1 Introduction

This set of notes helps in understanding and solving dynamic programming problems in economics. Dynamic programming is typically one branch of dynamic optimization techniques. The theory of optimal control (see Chiang, 1992) is the other main branch pointed at continuous models. Dynamic programming is especially interesting if time is discrete. As known calculus of variations is a special case of both techniques as we will show below.

It is useful to distinguish conceptual and methodological issues in dynamic programming. First we need the intuition and after that we can go for the direct solution of a problem. We will try to get the intuition from a basic model, the optimal growth model See section 2. After that we go the more formal derivations and solutions.

2 Optimal growth model

We consider the Cass-Koopmans optimal growth model. This model is an extension of the Solow growth model. In the Solow model the savings rate is imposed, and there is no representation of preferences. the optimal growth model adds preferences for households, and derives the optimal savings rate. Utility of consumers is optimized, given the technology the firms can offer. We can write the model in terms of decisions to be made by a social planner.

We model preferences in an instantaneous utility function:

\[ U(c_t), t = 0, \ldots, \infty \]

where \( c_t \) is per capita consumption. We assume a constant population which we normalize to 1. Utility is assumed to be separable in time: marginal utility of consumption only depends on today’s consumption. Households discount utility in the future by a constant factor \( \beta = 1/(1+r) < 1 \) for positive rates \( r \). So the objective of a consumer is to maximize the present discounted value of future utility:

\[ \sum_{t=0}^{\infty} \beta^t U(c_t) \]
Now we go to the technology. We assume that output is produced using capital as input:

\[ y_t = f(k_t) \]  

We assume that \( k_t \) is capital available at the beginning of period \( t \). This is capital that was accumulated in the previous period \( t - 1 \). Capital accumulation takes place through investment:

\[ k_{t+1} = k_t(1 - \delta) + i_t \]  

Depreciation \( \delta \) is assumed to be 100 per cent: \( \delta = 1 \), so we get from the resource constraint

\[ c_t + i_t = f(k_t) = y_t \]  

the budget constraint:

\[ c_t + k_{t+1} = f(k_t) \]  

So the easy way of thinking of this equality is seeing \( k_{t+1} \) as savings.

So the whole problem is how to optimize lifetime utility given the budget constraint and some initial value of \( k_0 \). The solution will be a sequence of \( c_t \) and \( k_{t+1} \) for all \( t \)'s. There is some structure though, since the problem is the same for all periods, so we can derive a nice policy rule. This policy rule tells us what is optimal to do, given the current state of the economy.

Back to the intuition. In economic models a subset of all variables often fully summarizes the state of the system at any point of time. The behavior of the other variables can be derived from the behavior of this subset. For instance in the optimal growth model the capital stock \( k_t \) fulfills this role. We call such a variable a state variable. For each non-state variable we have a behavioral equation relating its value at time \( t \) to the values of the state variables at \( t \). For example: \( y_t = f(k_t) \). Knowing the subset of state variables can be handy, since we reduce the complexity of the model to a large extent.

The evolution of the state variables over time determines the dynamics of the model. So we need to describe this using a so-called transition equation for each of the state variables. Another term used is law of motion. In the example above this is the equation that relates \( k_{t+1} \) to \( k_t \): \( k_{t+1} = k_t(1 - \delta) + i_t \).

In a model of explicit optimizing behavior we distinguish state variables from choice variables, or instruments. This choice variables is in the control of the agent and can be set as a function of the state variables at the beginning of the period in which the choice is made. In the example above consumption \( c_t \) plays this role. One should be careful here though, since sometimes state variables at time \( t + 1 \) can be in the instrument set.

So for our optimal growth model we need to find a sequence of \( k_t, k_{t+1}, \ldots \) that gives maximum discounted utility. In this line it is worthwhile to introduce the concept of a value function. A value function describes the maximum present discounted value of the objective function from a specific point in time as a function of the state variables at that date. So for the problem above we could define a value function by \( V(k_t) \). This can be seen as follows. \( k_t \) determines \( k_{t+1} \), which determines \( k_{t+2} \), etc. This series describes the whole sequence \( c_t, c_{t+1}, \ldots \).
These values for consumption define the maximum attainable utility. The value function concept argues that we do not need to know this sequence, but it is sufficient to know \( k_t \) and so \( V(k_t) \). We will use this idea in solving the dynamic programming problem using the so-called Bellman approach.

3 Finite horizon problems

It is easy to start with a finite horizon approach. Assume you die at time \( T \). We can consider an infinite horizon problem as the case \( T \to \infty \). For the problem above the assumption of a finite \( T \) affects the problem to some extent. One will consume all in \( T \) such that \( k_{T+1} = 0 \).

Let us now look at optimality. Consider the problem:

\[
\max \sum_{t=0}^{T} \beta^t U[f(k_t) - k_{t+1}] \tag{7}
\]

which we need to optimize for the path for \( k_t \). For one specific period this objective function is:

\[
\ldots + \beta^t U(f(k_t) - k_{t+1}) + \beta^{t+1} U(f(k_{t+1}) - k_{t+2}) + \ldots \tag{8}
\]

which includes all appearances of \( k_{t+1} \). Take the derivative with respect to \( k_{t+1} \) and divide by \( \beta^t \):

\[
U'(f(k_t) - k_{t+1}) = \beta U'(f(k_{t+1} - k_{t+2}) f'(k_{t+1}) \tag{9}
\]

which holds for all \( t < T \). This condition is an Euler-type condition. In equilibrium today’s marginal utility is in balance with discounted marginal utility of tomorrow. We discount by the rate of time preference and the interest rate (in equilibrium \( R = f'(k) \)). So for all periods this marginal necessary condition holds, except for the last period.

3.1 Special case

Consider the case of a logarithmic utility function \( U(c_t) = \log(c_t) \) and a Cobb-Douglas production function: \( y_t = k_t^\alpha \). For the first-order condition we get:

\[
\frac{1}{k_t^\alpha} - k_{t+1} = \beta \left( \frac{1}{k_{t+1}^\alpha} - k_{t+2} \right) \alpha k_{t+1}^{\alpha-1} \tag{10}
\]

As one can see this is a rather unpleasant second-order difference equation. Here we use a trick. define:

\[
z_t = \frac{k_{t+1}}{k_t} \tag{11}
\]

So we can simplify the first-order condition into:

\[
\frac{1}{k_t^\alpha} \left( \frac{1}{1 - z_t} \right) = \beta \frac{k_{t+1}^\alpha}{k_{t+1}^\alpha} \left( \frac{1}{1 - z_{t+1}} \right) \alpha k_{t+1}^{\alpha-1} \tag{12}
\]
which we can simplify into:

\[
\frac{z_t}{1 - z_t} = \alpha \beta \left( \frac{1}{1 - z_{t+1}} \right)
\]  

(13)

The solution might be characterized by:

\[
z_{t+1} = 1 + \alpha \beta - \frac{\alpha \beta}{z_t}
\]  

(14)

This is not an explicit solution per se. Note that in order to get \( z_t = z_{t+1} \), \( z_T = 1 \) and \( z_t = \alpha \beta \) are candidate solutions.

If we start from the terminal condition that \( z_T = 0 \) we will get \( z_t = \alpha \beta \) as the solution. One can see this as follows. Start from \( Z_T = 0 \) to get:

\[
z_T = 0 = 1 + \alpha \beta - \frac{\alpha \beta}{z_{T-1}}
\]  

(15)

So:

\[
z_{T-1} = \frac{\alpha \beta}{1 + \alpha \beta}
\]  

(16)

In a similar way we get:

\[
z_{T-2} = \frac{\alpha \beta (1 + \alpha \beta)}{1 + \alpha \beta (1 + \alpha \beta)}
\]  

(17)

In the limit we get \( z_t = \alpha \beta \). This is rather ad hoc way of solving a dynamic programming model. We leave the example right now and go to the infinite solution.

4 Deterministic infinite models

4.1 Intuition

In general terms it is rather difficult to get the infinite case as a limiting case from the finite problems. The main reason is that the limit and max operators cannot be interchanged. So the max of the limit is not the limit of the max. So we will follow another route and use the Bellman principle. Let’s return to the optimal growth model. If we can decide on today’s consumption and tomorrow’s beginning capital, the other decisions can wait until tomorrow to be made. So we will treat the problem in two separate terms. We are concerned about today’s choices for the choice variables and the inheritance for tomorrow’s state variables. Here it will be easy to use the value function. Suppose we start at time \( t = 0 \). in the optimal growth model the single state variable is \( k_t \), so we can express the value function of the problem as \( v(k_0) \).

\[
v(k_0) = \sum_{t=0}^{\infty} \beta^t U(c_t)
\]  

(18)
We can rewrite this equation into:

\[ v(k_0) = \max \left[ U(c_0) + \max \sum_{t=1}^{\infty} \beta^t U(c_t) \right] \quad (19) \]

or

\[ v(k_0) = \max \left[ U(c_0) + \beta v(k_1) \right] \quad (20) \]

If we know the true shape of the value function \( v(k) \) we can compute the optimal policy function \( k_{t+1} = g(k_t) \) for this problem. But we do not know the value function and we do not need to know it.

We can substitute the restriction \( c_0 = -k_1 + f(k_0) \) into the problem to get:

\[ v(k_0) = \max \left[ U(f(k_0) - k_1) + \beta v(k_1) \right] \quad (21) \]

This is the Bellman equation, whose solution \( v(.) \) this function is stationary. \( k_0 \) is the state variable, \( k_1 \) is the choice variable. If we would not have substituted consumption \( c_0 \) out of the problem, it would have been a choice variable as well.

A simple first-order condition is:

\[ U'(f(k_0) - k_1) = \beta v'(k_1) \quad (22) \]

So what this expression tell us? The marginal utility of consuming current output must be equal; to the marginal utility of allocating it to capital and enjoying consumption next period.

4.2 Using the envelope theorem

In the last condition expressed above we have a nasty \( v' \) it would be easy to get rid of this term and express it in other model terms. Here we use the envelope theorem. It argues that given an optimal choice of the other variables, an optimal choice of our instrument yields the optimal solution. Let us start from the policy rule we have in mind:

\[ k_1 = g(k_0) \quad (23) \]

Substitute this into the value function:

\[ v(k_0) = U(f(k_0) - g(k_0)) + \beta v(g(k_0)) \quad (24) \]

So now we expressed the value function totally in terms of the state variable \( k_0 \). Optimality with respect to \( k_0 \) will yield optimality for the whole problem. So differentiate the value function with respect to \( k_0 \):

\[ v'(k_0) = U'(f(k_0) - g(k_0)) f'(k_0) + \beta v'(g(k_0)) g'(k_0) \quad (25) \]

But above we showed that for the optimal solution we have \( U'(f(k_0) - k_1) = \beta v'(k_1) \), so:

\[ v'(k_0) = U'(f(k_0) - k_1) f'(k_0) \quad (26) \]

which we can also use for \( k_1 \): \( v'(k_1) = U'(f(k_1) - k_2) f'(k_1) \). So we get finally:

\[ U'[f(k_0) - k_1] = \beta U'[f(k_1) - k_2) f'(k_2) \quad (27) \]
4.3 Example

Suppose we have our example again: The Cobb-Douglas production function: \( y_t = f(k_t) = k_t^\alpha \) and the logarithmic utility function: \( u(c_t) = \log(c_t) \). So we get the Bellman equation:

\[
v(k_t) = \max \left( \log(c_t) + \beta v(k_{t+1}) \right) + \lambda_t (k_t^\alpha - c_t - k_{t+1})
\]  

(28)

Differentiate with respect to \( c_t \):

\[
\frac{1}{c_t} = \lambda_t
\]  

(29)

And with respect to \( k_{t+1} \):

\[
\beta v'(k_{t+1}) = \lambda_t
\]  

(30)

Let’s work out the envelope condition here. So we write the solution to all variables as functions of the state variables: \( c_t = c(k_t) \) and \( k_{t+1} = k(k_t) \). So the Bellman equation turns into:

\[
v(k_t) = \max \log(c(k_t)) + \beta v(k(k_t)) + \lambda_t (k_t^\alpha - c(k_t) - k(k_t))
\]  

(31)

Differentiate with respect to \( k_t \):

\[
v'(k_t) = \frac{c_t'}{c_t} + \beta v'(k_{t+1}) k_t^\alpha + \lambda_t [k_t^\alpha - c(k_t) + k_{t+1}] + \lambda_t [\alpha k_t^{\alpha-1} - c_t' - k_{t+1}]
\]  

(32)

The second term vanishes because of the constraint. So we regroup into:

\[
v'(k_t) = c_t' \left( \frac{1}{c_t} - \lambda_t \right) + k_{t+1}' (\beta v'(k_{t+1}) - \lambda_t) + \lambda_t \alpha k_t^{\alpha-1}
\]  

(33)

Because of the first-order conditions we get rid of the first two terms, so we have:

\[
v'(k_t) = \frac{1}{c_t} \alpha k_t^{\alpha-1}
\]  

(34)

So we get the first-order condition:

\[
\frac{1}{c_t} = \alpha \beta \frac{1}{c_{t+1}} k_{t+1}^{\alpha-1}
\]  

(35)

This is a difference equation, which we can solve using various methods. We will give an overview of these methods in the next section.

5 Solving the first-order conditions

We have two basic methods to solve difference equations like in the example above:

- Solution by conjecture;
- Solution by iteration.
5.1 Solving by conjecture

Suppose in the example on the optimal growth model with the log consumption function and the Cobb-Douglas production function we conjecture that households save a fixed fraction $\theta$ of income. This implies:

$$k_{t+1} = \theta k_t^\alpha$$  \hspace{1cm} (36)

Or equivalently $c_t = (1 - \theta)k_t^\alpha$. Substitute in the first-order condition:

$$\alpha \beta \frac{k_{t+1}^\alpha}{c_{t+1}} = \frac{\theta}{1 - \theta}$$  \hspace{1cm} (37)

Using the consumption function for $c_{t+1}$ it is easy to see that:

$$\alpha \beta \frac{k_{t+1}^\alpha}{(1 - \theta)k_{t+1}^\alpha} = \frac{\theta}{1 - \theta}$$  \hspace{1cm} (38)

or $\alpha \beta = \theta$. So we get the solution:

$$c_t = (1 - \alpha \beta)k_t^\alpha$$  \hspace{1cm} (39)

So this shows the ease of solving by conjecture in simple models.

5.2 Solving by iterating the value function

This principle is based on the notion that the value function of the problem will differ each period, but it will gradually converge the further away we get from the terminal period. The method is tedious, but must work in all cases. One needs to know a little about limit values (and don’t be afraid to optimize repeatedly).

Suppose we have a finite problem and we know that we die in period $T + 1$, so $v(k_{T+1}) = 0$. This implies that savings at time $T$ will be zero $k_T = 0$ and all income is consumed $c_T = k_T^\alpha$. So counting back we have that:

$$v_1(k_T) = \log k_T^\alpha$$  \hspace{1cm} (40)

So in the period before we have:

$$v_2(k_{T-1}) = \max(\log(c_{T-1}) + \beta \log(k_{T-1}^\alpha))$$  \hspace{1cm} (41)

subject to $c_{T-1} = k_{T-1}^\alpha - k_T$. This is a problem we can solve easily using a simple Lagrange step. We get the first-order conditions:

$$\lambda = \frac{1}{c_{T-1}}$$  \hspace{1cm} (42)

$$\lambda = \alpha \beta \frac{1}{k_T}$$  \hspace{1cm} (43)
With the budget constraint we get:

\[
k_T = \frac{\alpha \beta}{1 + \alpha \beta} k_{T-1}^\alpha \tag{44}
\]

\[
ce_{T-1} = \frac{1}{1 + \alpha \beta} k_{T-1}^{\alpha T - 1} \tag{45}
\]

We can plug these values into the expression for \(v_2(k_{T-1})\):

\[
v_2(k_{T-1}) = \alpha \beta \log(\alpha \beta) - (1 + \alpha \beta) \log(1 + \alpha \beta) + (1 + \alpha \beta) \log(k_{T-1}^{\alpha}) \tag{46}
\]

and we can proceed with \(v_3(k_{T-2})\). It can be shown that the sequence of value functions converges to:

\[
v(k_t) = \max \left( \log c_t + \beta \left( (1 - \beta)^{-1} \left( \log(1 - \alpha \beta) + \frac{\alpha \beta}{1 - \alpha \beta} \log \alpha \beta \right) + \frac{\alpha}{1 - \alpha \beta} \log k_{t+1} \right) \right) \tag{47}
\]

The first-order condition was:

\[
U'(c_t) = \beta v_t(k_{t+1}) \tag{48}
\]

We have:

\[
v_t(k_{t+1}) = \frac{\alpha}{1 - \alpha \beta} \frac{1}{k_{t+1}} \tag{49}
\]

So:

\[
\frac{c_t}{k_{t+1}} = \frac{1 - \alpha \beta}{\alpha \beta} \tag{50}
\]

Rewrite the resource constraint into:

\[
\frac{c_t}{k_{t+1}} = \frac{k_t^\alpha}{k_{t+1}^\alpha} - 1 \tag{51}
\]

So we get:

\[
k_{t+1} = \alpha \beta k_t^\alpha \tag{52}
\]

which we have seen before!

6 Further reading

There are some nice introductory texts on dynamic optimization. One can look at Dixit (1990), the appendix of Kreps (1990), and Simon and Blume (1994). A book on optimal control theory in economics is Chiang (1992).

Macroeconomic applications can be found everywhere. Sargent (1987), Blanchard and Fischer (1989), Heijdra and Van der Ploeg (2002), and Walsh (2003) are some nice examples.