Foundations of Modern Macroeconomics

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Solutions for problems to Chapter 7

Question 1

Part (a)

The average tax rate, t_A, is defined as:

$$t_A \equiv \frac{T}{WN} = \frac{t_M WN - \theta_0 P}{WN} = t_M - \frac{\theta_0 P}{WN}. \tag{A1}$$

The tax system in progressive if the average tax rate increases with the tax base (labour income in this case). By differentiating (A1) with respect to WN we obtain:

$$\frac{dt_A}{d(WN)} = \frac{\theta_0 P}{(WN)^2} > 0, \tag{A2}$$

where the sign follows from the fact that we assume $\theta_0 > 0$. Hence, the tax system is progressive.

The marginal tax rate, t_M, is defined as:

$$T'(WM) = t_M > 0. \tag{A3}$$

By assumption the marginal tax is constant (i.e. does not depend on income). The tax curve has been illustrated in Figure 1. The marginal tax rate is constant but the average tax rate increases with WN. In points A and B, t_M is the same but t_A is higher in the latter point ($t_A^B > t_A^A$).

Part (b)

The household chooses consumption, C, and labour supply, $1 - N$, in order to maximize the utility function (1) subject to the budget constraint (2). The Lagrangian expression is:

$$\mathcal{L} \equiv U(C, 1-N) + \lambda \left[(1-t_A)WN - PC\right], \tag{A4}$$

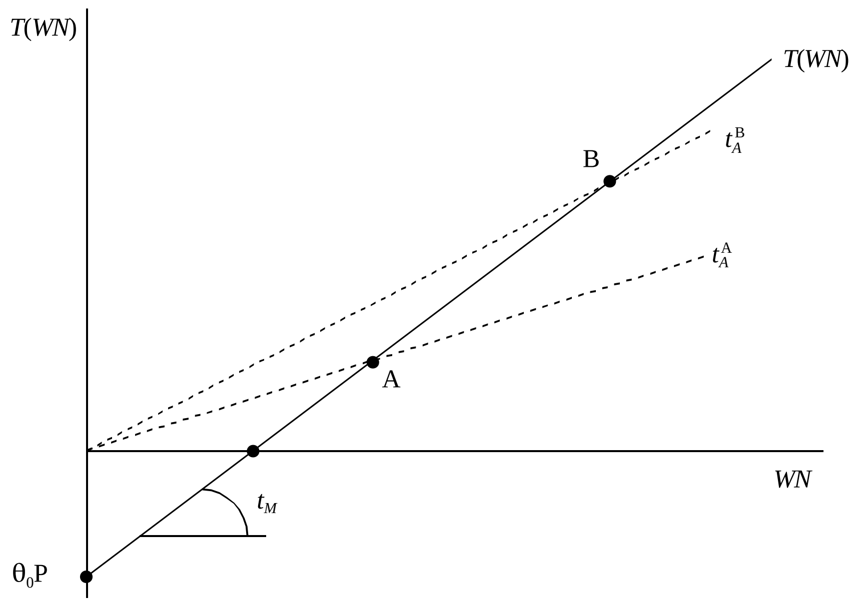

Figure 1: A progressive tax schedule

where we have used the fact that after-tax income $WN - T(WN)$ can be rewritten as $(1 - t_A)WN$ by making use of the definition for t_A. The first-order necessary conditions are the constraint and:

$$\frac{\partial \mathcal{L}}{\partial C} = 0: \qquad U_C - \lambda P = 0, \tag{A5}$$

$$\frac{\partial \mathcal{L}}{\partial N} = 0: \qquad -U_{1-N} + \lambda W \left[(1-t_A)\frac{dN}{dN} + N\frac{d(1-t_A)}{dN}\right] = 0. \tag{A6}$$

The second term in square brackets in (A6) incorporates the effect on the average tax rate due to a change in labour supply (and thus in wage income). It is not difficult to show that:

$$\begin{aligned} \frac{dt_A}{dN} &\equiv \frac{d\left(\frac{T(WN)}{WN}\right)}{dN} = \frac{WNT'(WN)W - T(WN)W}{(WN)^2} \\ &= \frac{W^2Nt_M}{(WN)^2} - \frac{WT(WN)}{(WN)^2} \\ &= \frac{1}{N}\left[t_M - t_A\right]. \end{aligned} \tag{A7}$$

By using (A7) in (A6) we obtain:

$$\begin{aligned} U_{1-N} &= \lambda W\left[(1-t_A) - (t_M - t_A)\right] \\ &= \lambda W(1-t_M). \end{aligned} \tag{A8}$$

Hence, the *marginal* tax rate features in the first-order condition for labour supply. Intuitively, this is because the household is making a marginal decision concerning consumption and labour supply. It will take into account that the average tax rates increases as more labour is supplied.

By combining (A5) and (A8) we obtain the expression for the marginal rate of substitution between leisure and consumption:

$$\lambda = \frac{U_C}{P} = \frac{U_{1-N}}{W(1-t_M)} \quad \Rightarrow \quad \frac{U_{1-N}}{U_C} = \frac{W(1-t_M)}{P}. \tag{A9}$$

Part (c)

The three cases correspond to, respectively, no substitution at all ($\sigma_{CM} = 0$), relatively easy substitution (the Cobb-Douglas case, with $\sigma_{CM} = 1$), and perfect substitution ($\sigma_{CM} \to \infty$). In Figures 2-4, the three cases have been illustrated. Consider first Figure 2. Along a given indifference curve, we obtain by differentiation:

$$dU = U_C dC + U_{1-N} d(1-N) = 0 \Rightarrow \quad \frac{dC}{d(1-N)} = -\frac{U_{1-N}}{U_C}. \tag{A10}$$

For points on the indifference curve U_0 that lie above point A, we have $U_C = 0$ (additional consumption gives no extra utility), i.e. $dC/(d(1-N) = -\infty$ there. For points on the indifference curve U_0 that lie to the right of point A we have $U_{1-N} = 0$ (additional leisure gives no extra utility), i.e. $dC/d(1-N) = 0$ there. The household will always choose to be in the kink. This means that, no matter what happens to the marginal rate of substitution between leisure and consumption, the ratio between C and $1-N$ is constant. This means that the numerator of (5) (and thus σ_{CM} itself) is always zero.

Next, we consider Figure 3, which assumes that the utility function is Cobb-Douglas:

$$U = C^{\alpha}(1-N)^{1-\alpha}, \tag{A11}$$

with $0 < \alpha < 1$. The marginal rate of substitution between leisure and consumption for the Cobb-Douglas utility function is:

$$\frac{U_{1-N}}{U_C} = \frac{(1-\alpha)C^{\alpha}(1-N)^{-\alpha}}{\alpha C^{\alpha-1}(1-N)^{1-\alpha}} = \frac{1-\alpha}{\alpha}\frac{C}{1-N}. \tag{A12}$$

By taking logarithms and totally differentiating we get:

$$d\log\left(\frac{U_{1-N}}{U_C}\right) = d\log\left(\frac{C}{1-N}\right). \tag{A13}$$

By using (A13) in the definition of σ_{CM} (in equation (5)) we find that $\sigma_{CM} = 1$ for the Cobb-Douglas utility function.

Finally, we consider Figure 4, which assume that the utility function is linear:

$$U = \beta_0 + \beta_1 C + \beta_2(1-N), \tag{A14}$$

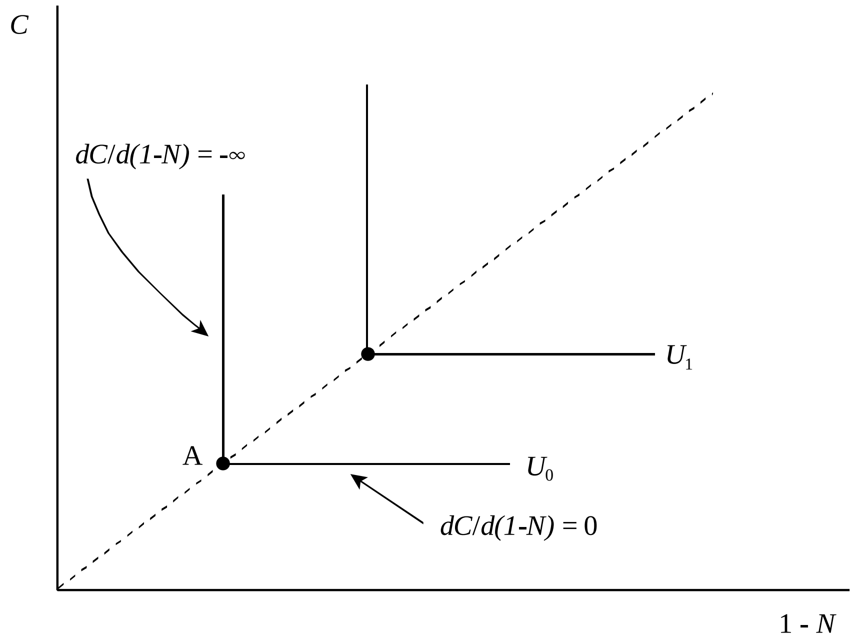

Figure 2: Leontief utility function

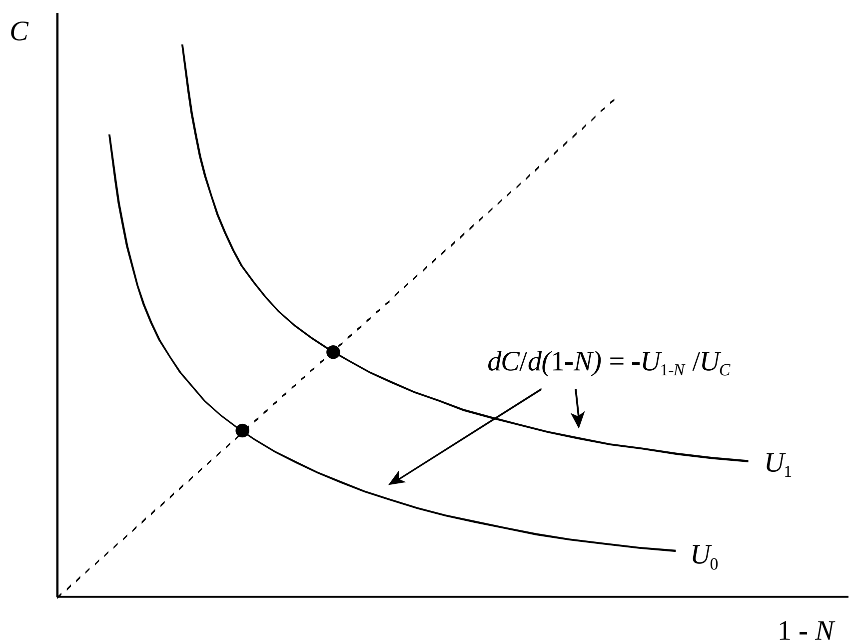

Figure 3: Cobb-Douglas utility function

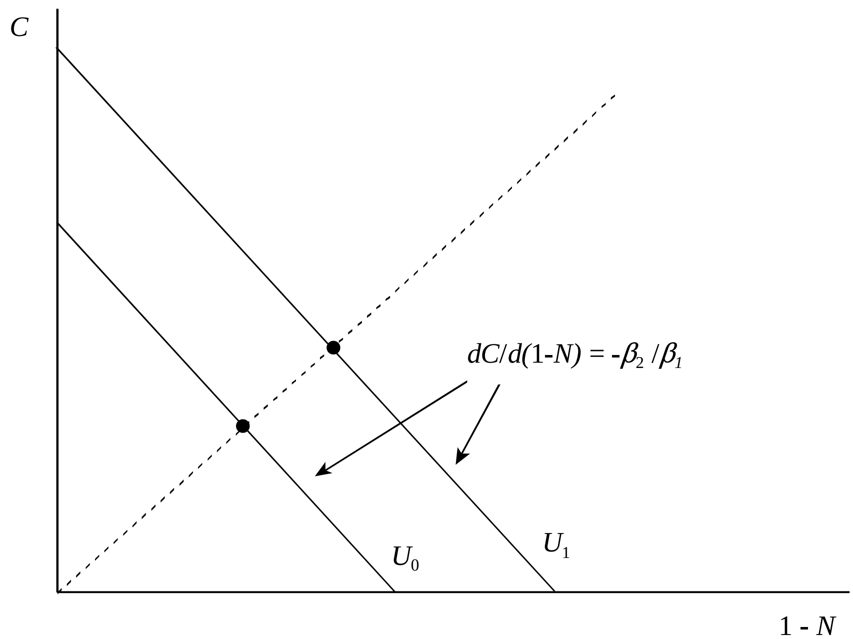

Figure 4: Linear utility function

where β_1 and β_2 are positive constants. For this utility function, U_C, U_{1-N}, and thus the marginal rate of substitution between leisure and consumption are all constant, i.e. $U_{1-N}/U_C = \beta_2/\beta_1$. It follows that $d\log(U_{1-N}/U_C) = 0$ regardless of $C/(1-N)$. Using this result in the definition of σ_{CM} (in equation (5)) shows that $\sigma_{CM} \to \infty$ for the linear utility function.

Part (d)

We start with (A9). By taking logarithms of both sides we obtain:

$$\log\left(\frac{U_{1-N}}{U_C}\right) = \log w + \log(1-t_M), \tag{A15}$$

where $w \equiv W/P$ is the gross real wage. By totally differentiating (A15) we obtain:

$$d\log\left(\frac{U_{1-N}}{U_C}\right) = d\log w + d\log(1-t_M). \tag{A16}$$

But we know from equation (5) that:

$$\sigma_{CM} d\log\left(\frac{U_{1-N}}{U_C}\right) = d\log\left(\frac{C}{1-N}\right) \equiv d\log C - d\log(1-N). \tag{A17}$$

By combining (A16)-(A17) we find:

$$\begin{aligned}
\frac{1}{\sigma_{CM}}\left[d\log C - d\log(1-N)\right] &= d\log w + d\log(1-t_M) &\Leftrightarrow \\
\frac{dC}{C} + \frac{dN}{1-N} &= \sigma_{CM}\left[\frac{dw}{w} - \frac{dt_M}{1-t_M}\right] &\Leftrightarrow \\
\tilde{C} + \left(\frac{N}{1-N}\right)\tilde{N} &= \sigma_{CM}\left[\tilde{w} - \tilde{t}_M\right], &\text{(A18)}
\end{aligned}$$

where $\tilde{C} \equiv dC/C$, $\tilde{N} \equiv dN/N$, $\tilde{w} \equiv dw/w$, and $\tilde{t}_M \equiv dt_M/(1-t_M)$.

Next we turn to the budget restriction, which can be written as $C = w(1-t_A)N$. In logarithms we have $\log C = \log w + \log(1-t_A) + \log N$. By totally differentiating this expression we obtain:

$$\begin{aligned}
d\log C &= d\log w + d\log(1-t_A) + d\log N &\Leftrightarrow \\
\tilde{C} &= \tilde{w} - \tilde{t}_A + \tilde{N}, &\text{(A19)}
\end{aligned}$$

where $\tilde{t}_A \equiv dt_A/(1-t_A)$. By combining (A18) and (A19) in a single matrix expression we obtain:

$$\begin{bmatrix} 1 & \frac{N}{1-N} \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \tilde{C} \\ \tilde{N} \end{bmatrix} = \begin{bmatrix} \sigma_{CM}\tilde{w} - \sigma_{CM}\tilde{t}_M \\ \tilde{w} - \tilde{t}_A \end{bmatrix}. \tag{A20}$$

The matrix on the left-hand side has a non-zero determinant (equal to $-1/(1-N)$) and can thus be inverted:

$$\begin{bmatrix} 1 & \frac{N}{1-N} \\ 1 & -1 \end{bmatrix}^{-1} = -(1-N)\begin{bmatrix} -1 & -\frac{N}{1-N} \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1-N & N \\ 1-N & -(1-N) \end{bmatrix}. \tag{A21}$$

By using (A21) in (A20) we find the solution for $\tilde{C}$ and $\tilde{N}$.

$$\begin{aligned}
\begin{bmatrix} \tilde{C} \\ \tilde{N} \end{bmatrix} &= \begin{bmatrix} 1-N & N \\ 1-N & -(1-N) \end{bmatrix} \begin{bmatrix} \sigma_{CM}\tilde{w} - \sigma_{CM}\tilde{t}_M \\ \tilde{w} - \tilde{t}_A \end{bmatrix} \\
&= \begin{bmatrix} \sigma_{CM}(1-N) + N \\ (\sigma_{CM}-1)(1-N) \end{bmatrix}\tilde{w} - \begin{bmatrix} \sigma_{CM}(1-N) \\ \sigma_{CM}(1-N) \end{bmatrix}\tilde{t}_M + \begin{bmatrix} -N \\ 1-N \end{bmatrix}\tilde{t}_A. \qquad \text{(A22)}
\end{aligned}$$

Part (e)

In the book we define the index of progressivity of the tax system as follows (see equation (7.46)):

$$s \equiv \frac{1-t_M}{1-t_A}. \tag{A23}$$

A decrease in s represents a move towards a more progressive tax system. By taking logarithms of both sides of (A23) and totally differentiating the resulting expression we find:

$$\begin{aligned}
d\log s &= d\log(1-t_M) - d\log(1-t_A) &\Leftrightarrow \\
\tilde{s} &= \tilde{t}_A - \tilde{t}_M, &\text{(A24)}
\end{aligned}$$

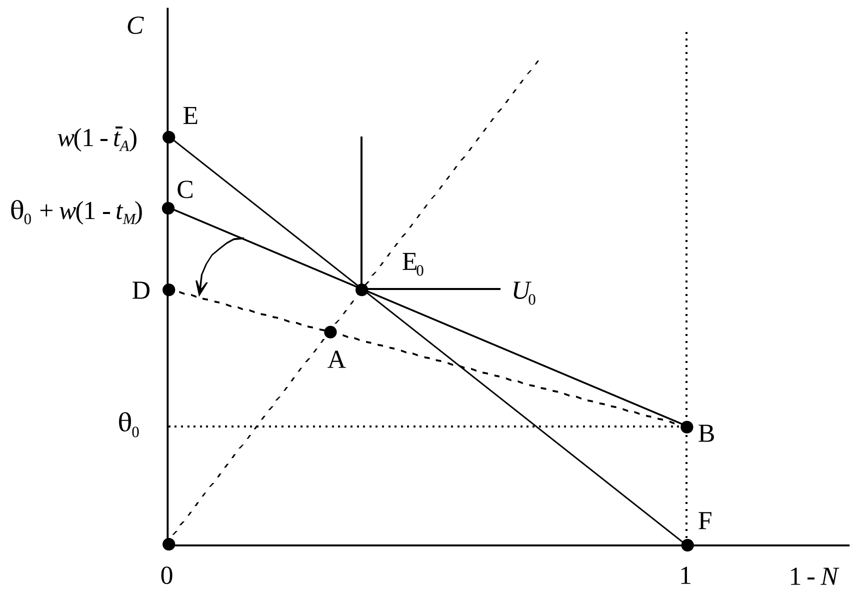

Figure 5: Increasing tax progressivity ($\sigma_{CM} = 0$)

where $\tilde{s} \equiv ds/s$. In the question we keep the average tax rate constant ($\tilde{t}_A = 0$) and increase the progressivity of the tax system by raising the marginal tax rate ($\tilde{s} = -\tilde{t}_M < 0$). It follows from (A22) that consumption and labour supply react according to:

$$\tilde{C} = \tilde{N} = -\sigma_{CM}(1-N)\tilde{t}_M, \tag{A25}$$

where we have used the fact that the wage is unchanged (i.e. $\tilde{w} = 0$ and we are only looking at the labour supply response for a given gross wage rate). We can conclude that labour supply falls unambiguously unless $\sigma_{CM} = 0$ in which case it stays the same. The labour supply effect is illustrated graphically in Figures 5-7.

Consider Figure 5, which is based on the assumption that $\sigma_{CM} = 0$ (no substitution). Since utility depends on (consumption and) leisure, it is helpful to rewrite the budget restriction in terms of leisure also. After some manipulations we find that (2)-(3) can be written as:

$$C = \theta_0 + (1-t_M)w - w(1-t_M)(1-N). \tag{A26}$$

Even if the household does not work at all ($N = 0$) it can still consume the lump-sum hand out from the government ($C = \theta_0$ in that case). The budget restriction is drawn in Figure 5 as the straight line BC. The household is initially at point E_0. If the marginal tax rate is increased, the budget constraint rotates in a counter-clockwise fashion and becomes the dashed line BD. In the absence of any counter-measures, the household would choose point A on the new budget line. In this experiment however, we must ensure that the average tax rate (evaluated at the initial optimum E_0) is not changed. We denote the initial average

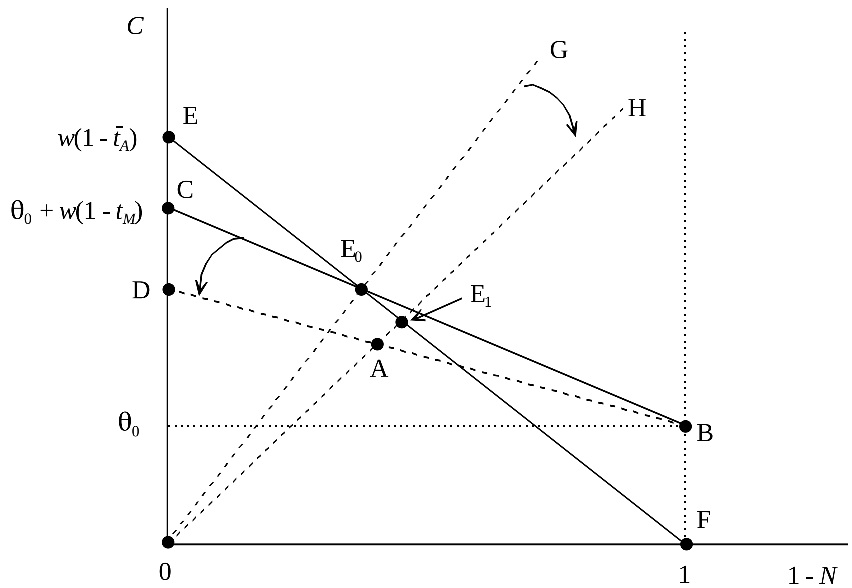

Figure 6: Increasing tax progressivity ($\sigma_{CM} = 1$)

tax rate by $\bar{t}_A$. Since, by definition, the household budget constraint can also be written as $C = w(1 - t_A)N$, it follows that the new choice point must lie along the following line:

$$C = w(1 - \bar{t}_A) - w(1 - \bar{t}_A)(1 - N). \tag{A27}$$

This line is drawn in Figure 5 as the straight line EF. Since $t_M > t_A$ for a progressive tax system, it is straightforward to show that EF is steeper than BC (compare equations (A26) and (A27)). Because both w and $\bar{t}_A$ are held constant, the position of the "alternative" budget line is not affected by the change in the marginal tax rate.

It is now clear where the new choice point must lie as it must satisfy the following criteria:

- It must lie on the line EF (so that the average tax is constant)
- It must be on the dashed line from the origin (because households want to consume goods and leisure in that proportion)
- There must be a tangency with a line parallel to BD.

It follows from these requirements (and indeed from (A25)) that the choice point must be E_0. In order to keep the average tax rate constant, the policy maker must increase θ_0 so that the budget line BD shifts in a parallel fashion such that it passes through point E_0. The household continues to choose E_0 and neither consumption or labour supply are changed.

Next we consider the Cobb-Douglas case ($\sigma_{CM} = 1$) in Figure 6. Equations (A26) and (A27) are drawn as, respectively, BC and FE, The initial equilibrium point is at E_0 (indifference curves are not drawn to avoid cluttering the diagram). The increase in the marginal tax rate rotates the budget line (A26) to the dashed line BD. In the absence of changes to θ_0, the household would choose point A. It follows from (A9) and (A12) that the household wants to choose a lower $C/(1-N)$ ratio, i.e. the expansion path rotates clockwise from 0G to 0H. Point A, however, does not satisfy the requirement that the average tax must remain constant as it doe not lie on the line EF. To satisfy that requirement the government must increase θ_0 such that the budget line shifts in a parallel fashion to intersect 0G and EF in point E_1. Because the utility function is homothetic, there is a tangency between the new budget line and an indifference curve at point E_1.

Question 2

Part (a)

Household i maximizes utility, $U^i = C^\alpha (1-N)^{\beta_i}$, subject to the budget constraint:

$$PC = \begin{cases} B & \text{if } N = 0 \\ \\ W\bar{N}(1-t) & \text{if } N = \bar{N} \end{cases} \tag{A1}$$

There are only two options for the household to check. It will choose $N = 0$ if this yields higher utility than $N = \bar{N}$ and vice versa. By substituting $N = 0$ and $N = \bar{N}$ into the utility function (4) we obtain:

$$U^i = \begin{cases} U^i_{N=0} \equiv \left(\frac{B}{P}\right)^\alpha & \text{if } N = 0 \\ \\ U^i_{N=\bar{N}} \equiv \left(\frac{W\bar{N}(1-t)}{P}\right)^\alpha (1-\bar{N})^{\beta_i} & \text{if } N = \bar{N} \end{cases} \tag{A2}$$

But, according to equation (3), we have $B = \gamma W\bar{N}(1-t)$ so the first expression in (A2) can be simplified to:

$$U^i_{N=0} = \left(\frac{B}{P}\right)^\alpha = \left(\frac{\gamma W\bar{N}(1-t)}{P}\right)^\alpha . \tag{A3}$$

By using (A3) and the second expression in (A2) we obtain:

$$\frac{U^i_{N=\bar{N}}}{U^i_{N=0}} = \frac{\left(\frac{W\bar{N}(1-t)}{P}\right)^\alpha (1-\bar{N})^{\beta_i}}{\left(\frac{\gamma W\bar{N}(1-t)}{P}\right)^\alpha} = \gamma^{-\alpha}(1-\bar{N})^{\beta_i}. \tag{A4}$$

It follows that the household makes the following labour supply choice:

$$N_i^S = \begin{cases} 0 & \text{if } \gamma^{-\alpha}(1-\bar{N})^{\beta_i} < 1 \\ \\ \bar{N} & \text{if } \gamma^{-\alpha}(1-\bar{N})^{\beta_i} > 1 \end{cases} \tag{A5}$$

The *marginal household* is indifferent between working and not working, i.e. it has a $\beta_i = \beta_M$ such that $\gamma^{-\alpha}(1-\bar{N})^{\beta_M} = 1$. By taking logarithms on both sides of this expression we can solve for β_M:

$$-\alpha \log \gamma + \beta_M \log(1-\bar{N}) = 0 \qquad \Leftrightarrow \qquad \beta_M = \frac{\alpha \log \gamma}{\log(1-\bar{N})} > 0, \tag{A6}$$

where the sign follows from the fact that $0 < \gamma < 1$ and $0 < \bar{N} < 1$ (so that $\log \gamma < 0$ and $\log(1-\bar{N}) < 0$). Households whose β_i exceeds β_M prefer not to work (they like leisure "too much") whereas households with a β_i smaller than β_M choose to work. (Someone with $\beta_i = 0$ is the proverbial *workaholic*.)

Part (b)

The β_i coefficients are distributed uniformly over the interval $[0, \bar{\beta}]$. The frequency distribution of β_i's in the population is drawn in Figure 7. All households with a $\beta_i \leq \beta_M$ are workers whereas all households with a $\beta_i > \beta_M$ are "loungers". Since the population size is Z, there are thus $(\bar{\beta} - \beta_M)Z/\bar{\beta}$ loungers and $\beta_M Z/\bar{\beta}$ workers (who each work $\bar{N}$ hours). Aggregate labour supply is thus:

$$N^S = \frac{\beta_M Z \bar{N}}{\bar{\beta}}. \tag{A7}$$

The macroeconomic labour supply curve is drawn in Figure 8. Note that this aggregate labour supply curve is vertical because β_M does not depend on the wage rate (due to the fact that unemployment benefits are linked to the wage rate).

If γ is increased, then it follows from (A6)-(A7) that:

$$\frac{\partial \beta_M}{\partial \gamma} = \frac{\alpha}{\gamma \log(1-\bar{N})} < 0, \qquad \frac{\partial N^S}{\partial \gamma} = \frac{Z\bar{N}}{\bar{\beta}} \frac{\partial \beta_M}{\partial \gamma} < 0, \tag{A8}$$

where the signs follow from the fact that $\log(1-\bar{N}) < 0$. The reduction in β_M causes the aggregate labour supply curve to shift to the left, as is indicated in Figure 8.

Part (c)

Now we assume that the unemployment benefits are linked to the gross wage, i.e. $B = \gamma W \bar{N}$. By substituting this into the first expression in (A2) we obtain:

$$U^i_{N=0} = \left(\frac{B}{P}\right)^\alpha = \left(\frac{\gamma W \bar{N}}{P}\right)^\alpha = \left(\frac{\gamma}{1-t}\right)^\alpha \left(\frac{W\bar{N}(1-t)}{P}\right)^\alpha. \tag{A9}$$

By using (A9) and the second expression in (A2) we find:

$$\frac{U^i_{N=\bar{N}}}{U^i_{N=0}} = \frac{\left(\frac{W\bar{N}(1-t)}{P}\right)^\alpha (1-\bar{N})^{\beta_i}}{\left(\frac{\gamma}{1-t}\right)^\alpha \left(\frac{W\bar{N}(1-t)}{P}\right)^\alpha} = \left(\frac{1-t}{\gamma}\right)^\alpha (1-\bar{N})^{\beta_i}. \tag{A10}$$

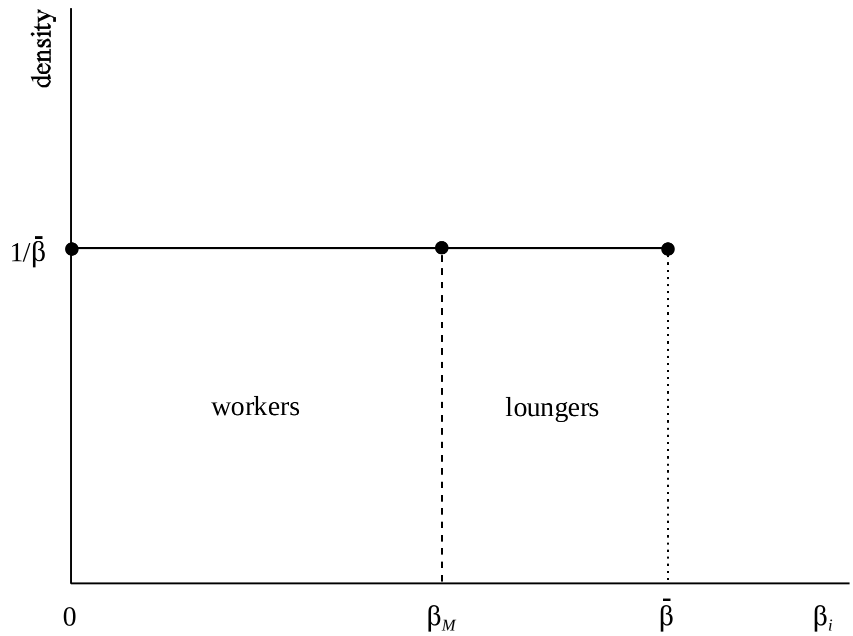

Figure 7: Distribution of β_i coefficients in the population

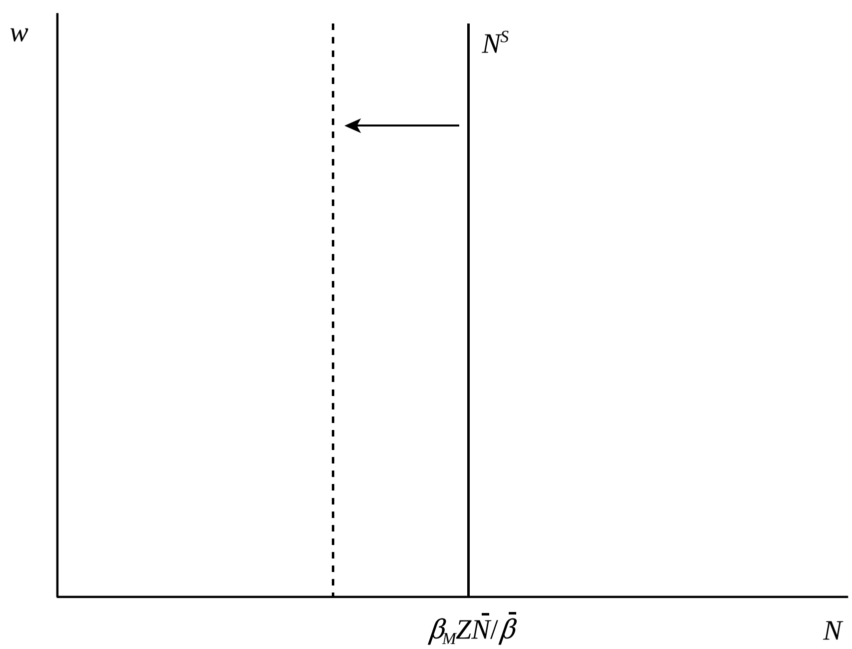

Figure 8: Aggregate labour supply with indivisible labour

The marginal household now has a $\beta_i = \beta_M$ equal to:

$$\alpha\left[\log(1-t) - \log\gamma\right] + \beta_M \log(1-\bar{N}) = 0 \qquad \Leftrightarrow$$
$$\beta_M = \frac{\alpha\left[\log\gamma - \log(1-t)\right]}{\log(1-\bar{N})} > 0, \tag{A11}$$

where the sign follows from the fact that $\log(1-\bar{N}) < 0$ and the assumption that $\gamma < 1-t$.

An increase in the tax rate, leads to an increase in $\gamma/(1-t)$ and thus to an increase in the *effective* replacement rate. This implies that β_M falls so that aggregate labour supply falls.

Question 3

Part (a)

The three magic words are recruit, retain, and motivate.

- Recruit. Make sure that the best workers choose to join your firm (rather than your competitor's firm).
- Retain. Make sure that your employees do not quit to go to another firm.
- Motivate. Make sure that your employees provide sufficient effort on the job.

Part (b)

In the Shapiro-Stiglitz model, unemployment acts as a worker discipline device. If they are caught shirking (not expending sufficient effort) then the firm can fire the worker. If there were no unemployment, then there would be no way to punish the worker because he/she would immediately find the same kind of job. In the internal solution of the Shapiro-Stiglitz model, there is non-zero unemployment and the threat of unemployment provides the firm with an effective instrument to limit shirking.