Foundations of Modern Macroeconomics

Ben J. Heijdra
University of Groningen

April 2002

Solutions for problems to Chapter 15

Question 1

Part (a)

In the unit-elastic model we have:

$$\begin{array}{llll} \sigma_X = 1 & \text{so that} & \Phi = \log U & \text{(a)} \\ \sigma_{CL} = 1 & \text{so that} & U = C^{\epsilon_C}(1-L)^{1-\epsilon_C} & \text{(b)} \\ \sigma_{KL} = 1 & \text{so that} & y = L^{\epsilon_L}K^{1-\epsilon_L} & \text{(b)} \end{array} \tag{A1}$$

There are thus only two types of results to prove, namely results of type (a) and of type (b).

To demonstrate Result (a), we can use footnote 14 in Chapter 14. Given the definition of Φ in equation (1), it follows that both the numerator and the denominator go to zero as $1/\sigma_X \to 1$. We must therefore use L'Hôpital's rule for evaluating limits of the $0 \div 0$ type. We find:

$$\lim_{1/\sigma_X \to 1} \Phi = \lim_{1/\sigma_X \to 1} \frac{-1 \times U^{1-1/\sigma_X} \log U}{-1} = \log U. \tag{A2}$$

To demonstrate Result (b) we first take the logarithm of equation (2):

$$\log U = \frac{\log\left[\epsilon_C C^x + (1-\epsilon_C)[1-L]^x\right]}{x}, \tag{A3}$$

where $x \equiv (\sigma_{CL} - 1)/\sigma_{CL}$. Both the numerator and the denominator in (A3) go to zero as $x \to 0$ so we must again use L'Hôpital's rule:

$$\begin{aligned} \lim_{x\to 0} \log U &= \lim_{x\to 0} \frac{\epsilon_C C^x \log C + (1-\epsilon_C)[1-L]^x \log(1-L)}{\epsilon_C C^x + (1-\epsilon_C)[1-L]^x} \\ &= \epsilon_C \log C + (1-\epsilon_C)\log(1-L), \end{aligned} \tag{A4}$$

where we have used the fact that $\lim_{x\to 0} C^x = \lim_{x\to 0}(1-L)^x = 1$ to get from the first to the second line of (A4). It follows from (A4) that $U = C^{\epsilon_C}(1-L)^{1-\epsilon_C}$.

Part (b)

According to the hint, given in equation (6), we can write $U(\tau) = X(\tau)/P_U(\tau)$. By using this result, the optimization problem for the household in *stage 1* is to choose paths for $X(\tau)$ and $A(\tau)$ such that:

$$\Lambda(t) = \int_t^\infty \left[\frac{(X(\tau)/P_U(\tau))^{1-1/\sigma_X} - 1}{1 - 1/\sigma_X} \right] e^{\rho(t-\tau)} d\tau, \tag{A5}$$

is maximized subject to the household budget identity:

$$\dot{A}(\tau) = r(\tau)A(\tau) + W(\tau) - T(\tau) - X(\tau), \tag{A6}$$

and the NPG condition (15.5). The household takes as given its initial level of financial assets, $A(t)$. Note that (A6) is obtained from (15.3) in the book by using the definition of full consumption given in equation (6).

The current-value Hamiltonian for this optimization problem is:

$$\mathcal{H} \equiv \frac{(X(\tau)/P_U(\tau))^{1-1/\sigma_X} - 1}{1 - 1/\sigma_X} + \mu(\tau) \Big[r(\tau)A(\tau) + W(\tau) - T(\tau) - X(\tau) \Big],$$

where $\mu(\tau)$ is the co-state variable, $A(\tau)$ is the state variable, and $X(\tau)$ is the control variable. The first-order conditions are $\partial H/\partial X = 0$ and $-\partial H/\partial A = \dot{\mu} - \rho\mu$, or:

$$\left(\frac{X(\tau)}{P_U(\tau)} \right)^{-1/\sigma_X} \frac{1}{P_U(\tau)} = \mu(\tau), \tag{A7}$$

$$-r(\tau)\mu(\tau) = \dot{\mu}(\tau) - \rho\mu(\tau). \tag{A8}$$

Combining these first-order conditions yields:

$$\begin{aligned}
\frac{\dot{\mu}(\tau)}{\mu(\tau)} &= -\frac{1}{\sigma_X} \left(\frac{\dot{X}(\tau)}{X(\tau)} - \frac{\dot{P}_U(\tau)}{P_U(\tau)} \right) - \frac{\dot{P}_U(\tau)}{P_U(\tau)} = \rho - r(\tau) \quad \Rightarrow \\
\frac{\dot{X}(\tau)}{X(\tau)} - \frac{\dot{P}_U(\tau)}{P_U(\tau)} + \sigma_X \frac{\dot{P}_U(\tau)}{P_U(\tau)} &= \sigma_X \left[r(\tau) - \rho \right] \quad \Rightarrow \\
\frac{\dot{X}(\tau)}{X(\tau)} &= \sigma_X \left[r(\tau) - \rho \right] + (1 - \sigma_X) \frac{\dot{P}_U(\tau)}{P_U(\tau)}. \qquad \text{(A9)}
\end{aligned}$$

Equation (A9) coincides with the expression in (4). We must now verify that (5) is the correct expression for the true cost-of-living index. We do so by solving *stage 2* of the optimization procedure.

In stage 2, the household maximizes subfelicity, U, subject to the constraint $X = C + W(1-L)$, with X given. The Lagrangian expression is:

$$\begin{aligned}
\mathcal{L} \equiv & \left[\epsilon_C C(\tau)^{(\sigma_{CL}-1)/\sigma_{CL}} + (1-\epsilon_C)[1 - L(\tau)]^{(\sigma_{CL}-1)/\sigma_{CL}} \right]^{\sigma_{CL}/(\sigma_{CL}-1)} \\
& + \lambda \left[X - C - W(1-L) \right], \qquad \text{(A10)}
\end{aligned}$$

where λ is the Lagrange multiplier. The first-order conditions are the constraint as well as $\partial\mathcal{L}/\partial C = \partial\mathcal{L}/\partial(1-L) = 0$. It follows from the latter two conditions that:

$$[\cdot]^{\sigma_{CL}/(\sigma_{CL}-1)-1}\,\epsilon_C C^{(\sigma_{CL}-1)/\sigma_{CL}-1} = \lambda, \tag{A11}$$

$$[\cdot]^{\sigma_{CL}/(\sigma_{CL}-1)-1}\,(1-\epsilon_C)(1-L)^{(\sigma_{CL}-1)/\sigma_{CL}-1} = \lambda W, \tag{A12}$$

where $[\cdot]$ is the term in square brackets in equation (2). By dividing (A12) by (A11) we obtain the expression for the marginal rate of substitution between leisure and consumption:

$$\frac{(1-\epsilon_C)(1-L)^{-1/\sigma_{CL}}}{\epsilon_C C^{-1/\sigma_{CL}}} = W. \tag{A13}$$

We can use (A13) and the constraint to express C and $1-L$ in terms of X, W, and the parameters. We show a few steps here:

$$\begin{aligned}
\left(\frac{\epsilon_C W}{1-\epsilon_C}\right)^{\sigma_{CL}}(1-L) + W(1-L) &= X \quad \Rightarrow \\
W(1-L)\left[1+\epsilon_C^{\sigma_{CL}}(1-\epsilon_C)^{-\sigma_{CL}}W^{\sigma_{CL}-1}\right] &= X \quad \Rightarrow \\
(1-\epsilon_C)^{-\sigma_{CL}}W^{\sigma_{CL}-1}W(1-L)\left[\epsilon_C^{\sigma_{CL}}+(1-\epsilon_C)^{\sigma_{CL}}W^{1-\sigma_{CL}}\right] &= X \quad \Rightarrow \\
W(1-L) &= (1-c_X)X,
\end{aligned} \tag{A14}$$

where $1-c_X$ is the full consumption share of spending on leisure. It is defined as:

$$1-c_X = \frac{(1-\epsilon_C)^{\sigma_{CL}}W^{1-\sigma_{CL}}}{\left[\epsilon_C^{\sigma_{CL}}+(1-\epsilon_C)^{\sigma_{CL}}W^{1-\sigma_{CL}}\right]}. \tag{A15}$$

Similarly, we can write $C = c_X X$, where c_X is defined as:

$$c_X = \frac{\epsilon_C^{\sigma_{CL}}}{\left[\epsilon_C^{\sigma_{CL}}+(1-\epsilon_C)^{\sigma_{CL}}W^{1-\sigma_{CL}}\right]}. \tag{A16}$$

We can now relate U and X and derive the expression for P_U. We find:

$$\begin{aligned}
U &= \left[\epsilon_C C^{(\sigma_{CL}-1)/\sigma_{CL}} + (1-\epsilon_C)[1-L]^{(\sigma_{CL}-1)/\sigma_{CL}}\right]^{\sigma_{CL}/(\sigma_{CL}-1)} \\
&= \left[\epsilon_C\,(c_X X)^{(\sigma_{CL}-1)/\sigma_{CL}} + (1-\epsilon_C)\left(\frac{(1-c_X)X}{W}\right)^{(\sigma_{CL}-1)/\sigma_{CL}}\right]^{\sigma_{CL}/(\sigma_{CL}-1)} \\
&= X\left[\epsilon_C c_X^{(\sigma_{CL}-1)/\sigma_{CL}} + (1-\epsilon_C)\left(\frac{1-c_X}{W}\right)^{(\sigma_{CL}-1)/\sigma_{CL}}\right]^{\sigma_{CL}/(\sigma_{CL}-1)}.
\end{aligned} \tag{A17}$$

By using the expressions for c_X and $1-c_X$, the complicated term in square brackets on the right-hand side of (A17) can be simplified:

$$
\begin{aligned}
[\cdot] &= \epsilon_C \left[\frac{\epsilon_C^{\sigma_{CL}}}{\left[\epsilon_C^{\sigma_{CL}} + (1-\epsilon_C)^{\sigma_{CL}} W^{1-\sigma_{CL}}\right]} \right]^{(\sigma_{CL}-1)/\sigma_{CL}} \\
&\quad + (1-\epsilon_C) \left[\frac{(1-\epsilon_C)^{\sigma_{CL}} W^{1-\sigma_{CL}}}{\left[\epsilon_C^{\sigma_{CL}} + (1-\epsilon_C)^{\sigma_{CL}} W^{1-\sigma_{CL}}\right]} \frac{1}{W} \right]^{(\sigma_{CL}-1)/\sigma_{CL}} \\
&= \frac{\epsilon_C^{\sigma_{CL}} + (1-\epsilon_C)^{\sigma_{CL}} W^{1-\sigma_{CL}}}{\left[\epsilon_C^{\sigma_{CL}} + (1-\epsilon_C)^{\sigma_{CL}} W^{1-\sigma_{CL}}\right]^{(\sigma_{CL}-1)/\sigma_{CL}}} \\
&= \left[\epsilon_C^{\sigma_{CL}} + (1-\epsilon_C)^{\sigma_{CL}} W^{1-\sigma_{CL}}\right]^{1/\sigma_{CL}} . \qquad \text{(A18)}
\end{aligned}
$$

By using (A18) in (A17) we obtain:

$$
U = \frac{X}{\left[\epsilon_C^{\sigma_{CL}} + (1-\epsilon_C)^{\sigma_{CL}} W^{1-\sigma_{CL}}\right]^{1-\sigma_{CL}}} = \frac{X}{P_U}, \qquad \text{(A19)}
$$

where P_U is thus:

$$
P_U \equiv \left[\epsilon_C^{\sigma_{CL}} + (1-\epsilon_C)^{\sigma_{CL}} W^{1-\sigma_{CL}}\right]^{1-\sigma_{CL}} . \qquad \text{(A20)}
$$

This expression coincides with the second line in (5). To obtain the first line in (5) we can let $\sigma_{CL} \to 1$ in (A20). This limit can again be determined by using L'Hôpital's rule.

Part (c)

Firm behaviour is still characterized by the usual rental expressions stated in (15.15). By using equation (3), we find that the marginal product of labour can be written as follows:

$$
\begin{aligned}
\frac{\partial F}{\partial L} &= \left[\epsilon_L L^{(\sigma_{KL}-1)/\sigma_{KL}} + (1-\epsilon_L) K^{(\sigma_{KL}-1)/\sigma_{KL}}\right]^{\sigma_{KL}/(\sigma_{KL}-1)-1} \epsilon_L L^{-1/\sigma_{KL}} \\
&= \left[Y^{(\sigma_{KL}-1)/\sigma_{KL}}\right]^{1/(\sigma_{KL}-1)} \epsilon_L L^{-1/\sigma_{KL}} \\
&= \epsilon_L \left(\frac{Y}{L}\right)^{1/\sigma_{KL}} . \qquad \text{(A21)}
\end{aligned}
$$

Similarly, we can write the marginal product of capital as:

$$
\begin{aligned}
\frac{\partial F}{\partial K} &= \left[\epsilon_L L^{(\sigma_{KL}-1)/\sigma_{KL}} + (1-\epsilon_L) K^{(\sigma_{KL}-1)/\sigma_{KL}}\right]^{\sigma_{KL}/(\sigma_{KL}-1)-1} (1-\epsilon_L) K^{-1/\sigma_{KL}} \\
&= \left[Y^{(\sigma_{KL}-1)/\sigma_{KL}}\right]^{1/(\sigma_{KL}-1)} (1-\epsilon_L) K^{-1/\sigma_{KL}} \\
&= (1-\epsilon_L) \left(\frac{Y}{K}\right)^{1/\sigma_{KL}} . \qquad \text{(A22)}
\end{aligned}
$$

Intuitively, σ_{KL} measures how easy it is to substitute the production factors for each other. By using (A21)-(A22) in (15.15) we find the demand functions for capital and labour:

$$
\frac{K}{Y} = \left(\frac{r+\delta}{1-\epsilon_L}\right)^{-\sigma_{KL}}, \qquad \frac{L}{Y} = \left(\frac{W}{1-\epsilon_L}\right)^{-\sigma_{KL}} . \qquad \text{(A23)}
$$

If σ_{KL} is very high then the demand functions are very sensitive to changes in factor prices, i.e. they are very flat. This is because substitution is very easy in that case. Conversely, if σ_{KL} is close to zero, then the demand functions are rather insensitive to changes in factors prices, i.e. they are very steep. Intuitively, this is because substitution is very difficult in that case and the production function features nearly constant input coefficients, i.e. it is close to a Leontief production function (and the isoquants are close to L-shaped).

Part (d)

The great ratios are determined as follows. In the steady state we have $\dot{X} = 0$, $\dot{K} = 0$ (and $\dot{P}_U = 0$) so that it follows from (4) that $r = \rho$ and from (15.14) that $I = \delta K$. Since $r + \delta = (1 - \epsilon_L)(Y/K)^{1/\sigma_{KL}}$ it follows that Y/K is constant. Hence, I/Y and (by the CRTS production function) K/L are also constant. Since F_L depends on the K/L ratio, it and the real wage are both constants. By (A13) we find that $C/(1-L)$ is also constant.

Part (e)

The long-run multiplier can be computed by noting that the supply side of the model fixes the great ratios. Equations (15.18)-(15.19) are all still valid so the multiplier is still as given in (15.20). The intertemporal substitution elasticity does not affect the great ratios at all because these are fixed by the supply side of the model. It does, however, affect the transition path towards the steady state.

Question 2

Part (a)

The representative household makes the consumption and accumulation decisions. Note that (5) can be rewritten in a more conventional form as:

$$C_\tau + I_\tau^M + I_\tau^H = W_\tau L_\tau + R_\tau^K K_\tau^M - T_\tau, \tag{A1}$$

where $I_\tau^M \equiv K_{\tau+1}^M - (1-\delta_M)K_\tau^M$ and $I_\tau^H \equiv K_{\tau+1}^H - (1-\delta_H)K_\tau^H$ represent *gross* investment in, respectively, business and home capital. Equation (A1) is thus a generalized version of equation (15.61) in the book. Since we are adding up the different types of capital in (5) and in (A1), we implicitly assume that the two types of capital are perfect substitutes as far as the household's investment decision is concerned.

Part (b)

The household chooses sequences $\{C_\tau\}_t^\infty$, $\{C_\tau^M\}_t^\infty$, $\{C_\tau^H\}_t^\infty$, $\{L_\tau\}_t^\infty$, $\{L_\tau^M\}_t^\infty$, $\{L_\tau^H\}_t^\infty$, $\{K_{\tau+1}^H\}_t^\infty$, and $\{K_{\tau+1}^M\}_t^\infty$ in order to maximize (1) subject to the constraints and definitions (2)-(5), taking as given the initial stock of total capital, $K_t^M + K_t^H$. It is thus assumed that household capital can costlessly and instantaneously be turned into business capital and vice versa. The Lagrangian expression for this problem is:

$$\begin{aligned}
\mathcal{L}_t^H \equiv\; & E_t \sum_{\tau=t}^{\infty} \left(\frac{1}{1+\rho}\right)^{\tau-t} \Big[\epsilon_C \log C_\tau + (1-\epsilon_C)\log[1-L_\tau^M - L_\tau^H] \\
& -\lambda_\tau \left(K_{\tau+1}^M + K_{\tau+1}^H + C_\tau^M - W_\tau L_\tau^M - (R_\tau^K + 1 - \delta_M)K_\tau^M - (1-\delta_H)\,K_\tau^H\right) \\
& -\mu_\tau \left(C_\tau - \left[\epsilon_H \left(C_\tau^H\right)^{(\sigma-1)/\sigma} + (1-\epsilon_H)\left(C_\tau^M\right)^{(\sigma-1)/\sigma}\right]^{\sigma/(\sigma-1)}\right) \\
& -\nu_t \left(C_\tau^H - Z_\tau^H \left(L_\tau^H\right)^{\eta_L} \left(K_\tau^H\right)^{1-\eta_L}\right)\Big],
\end{aligned} \tag{A2}$$

where we have substituted the time constraint (3) into the felicity function. The Lagrange multipliers are denoted by μ_τ, ν_τ, and λ_τ and the (interesting) first-order conditions are:

$$\frac{\partial \mathcal{L}_t^H}{\partial C_\tau} = \left(\frac{1}{1+\rho}\right)^{\tau-t} E_t\left[\frac{\epsilon_C}{C_\tau} - \mu_\tau\right] = 0, \tag{A3}$$

$$\frac{\partial \mathcal{L}_t^H}{\partial L_\tau^M} = -\left(\frac{1}{1+\rho}\right)^{\tau-t} E_t\left[\frac{1-\epsilon_C}{1-L_\tau} - \lambda_\tau W_\tau\right] = 0, \tag{A4}$$

$$\frac{\partial \mathcal{L}_t^H}{\partial L_\tau^H} = -\left(\frac{1}{1+\rho}\right)^{\tau-t} E_t\left[\frac{1-\epsilon_C}{1-L_\tau} - \nu_\tau \hat{W}_\tau\right] = 0, \tag{A5}$$

$$\frac{\partial \mathcal{L}_t^H}{\partial C_\tau^M} = \left(\frac{1}{1+\rho}\right)^{\tau-t} E_t\left[\mu_\tau(1-\epsilon_H)\left(\frac{C_\tau}{C_\tau^M}\right)^{1/\sigma} - \lambda_\tau\right] = 0, \tag{A6}$$

$$\frac{\partial \mathcal{L}_t^H}{\partial C_\tau^H} = \left(\frac{1}{1+\rho}\right)^{\tau-t} E_t\left[\mu_\tau \epsilon_H \left(\frac{C_\tau}{C_\tau^H}\right)^{1/\sigma} - \nu_\tau\right] = 0, \tag{A7}$$

$$\frac{\partial \mathcal{L}_t^H}{\partial K_{\tau+1}^M} = \left(\frac{1}{1+\rho}\right)^{\tau-t} E_t\left[-\lambda_\tau + \lambda_{\tau+1}\left(\frac{R_{\tau+1}^K + 1 - \delta_M}{1+\rho}\right)\right] = 0, \tag{A8}$$

$$\frac{\partial \mathcal{L}_t^H}{\partial K_{\tau+1}^H} = \left(\frac{1}{1+\rho}\right)^{\tau-t} E_t\left[-\lambda_\tau + \lambda_{\tau+1}\left(\frac{(\nu_{\tau+1}/\lambda_{\tau+1})\,\hat{R}_{\tau+1}^K + 1 - \delta_H}{1+\rho}\right)\right] = 0, \tag{A9}$$

where $\hat{W}_\tau$ is the *imputed* home wage, i.e. the marginal product of working at home (producing home goods):

$$\hat{W}_\tau \equiv \eta_L Z_\tau^H \left(L_\tau^H\right)^{\eta_L - 1} \left(K_\tau^H\right)^{1-\eta_L} = \eta_L \left(\frac{C_\tau^H}{L_\tau^H}\right), \tag{A10}$$

and $\hat{R}_\tau^K$ is the imputed rental charge on home capital, which is defined as:

$$\hat{R}_\tau^K \equiv (1-\eta_L) Z_\tau^H \left(L_\tau^H\right)^{\eta_L} \left(K_\tau^H\right)^{-\eta_L} = (1-\eta_L)\left(\frac{C_\tau^H}{K_\tau^H}\right). \tag{A11}$$

For the planning period, $\tau = t$, we can simplify these first-order conditions into a number of static conditions and a dynamic condition. By using (A4)-(A5) we find:

$$\pi_t \equiv \frac{\nu_t}{\lambda_t} = \frac{W_t}{\hat{W}_t}, \tag{A12}$$

where π_t is the relative price of home goods. According to (A12), labour must be allocated across the two activities such that it yields the same real (imputed) wage. By substituting (A3)-(A4) into (A6) we find:

$$\begin{aligned}
\lambda_t &= \frac{\epsilon_C(1-\epsilon_H)}{C_t}\left(\frac{C_t}{C_t^M}\right)^{1/\sigma} = \frac{1-\epsilon_C}{1-L_t}\frac{1}{W_t} \qquad \Rightarrow \\
W_t &= \frac{(1-\epsilon_C)/(1-L_t)}{[\epsilon_C/C_t]\,(1-\epsilon_H)\left(C_t/C_t^M\right)^{1/\sigma}},
\end{aligned} \tag{A13}$$

i.e. the marginal rate of substitution between leisure and consumption of the home good must be equated to the wage rate. Similarly, by using (A3) and (A5) in (A7) we find:

$$\begin{aligned}
\nu_t &\equiv \lambda_t\pi_t = \frac{\epsilon_C\epsilon_H}{C_t}\left(\frac{C_t}{C_t^H}\right)^{1/\sigma} = \frac{1-\epsilon_C}{1-L_t}\frac{1}{\hat{W}_t} \qquad \Rightarrow \\
\hat{W}_t &\equiv \frac{W_t}{\pi_t} = \frac{(1-\epsilon_C)/(1-L_t)}{[\epsilon_C/C_t]\,\epsilon_H\left(C_t/C_t^H\right)^{1/\sigma}}.
\end{aligned} \tag{A14}$$

Again, the optimality condition calls for an equalization of the marginal rate of substitution between leisure and home consumption with the relevant wage rate, $\hat{W}_t$. Equations (A13)-(A14) determine the optimal division between home consumption and consumption of market goods as a function of the relative price, π_t.

Next we use (A6)-(A7) to deduce a relationship between μ_t and λ_t. Equations (A6)-(A7) imply:

$$C_t^M = \left(\frac{\mu_t(1-\epsilon_H)}{\lambda_t}\right)^{\sigma} C_t, \qquad C_t^H = \left(\frac{\mu_t\epsilon_H}{\lambda_t\pi_t}\right)^{\sigma} C_t, \tag{A15}$$

where π_t is defined in (A12) above. By substituting these expressions into equation (2) we find:

$$\begin{aligned}
C_t^{(\sigma-1)/\sigma} &= \epsilon_H\left[\left(\frac{\mu_t\epsilon_H}{\lambda_t\pi_t}\right)^{\sigma} C_t\right]^{(\sigma-1)/\sigma} + (1-\epsilon_H)\left[\left(\frac{\mu_t(1-\epsilon_H)}{\lambda_t}\right)^{\sigma} C_t\right]^{(\sigma-1)/\sigma} \qquad \Leftrightarrow \\
1 &= \epsilon_H\left(\frac{\mu_t\epsilon_H}{\lambda_t\pi_t}\right)^{\sigma-1} + (1-\epsilon_H)\left(\frac{\mu_t(1-\epsilon_H)}{\lambda_t}\right)^{(\sigma-1)} \qquad \Leftrightarrow \\
\frac{\mu_t}{\lambda_t} &= \left[\epsilon_H^{\sigma}\pi_t^{1-\sigma} + (1-\epsilon_H)^{\sigma}\right]^{1/(1-\sigma)} \equiv P(\pi_t),
\end{aligned} \tag{A16}$$

where $P(\pi_t)$ is the true price index of composite consumption. Since (A6)-(A7) are essentially static decisions, (A16) holds not only for period t but also for all other periods, i.e. $\mu_\tau/\lambda_\tau = P(\pi_\tau)$. Note that (A3)-(A4) in combination with (A16) imply:

$$\frac{(1-\epsilon_C)/(1-L_\tau)}{\epsilon_C/C_\tau} = \frac{W_\tau}{P(\pi_\tau)}. \tag{A17}$$

The marginal rate of substitution between leisure and composite consumption is equated to the real wage rate, using the true price index for composite consumption as the deflator. Equation (A17) in the counterpart to (15.66) in the book.

Equations (A8)-(A9) state that the two capital stocks should yield the same *ex ante* rate of return. For period t we find from (A8)-(A9):

$$\lambda_t = E_t\left[\lambda_{t+1}\left(\frac{1+r_{t+1}}{1+\rho}\right)\right] = E_t\left[\lambda_{t+1}\left(\frac{1+\hat{r}_{t+1}}{1+\rho}\right)\right], \tag{A18}$$

where $r_{t+1} \equiv R^K_{t+1} - \delta_M$ and $\hat{r}_{t+1} \equiv \pi_{t+1}\hat{R}^K_{t+1} - \delta_M$. By using (A3) and (A16) we find that (A18) can be rewritten as:

$$\frac{\epsilon_C}{C_t} = E_t\left[\frac{1+r_{t+1}}{1+\rho}\frac{\epsilon_C}{C_{t+1}}\frac{P(\pi_t)}{P(\pi_{t+1})}\right] \tag{A19}$$

$$= E_t\left[\frac{1+\hat{r}_{t+1}}{1+\rho}\frac{\epsilon_C}{C_{t+1}}\frac{P(\pi_t)}{P(\pi_{t+1})}\right]. \tag{A20}$$

Equation (A19) is the counterpart to (15.67) in the book. Of course, since capital is perfectly mobile across activities, *ex post* rates of return will also equalize, i.e. $r_\tau = \hat{r}_\tau$ for all τ.

Part (c)

The representative firm makes a static decision regarding output and input demands. In period τ, the realization of the technology shock, Z_τ is known and the firm maximizes Π_τ subject to the technology (6). The first-order conditions are:

$$\frac{\partial \Pi_\tau}{\partial L^M_\tau} = 0: \quad \epsilon_L\left(\frac{Y_\tau}{L^M_\tau}\right) = W_\tau, \tag{A21}$$

$$\frac{\partial \Pi_\tau}{\partial K^M_\tau} = 0: \quad (1-\epsilon_L)\left(\frac{Y_\tau}{K^M_\tau}\right) = R^K_\tau. \tag{A22}$$

Because the technology features CRTS excess profit is zero ($\Pi_\tau = 0$ for all τ).

Part (d)

Implicit in the formulation of the felicity function is the notion that the household derives disutility from its *total* work effort, $L^M_\tau + L^H_\tau$, regardless of where the work takes place. It follows that L^M_τ and L^H_τ are perfect substitutes to the household. Following Benhabib et al. (1991, p. 1171) we can change this aspect of the model by adopting the following felicity function:

$$U_\tau \equiv \epsilon_C \log C_\tau + (1-\epsilon_C)\log[1 - L^M_\tau - L^H_\tau] + \gamma_L L^M_\tau, \tag{A23}$$

with $0 < \epsilon_C < 1$ and $0 \leq \gamma_L < 1 - \epsilon_L$. With this formulation, the marginal disutility of working at home or in the market are:

$$-\frac{\partial U_\tau}{\partial L^M_\tau} = \frac{1-\epsilon_C}{1-L_\tau} - \gamma_L = -\frac{\partial U_\tau}{\partial L^H_\tau} - \gamma_L.$$

Provided γ_L is strictly positive, working in the market sector is preferred to working around the house. (Note that the sign restriction on γ_L ensures that $-\partial U_\tau/\partial L^M_\tau$ remains positive.)

References

Benhabib, J., Rogerson, R., and Wright, R. (1991). Homework in macroeconomics: Household production and aggregate fluctuations. *Journal of Political Economy*, 99:1166–1187.