left(\frac{\gamma_K}{r^*+\xi_G}\right)\left[\frac{\delta_{12}+(1-\omega_H)(\delta_{11}+\xi_G)}{\omega_H}+\frac{r\epsilon_L(\delta_{11}+\xi_G)}{\rho+\lambda+\xi_G}\right].
$$

In section 5 no bond policy is used ($\tilde{B}(t)=b_0=b_1=b_2=0$), so that the transition terms $\mathrm{T}(\xi_i,h^*,t)$ disappear from the various expressions. In section 6 of the paper, b_i and ξ_i are used as instruments to redistribute utility across generations, and the most general expressions are relevant.

5 Proof of Proposition 5

In the absence of bond policy, the change in private welfare experienced by very old generations ($v\to-\infty$) can be written as:

$$
(\rho+\lambda)dU_{NE}(-\infty,0)=r\mathcal{L}\{\tilde{r},\rho+\lambda\}.
$$

The proof of Proposition 5 proceeds as follows. Proposition 5(i) makes use of the following expression for $\mathcal{L}\{\tilde{r},\rho+\lambda\}$:

$$
\begin{aligned}
\mathcal{L}\{\tilde{r},\rho+\lambda\} &\equiv -\epsilon_L\mathcal{L}\{\tilde{K},\rho+\lambda\} \\
&= -\frac{\epsilon_L\gamma_K}{\rho+\lambda+h^*}\left[\frac{\delta_{11}}{r^*(\rho+\lambda)}-\frac{\delta_{11}+\xi_G}{(\rho+\lambda+\xi_G)(r^*+\xi_G)}\right]\tilde{G},
\end{aligned}
\tag{A.40}
$$

where we have used (A.13), (A.19), and (A.20). Designating the term in square brackets by Γ_1, we observe that $\Gamma_1>0$ if the policy is introduced instantaneously ($\xi_G\to\infty$):

$$
\lim_{\xi_G\to\infty}\Gamma_1=\frac{\delta_{11}}{r^*(\rho+\lambda)}>0.
$$

Hence, if $\xi_G\to\infty$, then $(\rho+\lambda)dU_{NE}(-\infty,0)=r\mathcal{L}\{\tilde{r},\rho+\lambda\}>0$ as ($\gamma_K<0$) so that old generations unambiguously gain. If the policy is introduced gradually ($\xi_G<\infty$), the sign of Γ_1 is ambiguous. For a low enough value of ξ_G, Γ_1 may become negative so that $(\rho+\lambda)dU_{NE}(-\infty,0)=r\mathcal{L}\{\tilde{r},\rho+\lambda\}<0$ and old generations lose as a result of the gradual introduction of the policy. The relevant condition for this case to obtain is that $\xi_G<\bar{\xi}_G$, where $\bar{\xi}_G$ is the non-zero value of ξ_G for which $\Gamma_1=0$:

$$
\bar{\xi}_G\equiv\frac{r^*(\rho+\lambda)-\delta_{11}(\rho+\lambda+r^*)}{\delta_{11}}.
$$

It is, of course, possible that $\bar{\xi}_G<0$ in which case $\Gamma_1>0$ for all positive values of ξ_G. This completes the proof of Proposition 5(i).

In order to prove Proposition 5(ii) we write $\rho+\lambda)dU_{NE}(0,0)=\tilde{C}(0)/\omega_H+r\mathcal{L}\{\tilde{r},\rho+\lambda\}$. We have already established the sign configuration of $\mathcal{L}\{\tilde{r},\rho+\lambda\}$. From (A.25) we know that:

$$\tilde{C}(0)=-\frac{\xi_G(r^*-\delta_{11})}{r^*\omega_C(r^*+\xi_G)}\tilde{G}=-\frac{\gamma_K(r^*-\delta_{11})\xi_G}{r^*\delta_{21}(r^*+\xi_G)}\tilde{G}<0.$$

By combining the relevant information, $(\rho+\lambda)dU_{NE}(0,0)$ can be written as:

$$(\rho+\lambda)dU_{NE}(0,0)\equiv-\frac{\gamma_K}{r^*}\left[\frac{r^*-\delta_{11}}{\omega_H\delta_{21}}\frac{\xi_G}{r^*+\xi_G}+\frac{r\epsilon_L r^*\Gamma_1}{\rho+\lambda+h^*}\right]\tilde{G}.$$

We designate the term in square brackets by Γ_2. It is possible to show that $\Gamma_2<0$ regardless of the speed at which the policy is introduced. By using the relations $(r^*-\delta_{11})/\delta_{21}=\delta_{12}/(r^*-\delta_{22})=\delta_{12}/(h^*+\delta_{11})$ and $\delta_{12}=-[r\epsilon_L+\delta_{11}]$, Γ_2 can be written in terms of two parts:

$$\begin{aligned}\Gamma_2 &\equiv \left[\frac{-\delta_{11}\xi_G}{\omega_H(h^*+\delta_{11})(r^*+\xi_G)}\right]+r\epsilon_L\left[\frac{\delta_{11}}{(\rho+\lambda)(\rho+\lambda+h^*)}\right.\\ &\quad\left.-\frac{r^*(\delta_{11}+\xi_G)}{(\rho+\lambda+\xi_G)(\rho+\lambda+h^*)(r^*+\xi_G)}-\frac{\xi_G}{\omega_H(\delta_{11}+h^*)(r^*+\xi_G)}\right].\end{aligned}$$

The first term in square brackets is unambiguously negative, so that a *sufficient* condition for $\Gamma_2<0$ is that the second term in square brackets, which we designate by Γ_3, be negative also. Note that Γ_3 can be written as: $\Gamma_3=\xi_G(\xi_G\Gamma_{31}+\Gamma_{32})/\Gamma_{33}$, where $\Gamma_{33}=\omega_H(\rho+\lambda)(\rho+\lambda+h^*)(\rho+\lambda+\xi_G)(r^*+\xi_G)(\delta_{11}+h^*)>0$ is the positive denominator. It is easy to show that Γ_{31} and Γ_{32} are both negative:

$$\begin{aligned}\Gamma_{31} &\equiv \delta_{11}\omega_H(\delta_{11}+h^*)-(\rho+\lambda)(\rho+\lambda+h^*)<0,\\ \Gamma_{32} &\equiv r^*\left[\delta_{11}-(\rho+\lambda)\right]\omega_H(\delta_{11}+h^*)+\delta_{11}(\rho+\lambda)\omega_H(\delta_{11}+h^*)\\ &- (\rho+\lambda)^2(\rho+\lambda+h^*)<0,\end{aligned}$$

where we have used the fact that $\delta_{11}=\lambda(1-\omega_H)<\rho+\lambda$ and $0<\omega_H<1$. Hence, $\Gamma_3<0$, so that the sufficient condition for $\Gamma_2<0$ is satisfied regardless of the magnitude of ξ_G. It has thus been established that $(\rho+\lambda)dU_{NE}(0,0)<0$. This completes the proof of Proposition 5(ii).

In order to prove Proposition 5(iii) we write steady-state utility as:

$$(\rho+\lambda)dU_{NE}(\infty,\infty)=\left[\frac{\tilde{C}(\infty)}{\omega_H}-\frac{(1-\omega_H)\tilde{K}(\infty)}{\omega_H}\right]+\left(\frac{r}{\rho+\lambda}\right)\tilde{r}(\infty).$$

Using (A.14), (A.15), and $\tilde{r}(\infty)=-\epsilon_L\tilde{K}(\infty)>0$, this expression can be re-written as:

$$(\rho+\lambda)dU_{NE}(\infty,\infty)=-\frac{\gamma_K}{r^*h^*}\left[\frac{\delta_{12}}{\omega_H}+\frac{\delta_{11}(1-\omega_H)}{\omega_H}+\frac{r\epsilon_L\delta_{11}}{\rho+\lambda}\right]\tilde{G}.$$

We designate the term in square brackets as Γ_4. By substituting $\delta_{12}=-[r\epsilon_L+\delta_{11}]$, Γ_4 can be rewritten as:

$$\Gamma_4\equiv-\delta_{11}-r\epsilon_L\left[\frac{1}{\omega_H}-\frac{\delta_{11}}{\rho+\lambda}\right]<0,$$

where we have used the fact that $0<\omega_H<1$ and $\delta_{11}=\lambda(1-\omega_H)<\rho+\lambda$. This completes the proof of Proposition 5(iii).

In order to prove Proposition 5(iv), write $dU_{NE}(0,0) - dU_{NE}(\infty,\infty)$ as follows:

$$\begin{aligned}
&\left(\frac{\omega_H(\rho+\lambda)(h^*+\delta_{11})r^*h^*}{\gamma_K\delta_{11}\tilde{G}}\right)[dU_{NE}(0,0) - dU_{NE}(\infty,\infty)] \\
= \;& (h^*-\lambda\omega_H)(1-\omega_H) - r\epsilon_L\left[\frac{(h^*-\lambda\omega_H)(1-\omega_H)+\rho+\lambda}{(\rho+\lambda+h^*)}\right].
\end{aligned} \tag{A.41}$$

Proposition 2 demonstrates that $h^* > \lambda\omega_H$. Hence, the first term on the right-hand side is positive but the term in square brackets is also positive, so that the two terms work in opposite directions. By bringing all terms on the same denominator we can rewrite the right-hand side of (A.41) as $\Gamma_5/(\rho+\lambda+h^*)$, where Γ_5 is defined as:

$$\begin{aligned}
\frac{\Gamma_5}{(1-\omega_H)} =& (h^*-\lambda\omega_H)(\rho+\lambda+h^*) - \left(\frac{(h^*+\delta_{11})(h^*+\delta_{22})}{\rho+\lambda} - \lambda\right) \times \\
& [((h^*-\lambda\omega_H)(1-\omega_H)+\rho+\lambda)]
\end{aligned}$$

and where we have used the relations $r\epsilon_L = -(\delta_{12}+\delta_{11})$ and $\delta_{12} = -(1-\omega_H)(h^*+\delta_{22})(h^*+\delta_{11})/(\rho+\lambda)$. After substituting $\delta_{11} = \lambda(1-\omega_H)$ and $\delta_{22} = \rho+\lambda(1-\omega_H)$ and simplifying, we can show that $\Gamma_5 < 0$:

$$\begin{aligned}
\Gamma_5 \;=\;& (1-\omega_H)(h^*-\lambda\omega_H)\left[(\rho+\lambda+h^*) - \left(\frac{(h^*-\lambda\omega_H)+(\rho+2\lambda)}{\rho+\lambda}\right)\times\right. \\
& \left.((h^*-\lambda\omega_H)(1-\omega_H)+\rho+\lambda)\right] \\
=\;& -(1-\omega_H)^2(h^*-\lambda\omega_H)\left[\lambda + \frac{(h^*-\lambda\omega_H)(h^*-\lambda\omega_H+\rho+2\lambda)}{\rho+\lambda}\right] < 0.
\end{aligned}$$

Since the term in round brackets on the left-hand side of (A.41) is negative, $\Gamma_5 < 0$ implies that $dU_{NE}(0,0) > dU_{NE}(\infty,\infty)$. This concludes the proof of Proposition 5(iv).

In order to prove Proposition 5(v) we write the utility of the representative agent at the time of the shock as $\rho dU_{NE}(0) = \tilde{C}(0) + \rho\mathcal{L}\{\tilde{r},\rho\}$ (the generations index is obviously not relevant here). We have already established the sign configuration of $\mathcal{L}\{\tilde{r},\rho+\lambda\}$. By combining the relevant information, $\rho dU_{NE}(0)$ can be written as:

$$\rho dU_{NE}(0) \equiv -\frac{\gamma_K\delta_{12}\xi_G\tilde{G}}{h^*(\rho+h^*)(\rho+\xi_G)} < 0,$$

where we have used $r^* = \rho+h^*$. Hence, regardless of the speed at which the policy is introduced (ξ_G), $\rho dU_{NE}(0) < 0$, and there are first-order welfare costs associated with the increase of the abatement activities. This completes the proof of Proposition 5(v). □

6 Environmental utility and the proof of Proposition 6

6.1 Environmental utility

The environmental component of total utility is given in equation (A.30). By linearising (A.30), and using Lemma A.7, the following expression for the change in environmental utility can be obtained:

$$\mathcal{L}\{dU_E,s\} = \frac{\mathcal{L}\{\tilde{E},\rho+\lambda\} - \mathcal{L}\{\tilde{E},s\}}{s-(\rho+\lambda)}. \tag{A.42}$$

By using the initial and final value theorems, respectively, on equation (A.42) we obtain:

$$dU_E(0) = \mathcal{L}\{\tilde{E}, \rho+\lambda\}, dU_E(\infty) = \frac{\tilde{E}(\infty)}{\rho+\lambda}.$$

6.2 Proposition 6

In order to prove Proposition 6(i), we write, by employing (A.17), $dU_E(0) = \mathcal{L}\{\tilde{E}, \rho+\lambda\} = [-\alpha_K\alpha_E\mathcal{L}\{\tilde{K}, \rho+\lambda\} + \alpha_G\alpha_E\mathcal{L}\{\tilde{G}, \rho+\lambda\}]/(\rho+\lambda+\alpha_E)$. By using (A.40) and the Laplace transform of (A.19) this can be rewritten as follows:

$$dU_E(0) = -\frac{\alpha_K\alpha_E}{\rho+\lambda+\alpha_E}\left(\frac{\gamma_K\Gamma_1}{\rho+\lambda+h^*}\right)\tilde{G} + \frac{\alpha_G\alpha_E}{\rho+\lambda+\alpha_E}\left(\frac{\xi_G}{(\rho+\lambda)(\rho+\lambda+\xi_G)}\right)\tilde{G}, \quad \text{(A.43)}$$

where Γ_1 is defined in section 5 above. We have already demonstrated that $\Gamma_1 > 0$ if the policy is introduced sufficiently quickly (i.e., if $\xi_G > \bar{\xi}_G$). Hence, since $\gamma_K < 0$, $dU_E(0) > 0$ in that case.

Environmental utility can decrease initially only if the policy is introduced slowly ($\xi_G < \bar{\xi}_G$ so that $\Gamma_1 < 0$) *and* abatement is not very effective at cleaning up the environment (α_G/α_K small). In terms of (A.43), a *sufficient* condition for the initial green welfare to deteriorate is therefore:

$$\Gamma_1 < -\left(\frac{\alpha_G}{\alpha_K}\right)\left(\frac{\rho+\lambda+h^*}{\rho+\lambda}\right)\left(\frac{\xi_G}{\rho+\lambda+\xi_G}\right)\left(\frac{1-\epsilon_L}{r}\right).$$

This completes the proof of Proposition 6(i).

The proof of Proposition 6(ii) is immediate:

$$dU_E(\infty) = \frac{\tilde{E}(\infty)}{\rho+\lambda} = \frac{1}{\rho+\lambda}\left[\alpha_G - \frac{\alpha_K\gamma_K\delta_{11}}{r^*h^*}\right]\tilde{G} > 0.$$

In order to prove Proposition 6(iii), we only need to consider the case of an abrupt introduction of the policy ($\xi_G \to \infty$) (since $dU_E(0)$ is largest for an instantaneous introduction of the policy). We can write:

$$\begin{aligned}(\rho+\lambda)\left[dU_E(\infty) - dU_E(0)\right] &= \left[\alpha_G - \frac{\alpha_K\gamma_K\delta_{11}}{r^*h^*}\right]\tilde{G} \\ &\quad - \left(\frac{\alpha_E}{\rho+\lambda+\alpha_E}\right)\left[\alpha_G - \frac{\alpha_K\gamma_K\delta_{11}}{r^*(\rho+\lambda+h^*)}\right]\tilde{G}\end{aligned}$$

$$= \alpha_G\left[1 - \frac{\alpha_E}{\rho+\lambda+\alpha_E}\right]\tilde{G} - \frac{\alpha_K\gamma_K\delta_{11}}{r^*h^*}\left[1 - \frac{\alpha_E}{\rho+\lambda+\alpha_E}\frac{h^*}{\rho+\lambda+h^*}\right]\tilde{G} > 0.$$

The long-term effect on environmental utility exceeds the short-term effect both because the environment regenerates slowly and because the economic system has a finite transition speed. Only if both α_E and h^* tend to infinity, do the two effects coincide. This completes the proof of Proposition 6(iii). □

In order to prove Proposition 6(iv), we use (A.17) and the Laplace transform of (A.19) (setting $b_i = 0$ and $\xi_G \to \infty$) to derive:

$$\begin{aligned}\mathcal{L}\{\dot{\tilde{E}}, s\} &= \frac{\alpha_E s}{s+\alpha_E}\left[-\alpha_K\mathcal{L}\{\tilde{K}, s\} + \alpha_G\mathcal{L}\{\tilde{G}, s\}\right] \\ &= \frac{\alpha_E}{s+\alpha_E}\left[\alpha_G + \frac{\alpha_K(-\gamma_K)\delta_{11}}{r^*}\frac{1}{s+h^*}\right]\tilde{G}.\end{aligned}$$

Since the term in square brackets is positive (as $\gamma_K < 0$), Lemma A.6 implies that $\tilde{E}(t)$ rises monotonically.□

7 Comparative static effects for the optimal abatement share

In the text below equation (21) the comparative static effects on the optimal share of abatement spending are discussed. These results can be obtained as follows. We assume that the policy is introduced instantaneously ($\xi_G \to \infty$). The first-order condition for an internal optimum is:

$$\rho dU(0) = \left[\frac{-\rho^2 \epsilon_L}{(1-\epsilon_L) h^* (\rho + h^*)} + \frac{\gamma_E \alpha_G \alpha_E}{\rho + \alpha_E} \right] \tilde{G} = 0. \tag{A.44}$$

Equation (A.7) implies that $r^* h^* = \rho^2 \epsilon_L \omega_C / (1 - \epsilon_L)$, and the trace condition shows that $r^* = \rho + h^*$. Hence, (A.44) can be solved for the optimal share of private consumption:

$$\hat{\omega}_C \equiv \frac{\rho + \alpha_E}{\gamma_E \alpha_G \alpha_E}. \tag{A.45}$$

The comparative static effects on $\hat{\omega}_G$ of α_G, α_K, α_E, and γ_E can be obtained by differentiating (A.45) with respect to these variables:

$$\begin{aligned}
\frac{d\hat{\omega}_C}{d\alpha_E} &\equiv -\frac{d\hat{\omega}_G}{d\alpha_E} = \frac{-\rho}{\gamma_E \alpha_G \alpha_E^2} < 0, \\
\frac{d\hat{\omega}_C}{d\alpha_G} &\equiv -\frac{d\hat{\omega}_G}{d\alpha_G} = \frac{-(\rho + \alpha_E)}{\gamma_E \alpha_G^2 \alpha_E} < 0, \\
\frac{d\hat{\omega}_C}{d\gamma_E} &\equiv -\frac{d\hat{\omega}_G}{d\gamma_E} = \frac{-(\rho + \alpha_E)}{\gamma_E^2 \alpha_G \alpha_E} < 0, \\
\frac{d\hat{\omega}_C}{d\alpha_K} &\equiv -\frac{d\hat{\omega}_G}{d\alpha_K} = 0.
\end{aligned}$$

With infinite lives and abrupt policy introduction, there is no effect on the capital stock. This explains why α_K does not affect the optimal share of abatement.

8 Redistribution issues

8.1 All current generations equally well off

In sub-section 6.1 of the paper we discuss a Pareto-improving policy which involves the neutralisation of intergenerational inequity of existing generations only. By imposing a once-off capital tax, all existing generations can be made to gain to the same extent, whilst future generations reap additional benefits due to the gradual improvement of the environment. In sub-section 6.1, τ_K is the only bond-policy instrument available to the policy maker.

It follows from equation (A.35) that the neutralisation of generational effects for existing generations requires $\tilde{C}(0) + \tau_K = 0$, or:

$$\tau_K = -\tilde{C}(0), \quad \tilde{B}(0) = (1 - \epsilon_L) \tilde{C}(0). \tag{A.46}$$

But the jump in consumption in its turn depends on the parameters of the bond path. By setting $\xi_G \to \infty$ and $b_1 = b_2 = 0$ in (A.25) we obtain:

$$\tilde{C}(0) = -\left(\frac{\gamma_K (r^* - \delta_{11})}{r^* \delta_{21}} \right) \tilde{G} - \left(\frac{\gamma_C}{r^*} \right) \tilde{B}(0). \tag{A.47}$$

By using (A.46) and (A.47) we find the expressions for $\tilde{C}(0)$, $\tilde{B}(0)$, and τ_K:

$$\tilde{C}(0) = -\tau_K = \tilde{B}(0)/(1 - \epsilon_L) = -\tilde{G}/\omega_C. \tag{A.48}$$

The bond-induced consumption jump eliminates all transitional dynamics from the consumption-capital system. In terms of Figure 2 of the paper, the bond policy shifts the MKR curve to intersect the new IS curve in a point vertically below the initial equilibrium E_0 (i.e. at point D). Hence, the macroeconomic variables satisfy:

$$\tilde{K}(t) = \tilde{Y}(t) = \tilde{r}(t) = \tilde{W}(t) = 0, \tilde{C}(t) = -\tilde{G}/\omega_C, \quad \text{(A.49)}$$

and the environment evolves according to:

$$\tilde{E}(t) = \alpha_G \tilde{G} \text{A}(\alpha_E, t). \quad \text{(A.50)}$$

By using equation (A.49) in (A.35) and (A.38), and equation (A.50) in (A.42) we obtain the expressions for, respectively, non-environmental and environmental utility:

$$\begin{aligned} dU_{NE}(v,0) &= dU_{NE}(t,t) = -\frac{\tilde{G}}{(\rho+\lambda)\omega_C}, \\ dU_E(t) &= \frac{\alpha_G \tilde{G}}{\rho+\lambda}\left[1 - \left(\frac{\rho+\lambda}{\rho+\lambda+\alpha_E}\right) e^{-\alpha_E t}\right], \end{aligned}$$

for $v \leq 0$ and $t \geq 0$. The total welfare gain to present generations equals:

$$(\rho+\lambda) dU(v,0) = \left[-\frac{1}{\omega_C} + \frac{\gamma_E \alpha_G \alpha_E}{\rho+\lambda+\alpha_E}\right]\tilde{G}, \text{ for } v \leq 0.$$

By setting ω_G such that $dU(v,0) = 0$, we obtain the expression for the optimal share of abatement which makes all present generations equally well off (and thus all future generations strictly better off):

$$\omega_C^{PI} = \frac{\rho+\lambda+\alpha_E}{\gamma_E \alpha_G \alpha_E}.$$

8.2 All current and future generations equally well off

The simulations underlying Table 4 in the text are performed as follows. The policy maker uses the path of debt as parameterized in (A.19). This implies that the paths for the capital stock, consumption, and the environment are given by (A.23)-(A.26). The paths for private utility are given in (A.35) and (A.39) and the path for environmental utility is given in (A.42). Total utility is the sum of private and environmental utility. The requirements for the egalitarian policy are that all generations gain to the same extent. Denoting this common gain by π, the requirements are summarized by:

$$dU(v,0) = dU(t,t) = \pi, \ v \leq 0, t \geq 0.$$

The instruments at the disposal of the policy maker are a once-off tax on capital owners at the time of the shock (τ_K), plus the parameters influencing the shape of the anticipated part of the bond path (ξ_i and b_i, $i = 1, 2$). The values for these parameters are computed as follows.

As in the previous sub-section, the capital levy τ_K is used to eliminate the intergenerational inequity for existing generations. With the more general bond policy used here, equation (A.48) becomes:

$$\tilde{C}(0) = -\tau_K = \tilde{B}(0)/(1-\epsilon_L) = -\tilde{G}/\omega_C - \left(\frac{\gamma_C}{r^* - \delta_{11}}\right)\sum_{i=1}^{2} \frac{b_i \xi_i}{r^* + \xi_i}. \quad \text{(A.51)}$$

Similarly, the long-run effects on consumption and the capital stock can be expressed as follows:

$$\tilde{C}(\infty) = -\frac{\tilde{G}}{\omega_C} + \left(\frac{\gamma_K \delta_{11}}{(r^* - \delta_{11})h^*}\right) \sum_{i=1}^{2} \frac{b_i(r^* + \xi_i - \delta_{11})}{r^* + \xi_i}, \tag{A.52}$$

$$\begin{aligned} \tilde{K}(\infty) &= \left(\frac{\gamma_K \delta_{11} \omega_C}{(1-\epsilon_L)(r^* - \delta_{11})h^*}\right) \sum_{i=1}^{2} \frac{b_i(r^* + \xi_i - \delta_{11})}{r^* + \xi_i} \\ &= \frac{\omega_C}{1-\epsilon_L}\left[\tilde{C}(\infty) + \frac{\tilde{G}}{\omega_C}\right]. \end{aligned} \tag{A.53}$$

By using the Laplace transforms of the bond path and the implied solution paths for capital and consumption, equation (A.38) can be written as follows:

$$\mathcal{L}\{dU_{NE}, s\} = p_0\left(\frac{1}{s}\right) + p_H\left(\frac{1}{s+h^*}\right) + \sum_{i=1}^{2} p_{Xi}\left(\frac{1}{s+\xi_i}\right), \tag{A.54}$$

where p_0, p_H, and p_{Xi} are defined as follows:

$$p_0 \equiv \frac{1}{(\rho+\lambda)\omega_H}\left[\tilde{C}(\infty) - (1-\omega_H)\left[\tilde{K}(\infty) + \frac{\tilde{B}(\infty)}{1-\epsilon_L}\right]\right] - \frac{r\epsilon_L}{(\rho+\lambda)^2}\tilde{K}(\infty),$$

$$\begin{aligned} p_H \equiv\; & \frac{1}{(\rho+\lambda)\omega_H}\left[\tilde{C}(0) - \tilde{C}(\infty) + (1-\omega_H)\tilde{K}(\infty) - \gamma_C \sum_{i=1}^{2}\left[\frac{b_i\left[\delta_{22} + \xi_i + \delta_{21}(1-\omega_H)\right]}{(r^*+\xi_i)(\xi_i - h^*)}\right]\right] \\ & + \frac{r\epsilon_L}{(\rho+\lambda)(\rho+\lambda+h^*)}\left[\tilde{K}(\infty) - \gamma_C\delta_{21}\sum_{i=1}^{2}\frac{b_i}{(r^*+\xi_i)(\xi_i-h^*)}\right], \end{aligned}$$

$$\begin{aligned} p_{Xi} \equiv\; & \frac{1}{(\rho+\lambda)\omega_H}\left[\left(\frac{1-\omega_H}{1-\epsilon_L}\right)b_i + \gamma_C\left(\frac{b_i\left[\delta_{22} + \xi_i + \delta_{21}(1-\omega_H)\right]}{(r^*+\xi_i)(\xi_i-h^*)}\right)\right] \\ & + \frac{r\epsilon_L\gamma_C\delta_{21}b_i}{(\rho+\lambda)(\rho+\lambda+\xi_i)(r^*+\xi_i)(\xi_i-h^*)}, \end{aligned}$$

$(i = 1, 2)$,where $\tilde{C}(0)$, $\tilde{C}(\infty)$, and $\tilde{K}(\infty)$ are defined, respectively, in (A.51), (A.52), and (A.53).

In a similar fashion, the Laplace transform of environmental utility can be written as follows:

$$\mathcal{L}\{dU_E, s\} = e_0\left(\frac{1}{s}\right) + e_H\left(\frac{1}{s+h^*}\right) + e_A\left(\frac{1}{s+\alpha_E}\right) + \sum_{i=1}^{2} e_{Xi}\left(\frac{1}{s+\xi_i}\right), \tag{A.55}$$

where e_0, e_H, e_A, and e_{Xi} are defined as follows:

$$e_0 \equiv \frac{\alpha_G\tilde{G} - \alpha_K\tilde{K}(\infty)}{\rho+\lambda},$$

$$e_H \equiv \frac{-\alpha_E\alpha_K}{(h^*-\alpha_E)(\rho+\lambda+h^*)}\left[\tilde{K}(\infty) - \gamma_C\delta_{21}\sum_{i=1}^{2}\frac{b_i}{(r^*+\xi_i)(\xi_i-h^*)}\right],$$

$$(\rho+\lambda+\alpha_E)e_A \equiv -\alpha_G\tilde{G} + \left(\frac{\alpha_K}{h^*-\alpha_E}\right)\left[h^*\tilde{K}(\infty) - \gamma_C\delta_{21}\alpha_E\sum_{i=1}^{2}\frac{b_i}{(r^*+\xi_i)(\xi_i-\alpha_E)}\right],$$

$$e_{Xi} \equiv \frac{-\alpha_K \alpha_E \gamma_C \delta_{21} b_i}{(\rho+\lambda+\xi_i)(r^*+\xi_i)(\xi_i-h^*)(\xi_i-\alpha_E)}, \;(i=1,2).$$

By combining (A.54) and (A.55), the Laplace transform of the path of total utility is obtained:

$$\begin{aligned}\mathcal{L}\{dU,s\} &= [p_0+\gamma_E e_0]\left(\frac{1}{s}\right)+[p_H+\gamma_E e_H]\left(\frac{1}{s+h^*}\right)\\ &\quad +\gamma_E e_A\left(\frac{1}{s+\alpha_E}\right)+\sum_{i=1}^{2}[p_{Xi}+\gamma_E e_{Xi}]\left(\frac{1}{s+\xi_i}\right).\end{aligned}$$

An egalitarian policy is such that all generations enjoy the same utility change, i.e. all terms involving time-variation are set to zero:

$$p_H+\gamma_E e_H = \gamma_E e_A = p_{Xi}+\gamma_E e_{Xi} = 0, \;(i=1,2).$$

The common gain to all generations is then given by:

$$\pi \equiv p_0+\gamma_E e_0.$$

Values for π are reported in Table 4. Finally, an optimal egalitarian policy is such that $\pi=0$.

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