# *A Life-Cycle Overlapping-Generations Model of the Small Open Economy*

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# Overview of this lecture

- motivation
- model (brief)
- macroeconomic shocks
  - balanced-budget fiscal policy
  - temporary tax cut (Ricardian experiment)
  - world interest rate change
- welfare effects
- does demography matter in the aggregate?
- extensions
- concluding remarks

# Motivation

- Gompertz-Makeham Law of Mortality:

  "It is possible that death may be the consequence of two generally co-existing causes; the one, chance, without previous disposition to death or deterioration; the other, a deterioration or an increased inability to withstand destruction."

  (Benjamin Gompertz, 1825)

- How have macroeconomists incorporated this "fact of life" into their models so far?
  - Barro (1974) and many others:
    - ∗ connected finite-lived generations
    - ∗ operative bequests lead to Ricardian equivalence

- Yaari (1965):
  - ∗ disconnected agents
  - ∗ heavier discounting of future felicity due to uncertainty of survival
  - ∗ actuarially fair life insurance opportunities
- Blanchard (1985)-Buiter (1988)-Weil (1989) add:
  - ∗ general equilibrium representation
  - ∗ constant death rate: all living “dynasties” have same expected remaining lifetime
  - ∗ aggregation possible
  - ∗ cannot capture life-cycle pattern
- Calvo & Obstfeld (1988):
  - ∗ general mortality process
  - ∗ focus on optimal time-consistent policy

- Recent related work in this area:
  - de la Croix & Licandro (1999); Boucekkine, de la Croix, and Licandro (2002):
    - ∗ human capital and endogenous growth
    - ∗ infinite intertemporal substitution elasticity
  - d'Albis (2004)
    - ∗ model similar to ours
    - ∗ focusses on different issues [e.g. efficiency property of steady state]
  - Rios-Rull (1996)
    - ∗ calibrated stochastic RBC model of the Auerbach-Kotlikoff OLG type
    - ∗ ....OLG feature does not matter to impulse-response functions with respect to technology shocks

- Hansen & Imrohoroglu (2005)
  * what if annuities markets do not exist?
  * absence of annuities markets can account for hump-shaped consumption pattern

- Focus of this paper
  - realistic demography in a small open economies
  - factor prices exogenous (and typically constant)
  - aggregation not necessary
  - model can be solved analytically: complementary to large-scale CGE models
  - demographic realism matters!
  - maintained assumption: actuarially fair annuities

# Model: Key Assumptions

- small open economy facing constant world interest rate
- labour only factor of production (capital could be added easily)
- savings instruments:
  - foreign assets
  - government debt
  - perfect substitutes: same rate of return
- life-time uncertainty; actuarially fair life insurance
- no aggregate uncertainty
- rational agents blessed with perfect foresight

# Model: Key Equations

- expected remaining lifetime utility at time $t$ of agent born at time $v$ ($t \geq v$)

$$\Lambda(v,t) \equiv \int_t^\infty \underbrace{\ln \bar{c}\left(v,\tau\right)}_{(a)} \underbrace{e^{M(t-v)-M(\tau-v)}}_{(b)} \underbrace{e^{\theta(t-\tau)}}_{(c)} d\tau \tag{2.6}$$

(a) felicity: unitary intertemporal substitution elasticity

(b) lifetime uncertainty: Probability that household of age $t-v$ reaches age $\tau - v$. Process not memoryless, i.e. $M\left(t-v\right) - M\left(\tau - v\right) \neq M\left(t-\tau\right)$.

(c) pure discounting ($\theta > 0$): impatience

- mortality factor and mortality rate:

$$M(\tau - v) \equiv \int_0^{\tau - v} m(s)\, ds \tag{2.4}$$

  - $m(s)$ is instantaneous mortality rate, i.e. hazard rate of hazard rate of the stochastic distribution of the date of death:

$$m(s) \equiv \frac{\phi(s)}{1 - \Phi(s)}$$

  - $\phi(s)$ = density function
  - $\Phi(s)$ = distribution (or cumulative density) function
  - in **this** paper: $m(s)$ depends only on household age [stationary demography]

- budget identity:

$$\dot{\bar{a}}(v,\tau) = [r + m(\tau - v)]\,\bar{a}(v,\tau) + \bar{w}(\tau) - \bar{z}(\tau) - \bar{c}(v,\tau) \qquad (2.7)$$

  – $\bar{a}(v,\tau)$ = financial assets
  – $r$ = world interest rate [patient country, $r > \theta$]
  – $r + m(\tau - v)$ = annuity rate of interest
  – $\bar{w}(\tau)$ = wage rate
  – $\bar{z}(\tau)$ = lump-sum tax
  – $\bar{c}(v,\tau)$ = consumption

- optimal choices of household with age $u \equiv t - v$:

$$\frac{\dot{\bar{c}}(v,\tau)}{\bar{c}(v,\tau)} = r - \theta > 0 \qquad (2.9)$$

$$\bar{c}(v,t) = \frac{1}{\Delta(u,\theta)}\left[\bar{a}(v,t) + \bar{h}(v,t)\right] \qquad (2.10)$$

$$\bar{h}(v,t) \equiv e^{ru+M(u)} \int_u^\infty \left[\bar{w}(s+v) - \bar{z}(s+v)\right] e^{-[rs+M(s)]} ds \qquad (2.11)$$

$$\Delta(u,\lambda) \equiv e^{\lambda u+M(u)} \int_u^\infty e^{-[\lambda s+M(s)]} ds, \quad (u \geq 0,\ \lambda > 0) \qquad (2.12)$$

  – $\bar{h}(v,t)$ = human wealth (market value of time endowment, using annuity rate of interest for discounting)

  – $\Delta(u,\lambda)$ = demographic factor (plays central role, e.g. $1/\Delta(u,\theta)$ is propensity to consume out of total wealth)

**Lemma 1** *Let* $\Delta(u,\lambda)$ *be defined as in (2.12) and assume that the mortality rate is non-decreasing, i.e.* $m'(s) \geq 0$ *for all* $s \geq 0$. *Then the following properties can be established for* $\Delta(u,\lambda)$*:*

*(i) decreasing in* $\lambda$, $\partial\Delta(u,\lambda)/\partial\lambda < 0$*;*

*(ii) non-increasing in household age,* $\partial\Delta(u,\lambda)/\partial u \leq 0$*;*

*(iii) upper bound,* $\Delta(u,\lambda) \leq 1/[\lambda + m(u)]$*;*

*(iv)* $\Delta(u,\lambda) > 0$ *for* $u < \infty$*;*

*(v) for* $\lambda \to \infty$, $\Delta(u,\lambda) \to 0$.

# Demographics: Theory

- birth process:

$$L(v,v) = bL(v) \tag{2.13}$$

  - $L(v,v)$ = newborn cohort at time $v$
  - $b$ = birth rate [constant]
  - $L(v)$ = total population at time $v$

- size of cohort over time:

$$L(v,\tau) = L(v,v)\, e^{-M(\tau-v)} \tag{2.14}$$

- aggregate mortality rate, $\bar{m}$:

$$\bar{m}L(t) = \int_{-\infty}^{t} m(t-v)\, L(v,t)\, dv \tag{2.15}$$

- relative cohort weights [needed for aggregation]:

$$l(v,t) \equiv \frac{L(v,t)}{L(t)} = be^{-[n(t-v)+M(t-v)]} \tag{2.16}$$

  - $n \equiv b - \bar{m}$ = aggregate population growth rate

- for given birth rate and mortality process, (2.15)-(2.16) imply implicit solution for $n$:

$$b = \frac{1}{\Delta(0,n)} \tag{3.2}$$

# Demographics: Estimates

- use actual demographic data for the United Stated
- projections on expected survival rates for people born in 2001
- four parametric models are estimated with nonlinear least squares:
  - constant mortality rate [Blanchard]
  - linear-in-age mortality rate
  - piece-wise linear mortality rate
  - Gompertz-Makeham
- Estimation results in **Table 1**.
- Visualisation of fit in **Figure 1**.

# Table 1: Estimated Survival Functions

| | $\hat{\mu}_0$ | $\hat{\mu}_1$ | $\hat{\mu}_2$ | $\hat{\bar{u}}$ | $\hat{\sigma}$ | $\hat{n}(b)$ | $1-\widehat{\Phi(100)}$ |
|---|---|---|---|---|---|---|---|
| 1. **Constant** | $0.7026\times10^{-2}$ | – | – | – | 0.2277 | 0.80 | **49.53** |
| $M(u)=\mu_0 u$ | (4.92) | | | | | | |
| 2. **Linear in age** | $-0.8970\times10^{-2}$ | 0.0152 | – | – | 0.1199 | – | – |
| $M(u)=\mu_0 u+\mu_1^2 u^2$ | $(-3.83)$ | (12.29) | | | | | |
| | – | 0.0104 | – | – | 0.1595 | 0.49 | 34.05 |
| | | (13.66) | | | | | |
| 3. **Piece-wise linear (PWL) in age** | $0.1544\times10^{-2}$ | 0.0410 | – | 60.85 | 0.0294 | 0.37 | 6.57 |
| $M(u)=\mu_0 u+\delta(u)\mu_1^2(u-\bar{u})^2$ | (6.41) | (16.12) | | (43.08) | | | |
| $\delta(u)=\begin{cases}0 & \text{for } 0<u<\bar{u}\\ 1 & \text{for } u\geq\bar{u}\end{cases}$ | | | | | | | |
| 4. **Gompertz-Makeham (GM)** | $0.5834\times10^{-3}$ | $0.3419\times10^{-4}$ | 0.0928 | – | 0.0018 | 0.37 | **1.69** |
| $M(u)=\mu_0 u+(\mu_1/\mu_2)\left[e^{\mu_2 u}-1\right]$ | (24.76) | (27.01) | (193.71) | | | | |

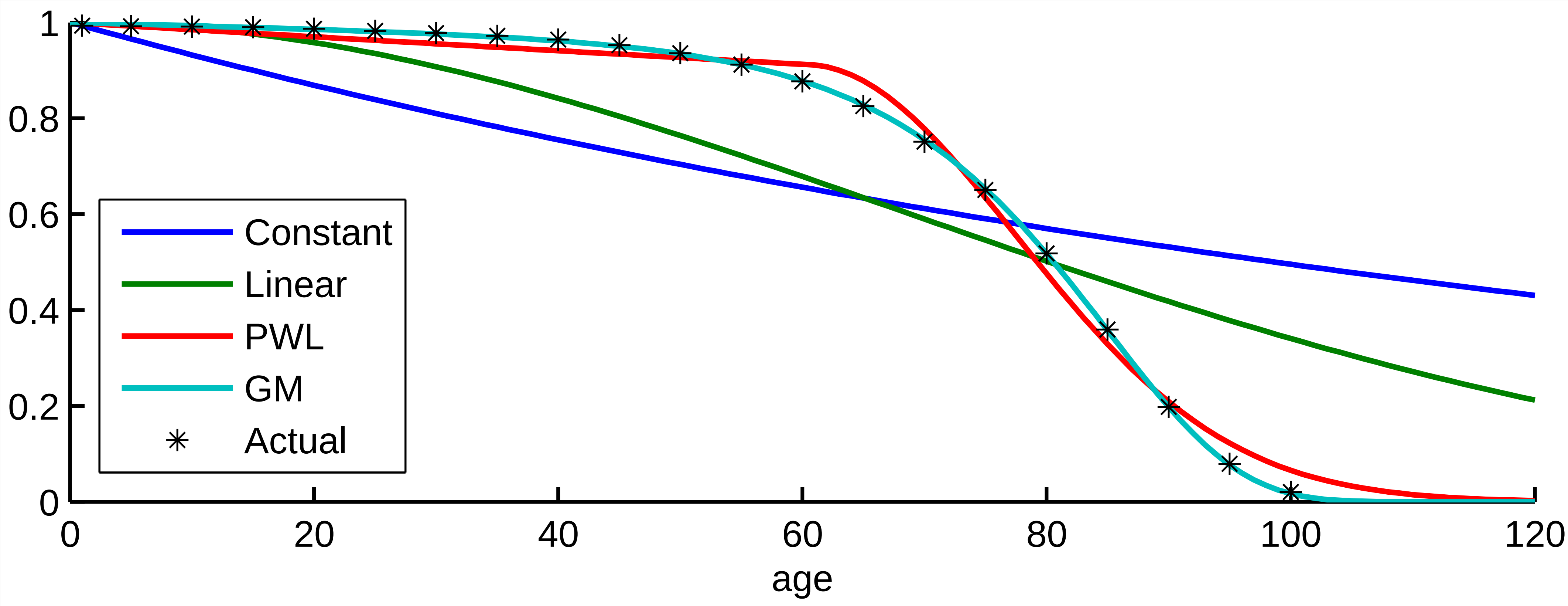


Figure 1: (a) Surviving Fraction of the Population

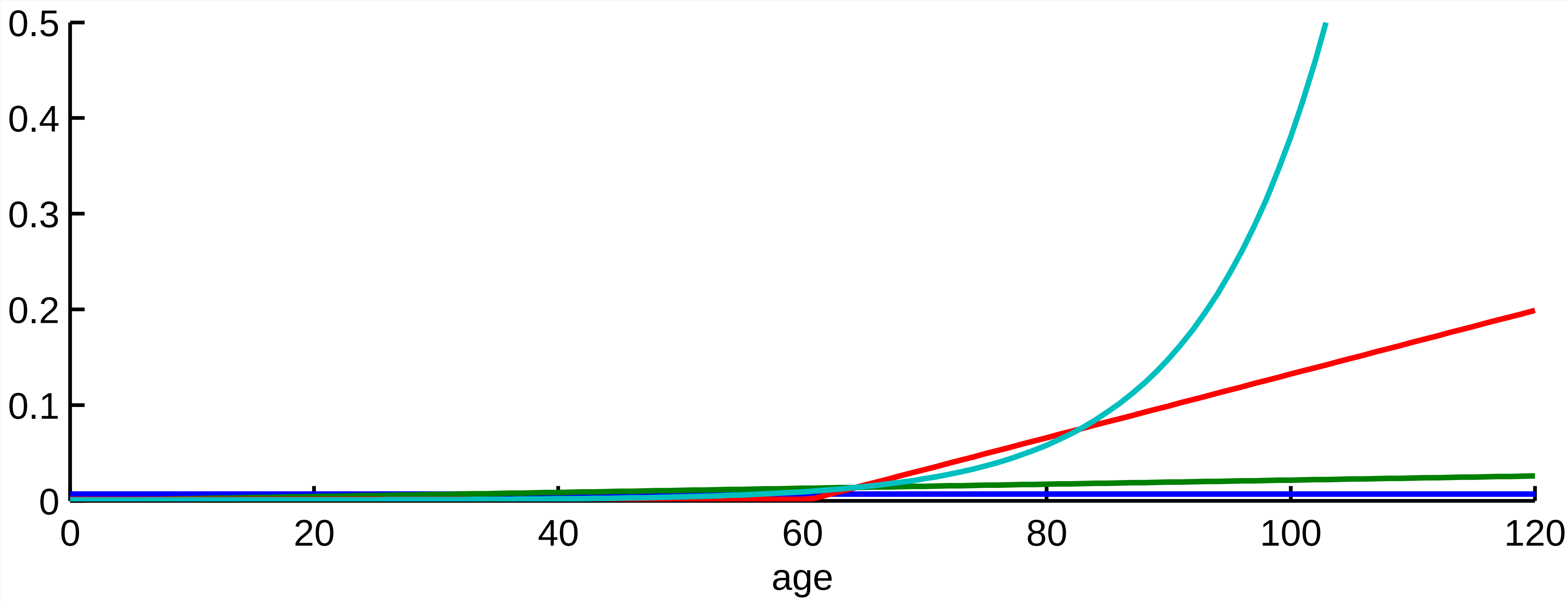


Figure 1: (b) Mortality Rate of the Population

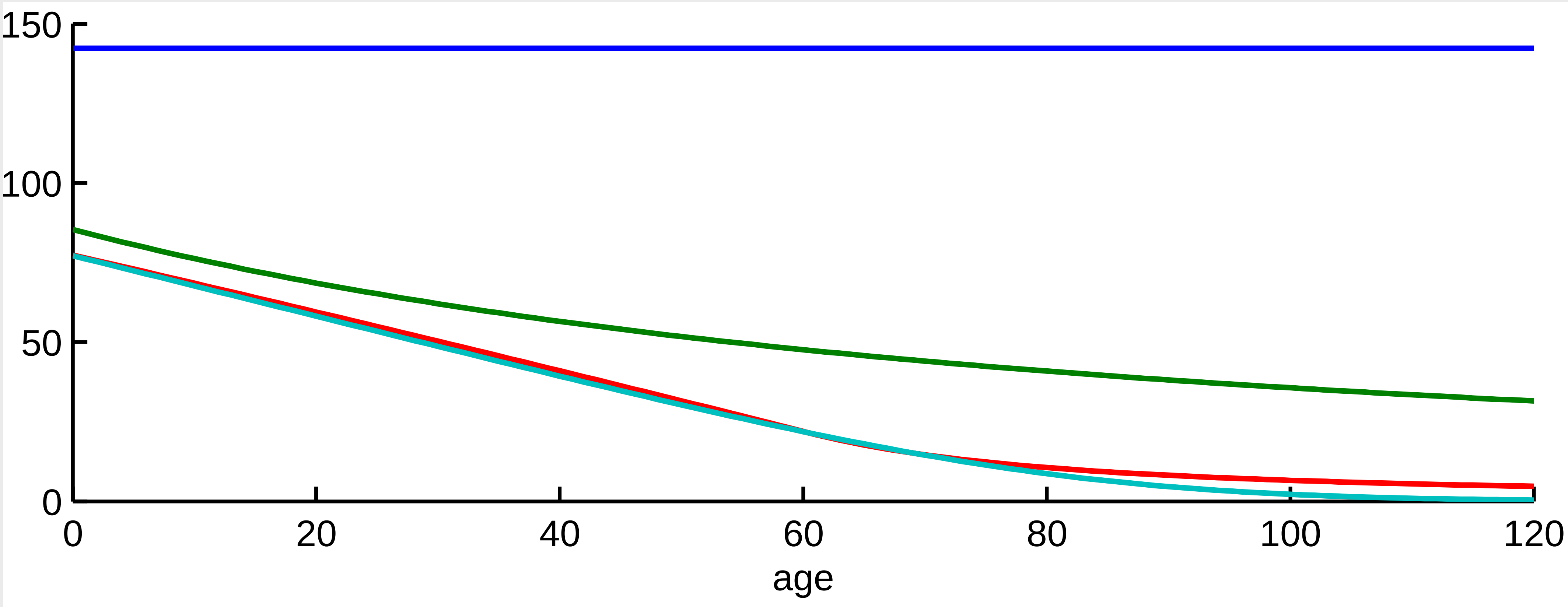


Figure 1: (c) Expected Remaining Lifetime

# Steady-State Profiles

$$\frac{1}{\Delta(u,\theta)} = \left[ e^{\theta u + M(u)} \int_u^{\infty} e^{-[\theta s + M(s)]} ds \right]^{-1}$$

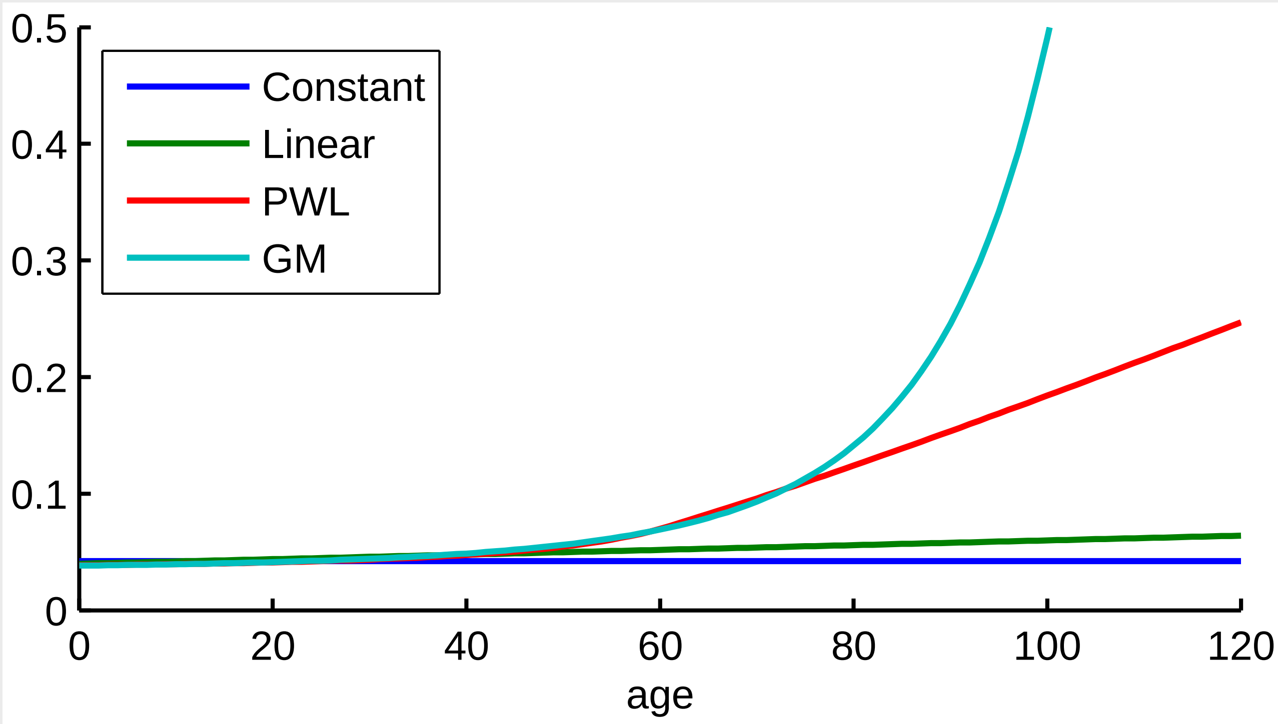


Figure 2: (a) Propensity to Consume

$$\hat{\bar{h}}(v,t) \equiv \Delta(u,r)\left[\hat{w} - \hat{z}\right]$$

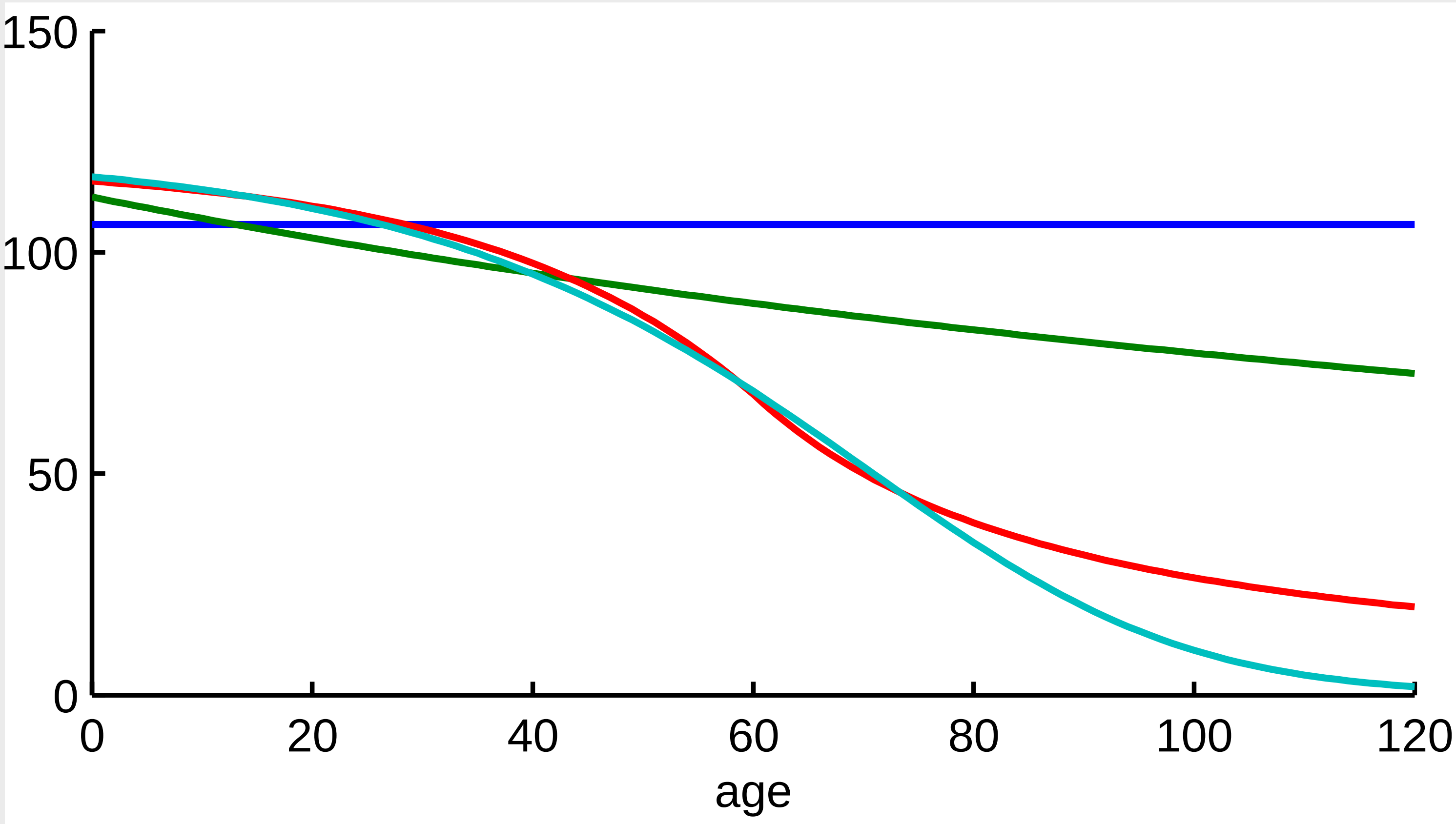


Figure 2: (b) Human Wealth

$$\hat{\bar{c}}(u) = \frac{\hat{\bar{h}}(0)}{\Delta(0,\theta)} e^{(r-\theta)u}$$

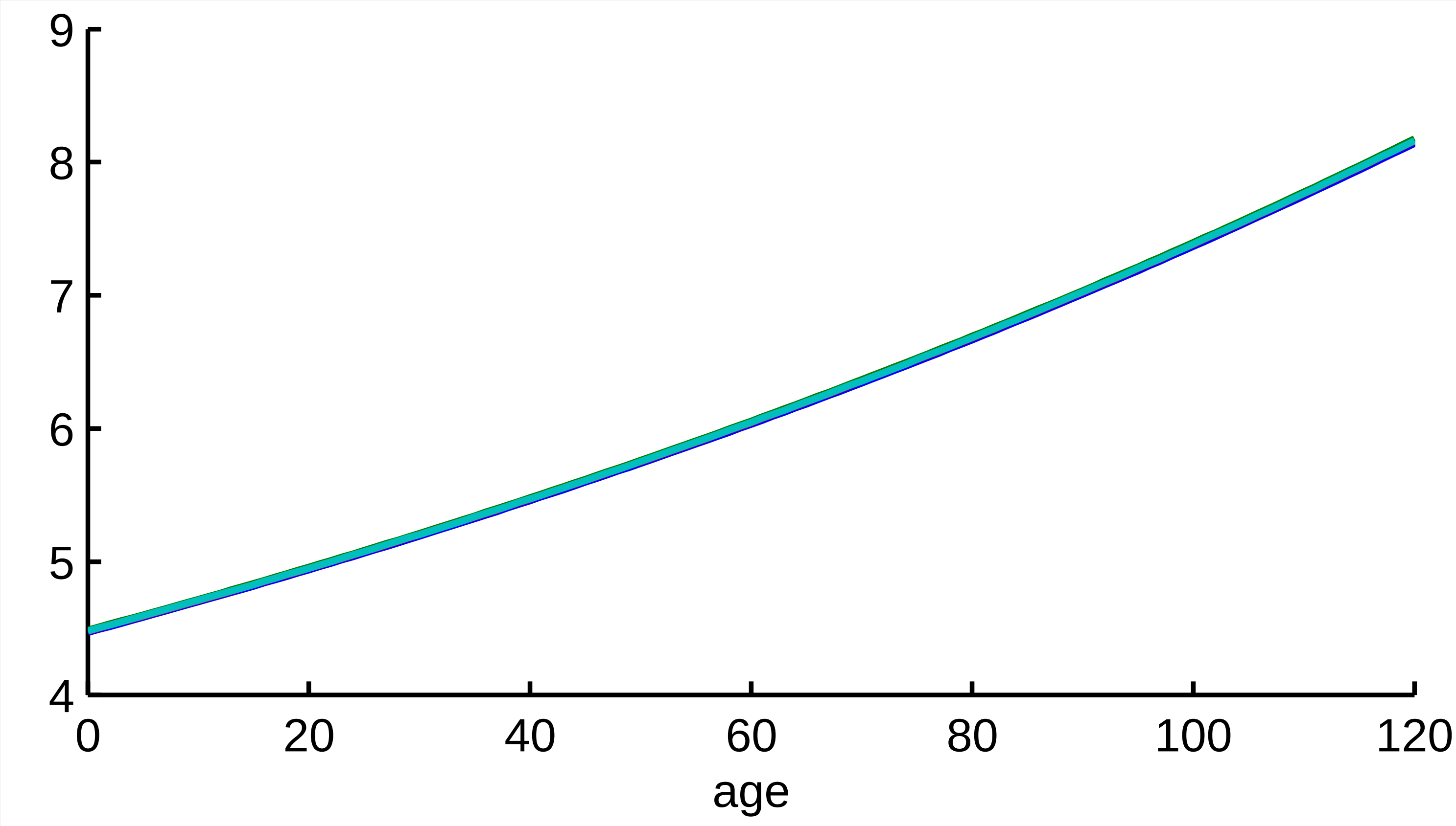


Figure 2: (c) Consumption

$$\hat{\bar{a}}(u) = \Delta(u,\theta)\,\hat{\bar{c}}(u) - \hat{\bar{h}}(u)$$

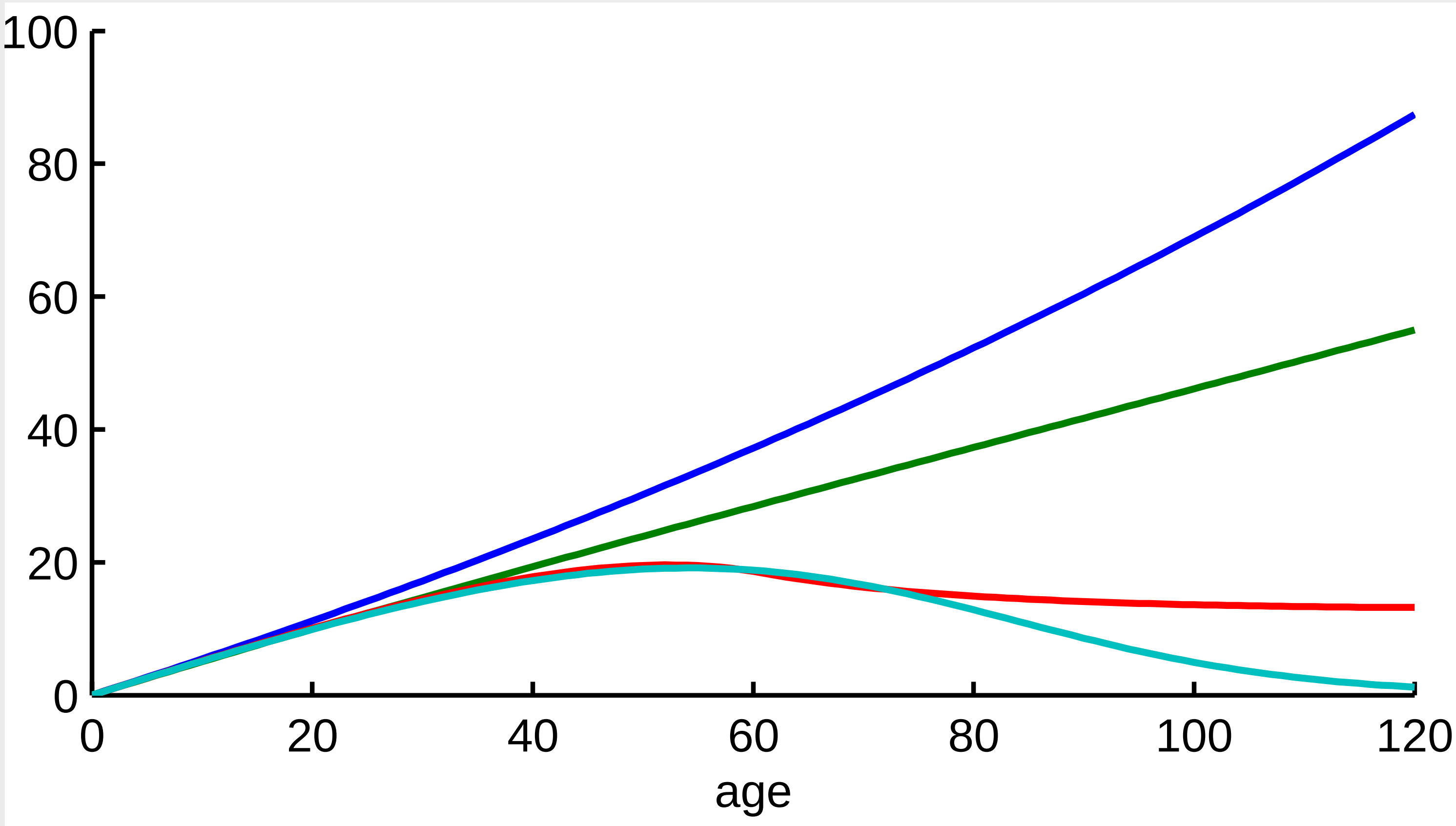


Figure 2: (d) Financial Assets

# Macroeconomic Shocks

- balanced-budget fiscal policy
  - once-off increase in government consumption and lump-sum taxes
- temporary tax cut
  - short-run tax cut financed with debt
  - gradual increase lump-sum tax
  - long-run debt positive
- interest rate shock
  - once-off increase in world interest rate

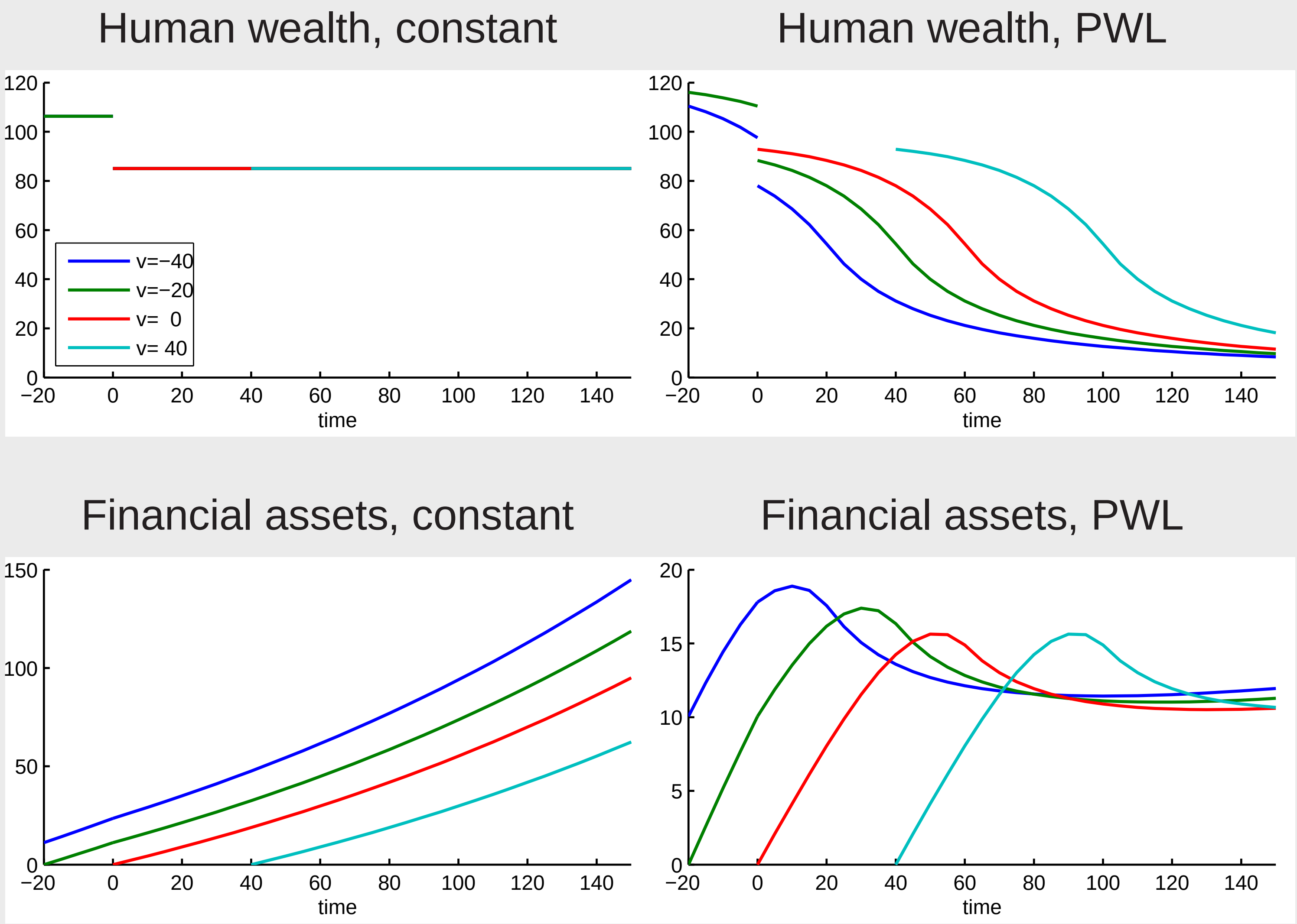


Figure 4: Balanced-Budget Fiscal Policy

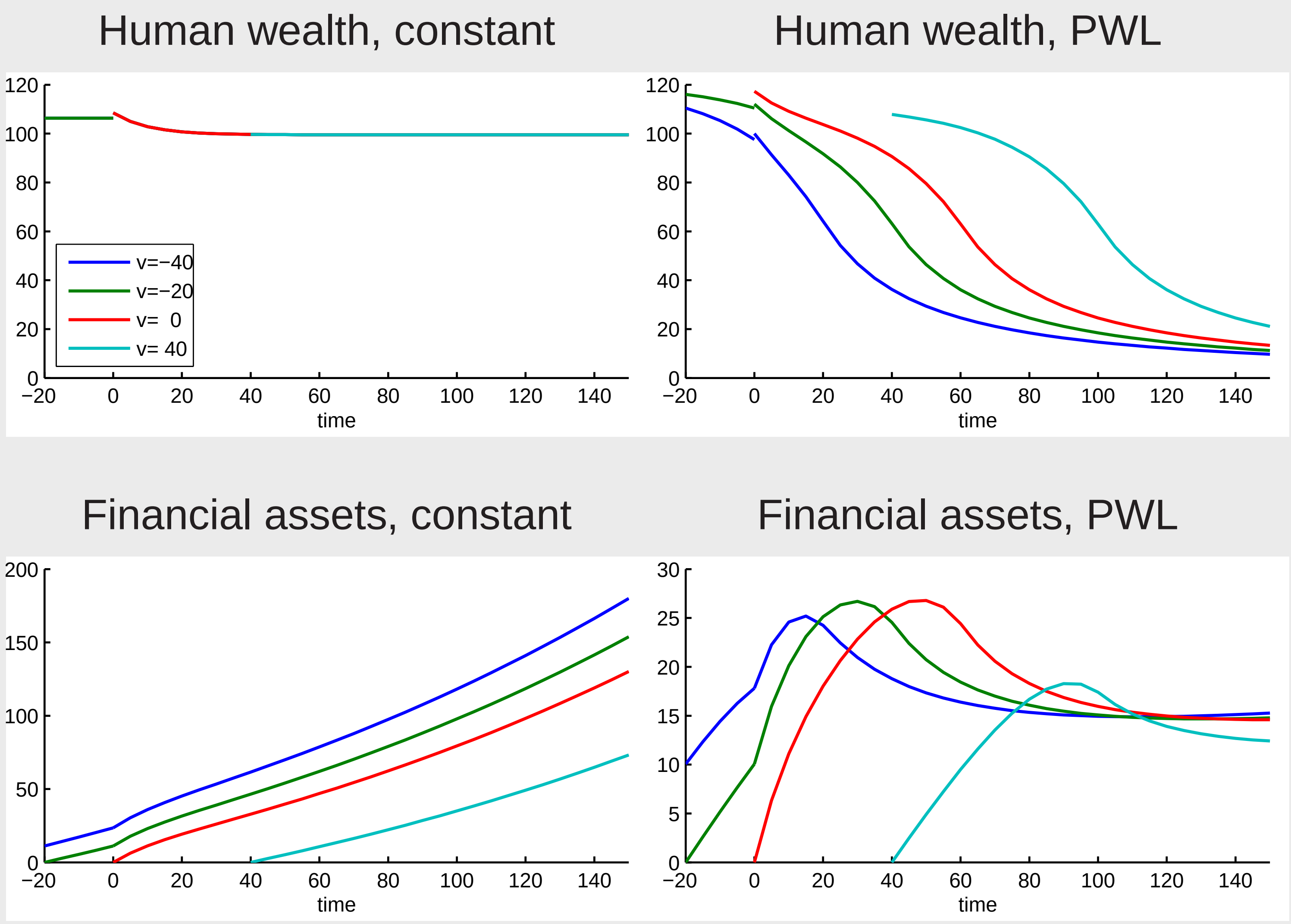


Figure 5: Ricardian Equivalence Experiment: Temporary Tax Cut

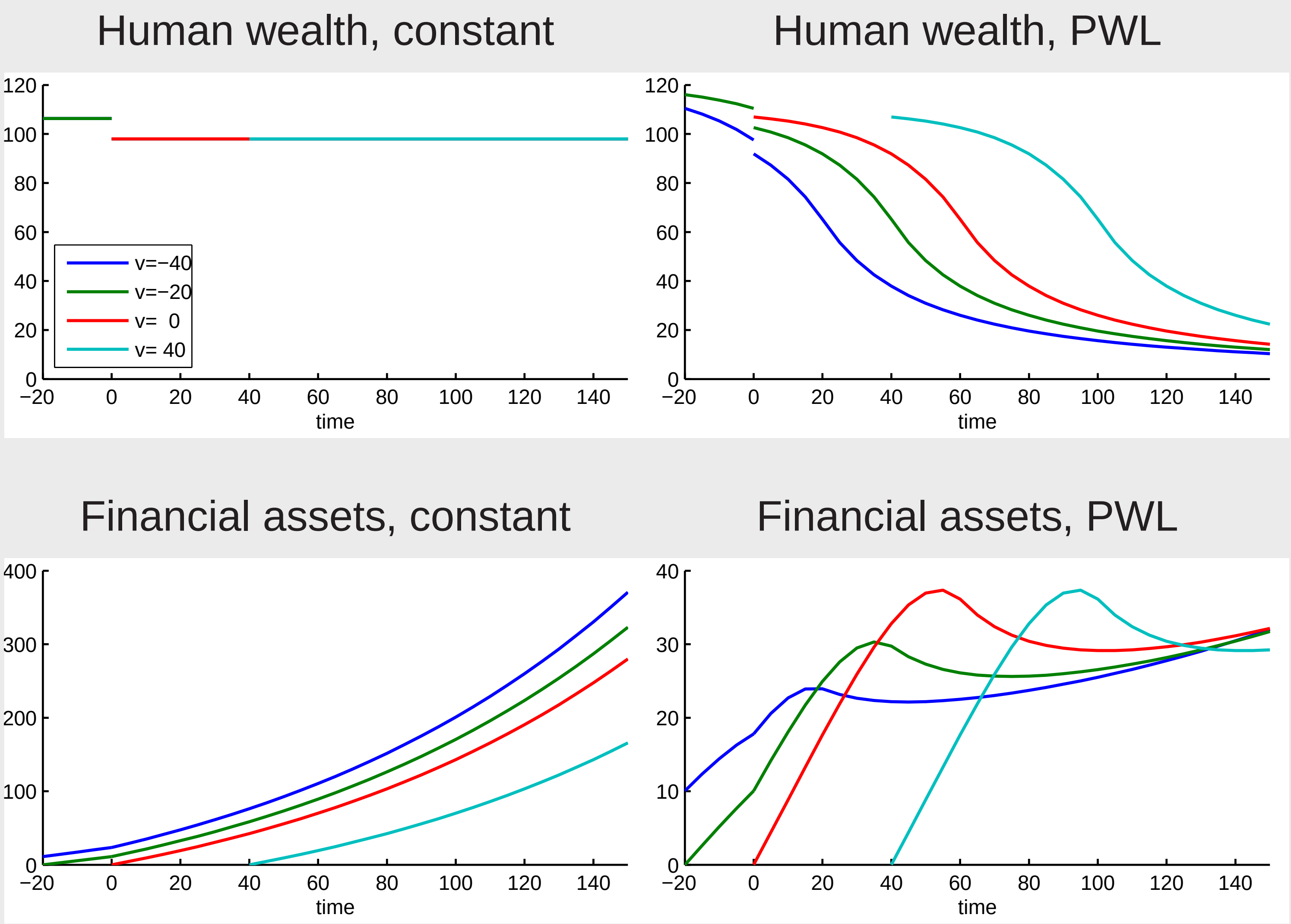


Figure 6: Increase in the World Interest Rate

# Welfare effects

- change in welfare from shock at time $t = 0$
  - existing agents ($v \leq 0$): evaluate $d\Lambda\left(v, 0\right)$:

$$d\Lambda(v,0) = dr \int_0^\infty \tau e^{-\theta\tau - M(\tau - v) + M(-v)} d\tau + \Delta(-v,\theta) \ln \Gamma_E(v) \quad (4.2)$$

$$\Gamma_E(v) \equiv \frac{\hat{\bar{a}}(-v) + \bar{h}(v,0)}{\hat{\bar{a}}(-v) + \hat{\bar{h}}(-v)} \quad (4.3)$$

  - future agents ($v > 0$): evaluate $d\Lambda\left(v, v\right)$:

$$d\Lambda(v,v) = dr \int_0^\infty s e^{-[\theta s + M(s)]} ds + \Delta(0,\theta) \ln \Gamma_F(v) \quad (4.4)$$

$$\Gamma_F(v) \equiv \frac{\bar{h}(v,v)}{\hat{\bar{h}}(0)} \quad (4.5)$$

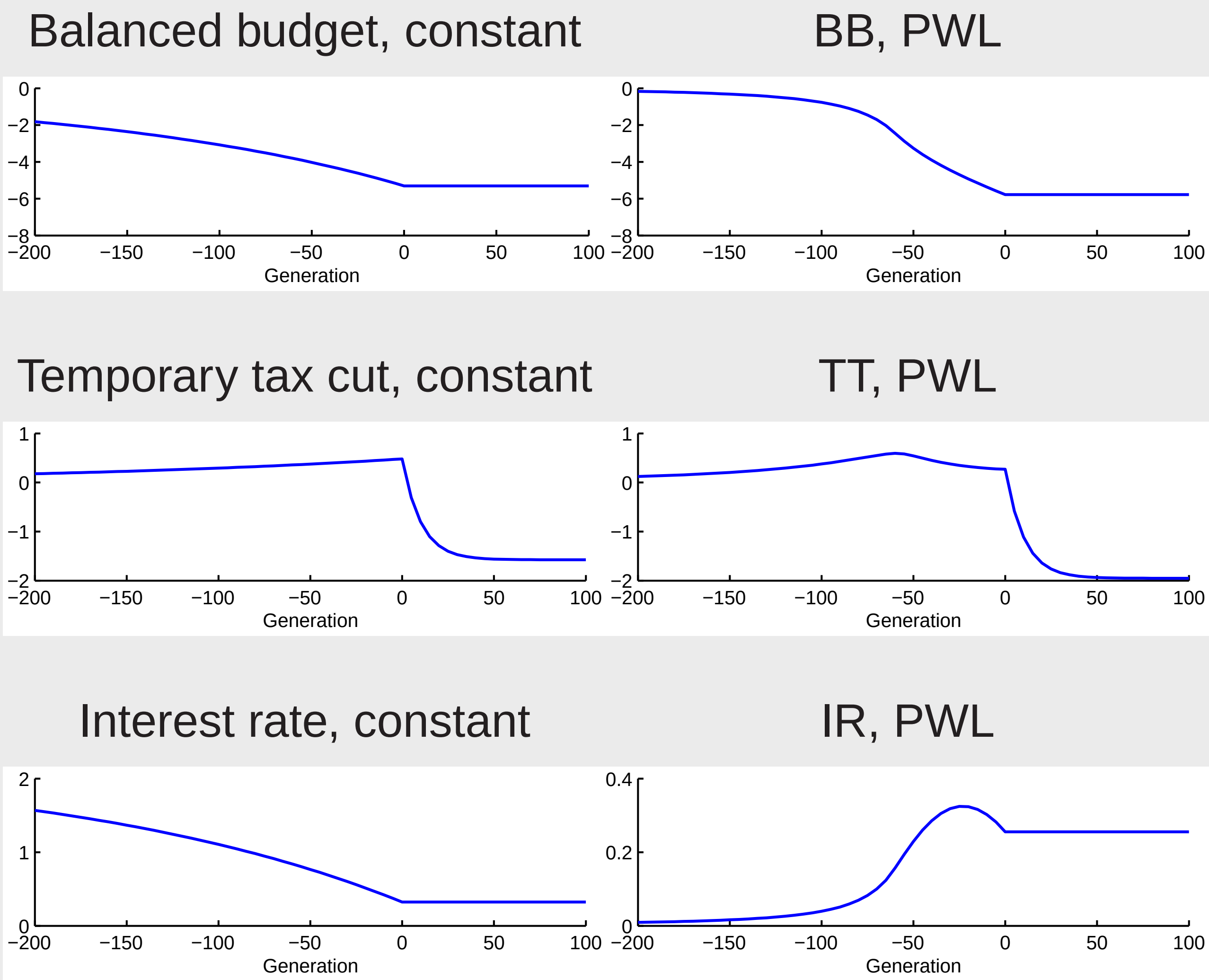


Figure 7: Welfare Effects

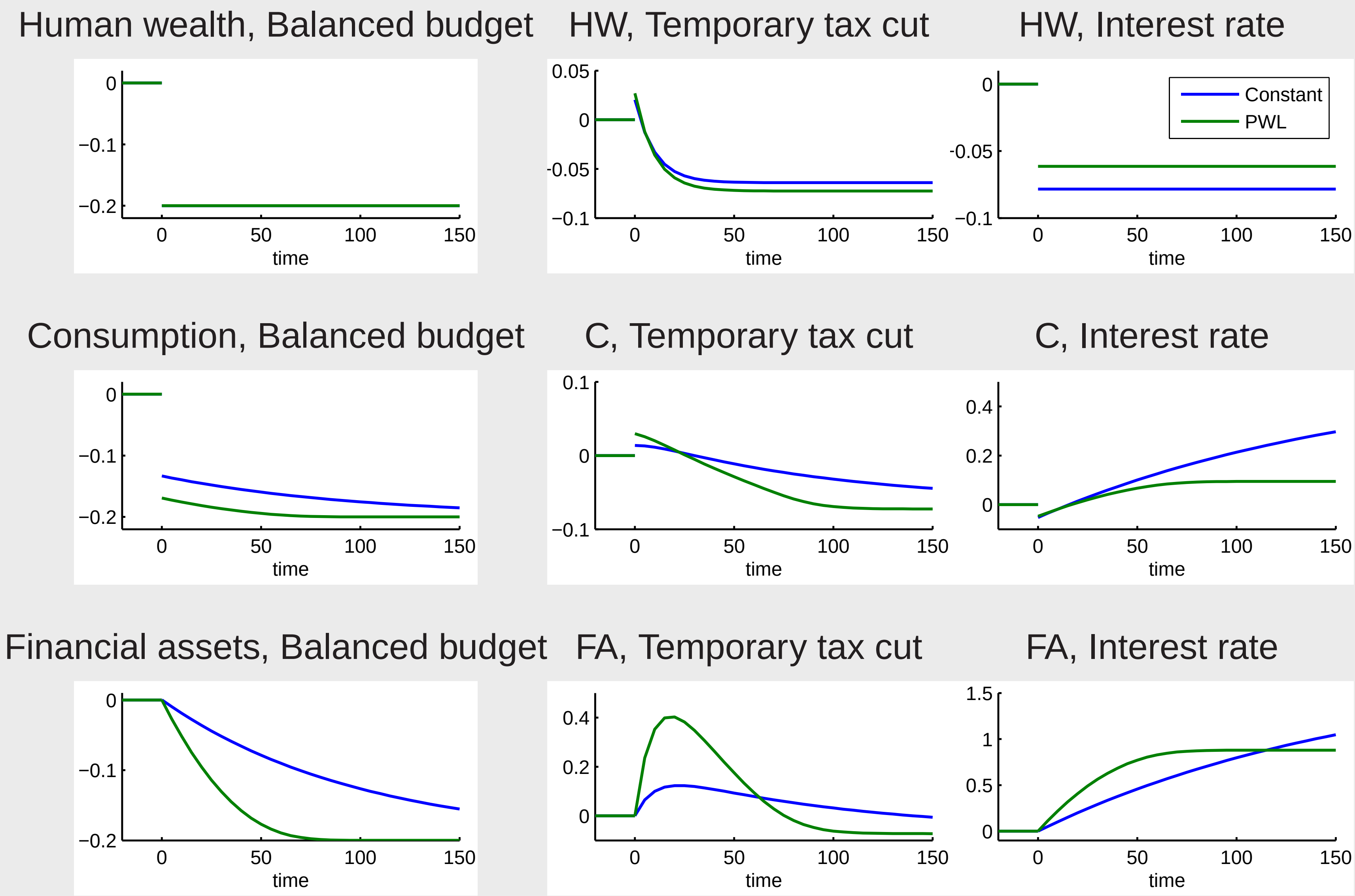


Figure 8: Relative Effect of the Shocks on Aggregate Variables

# Extensions

- effects of demographic change:
  - embodied: mortality rate depends on date of birth $m(v, s)$
  - disembodied: mortality rate depends on calender date $m(t, s)$
- hump-shaped consumption:
  - absent annuity markets [cf. Hansen & Imrohoroglu (2005)]
    * Euler equation becomes:

$$\frac{\dot{\bar{c}}(v,\tau)}{\bar{c}(v,\tau)} = r - (\theta + m(\tau - v)) \tag{1}$$

    * $\dot{\bar{c}}(v,\tau) > 0$ for young and $\dot{\bar{c}}(v,\tau) < 0$ for old agents

– diminishing needs as one gets older:

$$\Lambda(v,t) \equiv e^{M(t-v)} \int_t^{\infty} \left[ \frac{\bar{e}\left(v,\tau\right)^{1-1/\sigma} - 1}{1-1/\sigma} \right] e^{-[\theta(\tau-t)+M(\tau-v)]} d\tau \quad (5.1)$$

$$\bar{e}\left(v,\tau\right) \equiv \bar{c}\left(v,\tau\right) \exp\left\{ \frac{\zeta_0\left(\tau-v\right)^{1+\zeta_1}}{1+\zeta_1} \right\}, \quad \zeta_0 > 0,\ \zeta_1 > 0 \quad (5.2)$$

∗ $\sigma$ = intertemporal substitution elasticity

∗ $\bar{e}\left(v,\tau\right)$ = effective consumption

∗ Euler equation becomes:

$$\frac{\dot{\bar{c}}\left(v,\tau\right)}{\bar{c}\left(v,\tau\right)} = \sigma\left(r-\theta\right) - \left(1-\sigma\right)\zeta_0\left(\tau-v\right)^{\zeta_1} \quad (5.3)$$

∗ for $0 < \sigma < 1$, $\dot{\bar{c}}\left(v,\tau\right) > 0$ for young and $\dot{\bar{c}}\left(v,\tau\right) < 0$ for old agents

- endogenous labour supply and retirement decision
  - in progress
  - application: ageing, PAYG pension reform, and the retirement decision
- endogenous education decision
  - in progress
  - education at start of life [cf. de la Croix, Licandro, and Boucekkine papers mentioned above]
  - application: growth effects of ageing

- realistic demography in a closed economy
  - steady state easy
  - difficult to get analytical results for transitional dynamics
  - approximate solutions may be attainable

# Concluding Remarks

- in the context of a small open economy [or with constant marginal product of capital] there is no need to use models based on an unrealistic description of the demographic process
- using a realistic demographic process matters because.....
  - individual behaviour is different
  - impulse-response functions are different
  - transition speed is affected
  - welfare effects may be non-monotonic