Lecture 2: The Continuous-Time Overlapping-Generations Model: Basic Theory and Applications

Ben J. Heijdra

Department of Economics, Econometrics & Finance
University of Groningen

16 December 2011

Outline

1. Introduction
2. Individual behaviour under lifetime uncertainty
   - Yaari's lessons
   - Realistic mortality profile
   - The role of annuities
3. Macroeconomic consequences of lifetime uncertainty
   - Blanchard's model
   - Basic model properties

Aims of this lecture (1)

- Study "work-horse" model of modern macroeconomics which is based on overlapping generations. Motivation for doing this:
  - Ricardian equivalence may be inappropriate (the chain of bequests may not be fully operational).
  - Tractable way to introduce (and study consequences of) heterogeneous agents.
  - Contains Ramsey model as a special case.
- Show some applications of the Blanchard-Yaari model.
  - Fiscal policy (crowding out effects of public consumption).
  - Debt neutrality revisited.

Aims of this lecture (2)

- Extend the BY model in a number of minor/major directions:
  - Embed it in an endogenous growth model (how do a country's demography and economic growth interact?)
  - Age-dependent productivity (mimic life-cycle; reintroduces possibility of dynamic inefficiency – oversaving?).
  - Apply model to the small open economy (well-defined dynamics for the current account and consumption).
  - Endogenous labour supply (distorting aspects of taxation).
  - Life-cycle labour supply and retirement (ageing and retirement).
- Punchlines.

Yaari's lessons (1)

- *Key questions studied by Yaari*:
  - How does a household behave if it faces *lifetime uncertainty*?
  - What kind of institutions exist to insure oneself against risk of death?
- Up to now we have only studied models without lifetime uncertainty:
  - In the two-period consumption model the agent knows he/she will expire at the end of period 2 (certain death).
  - In the Ramsey model the agent has an infinite horizon (certain eternal life).

Yaari's lessons (2)

- A more realistic scenario:
  - Agent has a finite life.
  - Date of death is uncertain (but demographic data exist).
- Model complications: if date of death is uncertain then...
  - Complication (A): The agent faces a stochastic decision problem. Hence, the *expected utility hypothesis* must be used.
  - Complication (B): The restriction on terminal assets becomes more complicated. If $A(D)$ is real assets at time D and D is the (stochastic) time of death, then the terminal condition is that $A(D) \geq 0$ with probability one.

Complication (A) solved by Yaari (1)

- Even though D is stochastic we have a good idea about the distribution of D in the population (ask the demographers). See **Figures 16.1 – 16.2**. The probability density function (PDF) of D is:

$$\phi(D) \geq 0, \ \forall D \geq 0, \ \ \Phi(\bar{D}) = \int_0^{\bar{D}} \phi(D) dD = 1 \qquad \text{(S1)}$$

  - Densities are non-negative.
  - D is non-negative.
  - $\bar{D}$ is the maximum lifetime.

- Define (stochastic) lifetime utility as:

$$\Lambda(D) \equiv \int_0^D U\left(C(\tau)\right) e^{-\rho\tau} d\tau \qquad \text{(S2)}$$

Complication (A) solved by Yaari (2)

- But since D is stochastic, an agent has the following objective function:

$$E(\Lambda(D)) \equiv \int_0^{\bar{D}} \phi(D)\Lambda\left(D\right) dD$$

- Using (S1) and (S2) we can derive a simple expression for expected lifetime utility:

$$\begin{aligned} E(\Lambda(D)) &\equiv \int_0^{\bar{D}} \phi(D) \left[\int_0^{D} U\left(C(\tau)\right) e^{-\rho\tau} d\tau \right] dD \\ &= \int_0^{\bar{D}} \left[\int_\tau^{\bar{D}} \phi(D) dD \right] U\left(C(\tau)\right) e^{-\rho\tau} d\tau \end{aligned}$$

Complication (A) solved by Yaari (3)

- In compact form we write:

$$E(\Lambda(D)) \equiv \int_0^{\bar{D}} [1 - \Phi(\tau)] \cdot U\left(C(\tau)\right) e^{-\rho\tau} d\tau \qquad \text{(S3)}$$

- In (S3), the term $1 - \Phi(\tau)$ is the probability that the consumer will still be alive at time τ:

$$1 - \Phi(\tau) \equiv \int_\tau^{\bar{D}} \phi(D) dD$$

- The key thing to note about (S3) is that **lifetime uncertainty merely affects the rate at which felicity is discounted! This is Yaari's first lesson**

Complication (B) solved by Yaari (1)

- Let's solve the next complication – dealing with the time-of-death wealth constraint.
- First he derives the appropriate terminal condition on real assets in the presence of lifetime uncertainty (but in the absence of insurance opportunities):

$$A(\bar{D}) = 0 \qquad \text{(S4)}$$

$$C(\tau) \leq w(\tau) \text{ whenever } A(\tau) = 0 \qquad \text{(S5)}$$

- (S4): Assets must be zero if agent reaches maximum age.
- (S5): If agent hits constraint in period τ then he/she must start saving ($\dot{A} > 0$) immediately to avoid defaulting.

Complication (B) solved by Yaari (2)

- Second he shows that the consumption Euler equation is:

$$\frac{\dot{C}(\tau)}{C(\tau)} = \sigma\left(C(\tau)\right) \cdot \left[r(\tau) - \rho - \mu(\tau)\right] \qquad \text{(S6)}$$

where $\mu(\tau)$ is the instantaneous probability of death at time τ:

$$\mu(\tau) \equiv \frac{\phi(\tau)}{1 - \Phi(\tau)} \qquad \text{(S7)}$$

Note: As we saw above, the lifetime uncertainty shows up as a heavier discounting of future felicity (one may not be around to enjoy felicity!). This is Yaari's first lesson again.

Complication (B) solved by Yaari (3)

- Third, he argues that in reality all kind of insurance instruments exist. He introduces the so-called *actuarial note*.
  - Carries instantaneous yield $r^A(\tau)$.
  - If you buy €1 of such notes: yield of $r^A(\tau)$ while you are alive; you lose the principal when you die; yield must be higher than yield on other instruments ($r^A > r$) **ANNUITY**.
  - If you sell such a note: get €1 from life insurance company; pay premium of r^A while you are alive; debt is cancelled when you die; premium must compensate risk of the LIC ($r^A > r$) **LIFE-INSURED LOAN**.

Complication (B) solved by Yaari (4)

- Under *actuarial fairness* the rate of return on the two types of instruments satisfy a no-arbitrage condition:

$$r^A(\tau) = r(\tau) + \mu(\tau) \tag{S8}$$

  The yield on actuarial notes equals the interest rate on traditional assets plus the instantaneous probability of death.
- Fourth, Yaari shows that the household will always fully insure, i.e. will hold real wealth in the form of actuarial notes. This means that...
  - The budget identity is:

$$\dot{A}(\tau) = r^A(\tau)A(\tau) + w(\tau) - C(\tau)$$

  - The terminal asset condition is trivially met (WHY?):
  - The consumption Euler equation is:

$$\frac{\dot{C}(\tau)}{C(\tau)} = \sigma\left[C(\tau)\right] \cdot \left[r^A(\tau) - \rho - \mu(\tau)\right] \tag{S9}$$

Complication (B) solved by Yaari (5)

- Fifth, combining (S8) and (S9) we derive **Yaari's second lesson**:

$$\frac{\dot{C}(\tau)}{C(\tau)} = \sigma\left(C(\tau)\right) \cdot \left[r(\tau) - \rho\right] \tag{S10}$$

With fully insured (actuarially fair) lifetime uncertainty, the death rate drops out of the consumption Euler equation altogether! (**Note**: The level of consumption is affected by the death rate.)

Figure 16.1: Cumulative distribution function

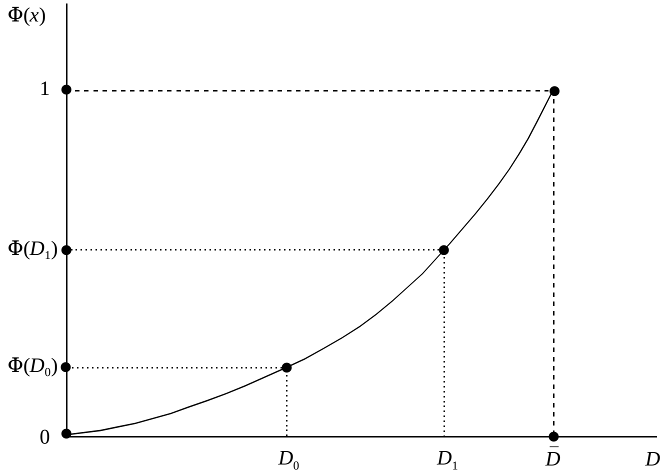
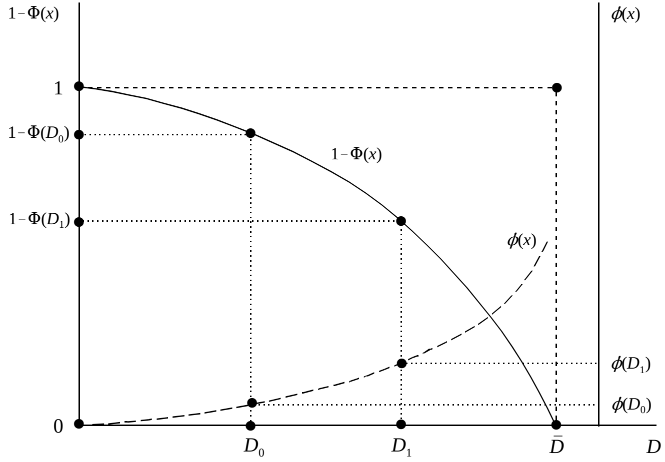


Figure 16.2: Density function and survival probability



Visualization using Dutch demographic data (1)

- Use specific functional form for the demographic process (for $0 \leq u \leq \bar{D} \equiv \frac{\ln \mu_0}{\mu_1}$):

$$\Phi(u) \equiv \frac{e^{\mu_1 u} - 1}{\mu_0 - 1}, \qquad 1 - \Phi(u) \equiv \frac{\mu_0 - e^{\mu_1 u}}{\mu_0 - 1} \qquad \text{(S11)}$$

- Estimate parameters μ_0 and μ_1 using actual demographic data (Netherlands cohort born in 1920): $\hat{\mu}_0 = 41.06$ and $\hat{\mu}_1 = 0.0429$
- Estimated maximum age is $\bar{D} = 86.6$ years
- Life expectancy at birth of 65.4 years.

Visualization using Dutch demographic data (2)

- Recall (S11):

$$\Phi(u) \equiv \frac{e^{\mu_1 u} - 1}{\mu_0 - 1}, \qquad 1 - \Phi(u) \equiv \frac{\mu_0 - e^{\mu_1 u}}{\mu_0 - 1}$$

- From this expression we find (for $0 < u < \bar{D}$):

$$\phi(u) \equiv \frac{d\Phi(u)}{du} = \frac{\mu_1 e^{\mu_1 u}}{\mu_0 - 1} \qquad \text{(S12)}$$

$$\mu(u) \equiv \frac{\phi(u)}{1 - \Phi(u)} = \frac{\mu_1 e^{\mu_1 u}}{\mu_0 - e^{\mu_1 u}} \qquad \text{(S13)}$$

- See Figures 16.3 – 16.4.

Figure 16.3: Logarithm of the instantaneous mortality rate

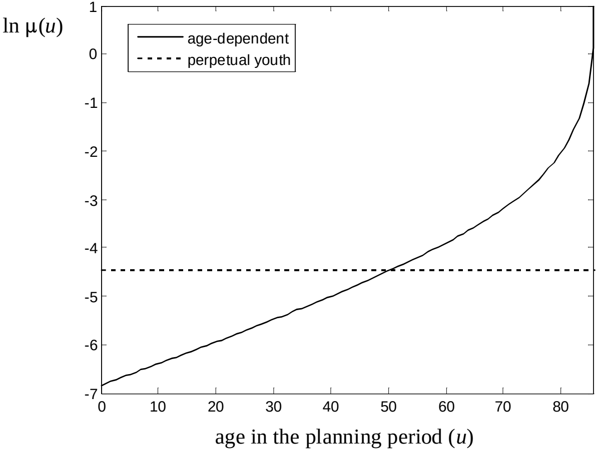
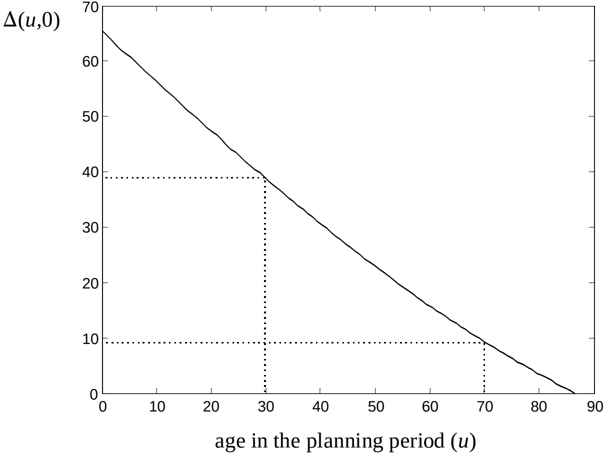


Figure 16.4: Expected remaining lifetime



With actuarially fair (perfect) annuities

- annuity rate facing age-u person is $r + \mu(u)$
- consumption growth is $\dot{C}(u)/C(u) = r - \rho > 0$
- consumption and assets over the life cycle:

$$\frac{C(u)}{w} = \frac{\Delta(0,r)}{\Delta(0,\rho)} e^{(r-\rho)u} \qquad \text{(S14)}$$

$$\frac{A(u)}{w} = e^{(r-\rho)u} \frac{\Delta(0,r)}{\Delta(0,\rho)} \Delta(u,\rho) - \Delta(u,r) \qquad \text{(S15)}$$

- demographic discount function:

$$\Delta(u,\psi) \equiv \frac{e^{\psi u}}{\mu_0 - e^{\mu_1 u}} \cdot \left[\mu_0 \cdot \frac{e^{-\psi u} - e^{-\psi \bar{D}}}{\psi} + \frac{e^{(\mu_1 - \psi)u} - e^{(\mu_1 - \psi)\bar{D}}}{\mu_1 - \psi} \right] \qquad \text{(S16)}$$

- See Figures 16.5 – 16.6.

Figure 16.5: Consumption

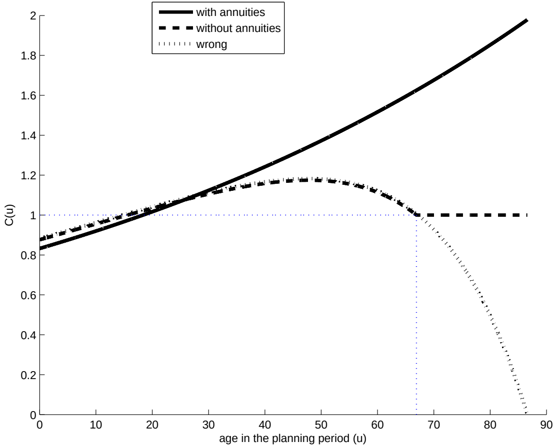
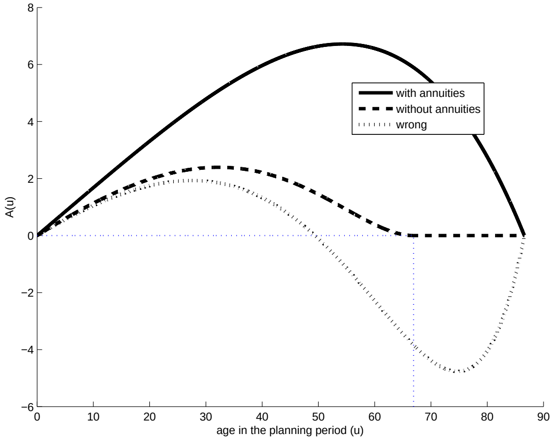


Figure 16.6: Financial assets



In the absence of annuities

- individual faces time-of-death borrowing constraint, $A(u) \geq 0$
- consumption growth is $\dot{C}(u)/C(u) = r - (\rho + \mu(u))$ until borrowing constraint is encountered
- individual runs out of financial assets and consumes wage income thereafter
- See **Figures 16.5 – 16.6**.

Bird's-eye view (1)

- Blanchard (1985): general equilibrium model with finite lives and overlapping generations
- *Key idea*: Blanchard embedded Yaari's approach in a general equilibrium framework. He simplified the approach by assuming that the planning horizon is *age-independent* and is distributed *exponentially* ("perpetual youth" assumption).
- Implications of this assumption:
  - The death rate equals μ (a constant, independent of age).
  - The expected planning horizon equals $1/\mu$ in that case. (**Note:** As $\mu = 0$ we have the Ramsey model again.)
  - Household decision rules linear in age parameter (see below).

Bird's-eye view (2)

- He furthermore assumed that at each instant a *large cohort* of agents in born (bare of any financial assets as they do not receive inheritance–unloved agents). Implications:
  - Denote the cohort born at time τ by $P(\tau,\tau) \equiv \mu P(\tau)$ (with $P(\tau)$ large): the first index is the birth date and the second index is time.
  - All agents face a probability of death of μ so $\mu P(\tau)$ agents die at each instant (#births equals #deaths so population size is constant and $P(\tau)$ can be normalized to unity).
  - With large cohorts "probabilities and frequencies coincide" and given the first assumption we can trace the size of each cohort over time:

$$
\begin{aligned}
P(v,\tau) &= P(v,v)e^{\mu(v-\tau)} \\
&= \mu e^{\mu(v-\tau)}, \quad \tau \geq v
\end{aligned}
$$

Bird's-eye view (3)

- Because we know cohort sizes we can aggregate all surviving households (nice for macro model).
- Eventually, as people die off the cohorts vanish.
- We can now derive the implications for individual and aggregate household behaviour. Details are in the chapter. Sketch of the outcome here.

Individual household behaviour (1)

- Expected lifetime utility of agent of cohort v in period t:

$$\begin{aligned} E(\Lambda(v,t)) &\equiv \int_t^\infty [1-\Phi(\tau-t)] \ln C(v,\tau) e^{\rho(t-\tau)} d\tau \\ &= \int_t^\infty \ln C(v,\tau) e^{(\rho+\mu)(t-\tau)} d\tau \end{aligned}$$

- Budget identity:

$$\dot{A}(v,\tau) = [r(\tau)+\mu]\, A(v,\tau) + w(\tau) - T(\tau) - C(v,\tau) \quad \text{(S18)}$$

- No Ponzi Game (NPG) condition:

$$\lim_{\tau\to\infty} e^{-R^A(t,\tau)} A(v,\tau) = 0, \quad R^A(t,\tau) \equiv \int_t^\tau [r(s)+\mu]\, ds$$

Individual household behaviour (2)

- Decision rule for consumption:

$$C(v,t) = (\rho+\mu)\left[A(v,t)+H(t)\right] \quad \text{(S19)}$$

$$H(t) \equiv \int_t^{\infty} \left[w(\tau)-T(\tau)\right] e^{-R^A(t,\tau)} d\tau \quad \text{(S20)}$$

- Notes:
  - Marginal propensity to consume out of total wealth is $\rho+\mu$ (does not feature an age index due to the perpetual youth assumption).
  - Human wealth discounted at the annuity rate of interest, $r(\tau)+\mu$.

Aggregate household behaviour (1)

- We know that the size of cohort v at time t is $\mu e^{\mu(v-t)}$. This means that we can define aggregate variables by aggregating over all existing agents at time t. For example, aggregate consumption is:

$$C(t) \equiv \mu \int_{-\infty}^{t} e^{\mu(v-t)} C(v,t) dv$$

- In view of (S19) *aggregate consumption* satisfies:

$$\begin{aligned} C(t) &\equiv \mu \int_{-\infty}^{t} e^{\mu(v-t)} (\rho+\mu) \left[A(v,t) + H(t) \right] dv \\ &= (\rho+\mu) \Big[\underbrace{\mu \int_{-\infty}^{t} e^{\mu(v-t)} A(v,t) dv}_{A(t)} + \underbrace{\mu \int_{-\infty}^{t} e^{\mu(v-t)} H(t) dv}_{H(t)} \Big] \\ &= (\rho+\mu) \left[A(t) + H(t) \right] \end{aligned}$$

Aggregate household behaviour (2)

- Similarly, the *aggregate budget identity* can be derived:

$$\dot{A}(t) = r(t)A(t) + w(t) - T(t) - C(t) \qquad \text{(S21)}$$

  The market rate of interest (**not** the annuity rate) features in the aggregate budget identity: the term $\mu A(t)$ is a transfer–via the life insurance companies–from agents who die to agents who stay alive.

- Recall (S18) (for period t):

$$\dot{A}(v,t) = [r(t) + \mu]\, A(v,t) + w(t) - T(t) - C(v,t)$$

Aggregate household behaviour (3)

- The consumption Euler equation for individual agents is:

$$\frac{\dot{C}(v,t)}{C(v,t)} = r(t) - \rho$$

The "aggregate Euler equation" satisfies:

$$\begin{aligned}\frac{\dot{C}(t)}{C(t)} &= [r(t) - \rho] - \mu(\rho + \mu)\frac{A(t)}{C(t)} \\ &= \frac{\dot{C}(v,t)}{C(v,t)} - \mu\frac{C(t) - C(t,t)}{C(t)}\end{aligned}$$

- Note: Aggregate consumption growth differs from individual consumption growth due to the turnover of generations. Newborns are poorer than the average household and therefore drag down aggregate consumption growth.

Phase diagram of the Blanchard-Yaari model (1)

- We now have all the ingredients of the BY model (firm behaviour is standard; we allow for debt creation in the GBC): see **Table 16.1**.
- In **Figure 16.7** we show the phase diagram for a special case of the BY model, under the assumption that there is no government at all ($T(t) = G(t) = B(t) = 0$).
- The $\dot{K} = 0$ line represents (C, K) combinations for which net investment is zero. It has the usual properties:
  - Golden rule point at A_2.
  - $\dot{K} > 0$ ($\dot{K} < 0$) for points below (above) the $\dot{K} = 0$ line (see horizontal arrows).

Phase diagram of the Blanchard-Yaari model (2)

- The $\dot{C} = 0$ line represents (C, K) combinations for which *aggregate* consumption is constant over time. It has some unusual properties:
  - It lies entirely to the left of the dashed line, representing the Keynes-Ramsey capital stock (for which $r^{KR} = \rho$). Why? Using the aggregate Euler equation for the BY model we get:

$$\begin{aligned} \frac{\dot{C}(t)}{C(t)} &= [r(t) - \rho] - \mu(\rho + \mu)\frac{K(t)}{C(t)} = 0 \quad \Rightarrow \\ r^{BY} - \rho &= \mu(\rho + \mu)\frac{K}{C}^{BY} \quad \Rightarrow \\ r^{BY} &> \rho \end{aligned}$$

The interest rate strictly higher than ρ (due to generational turnover). Hence, K^{BY} strictly smaller than K^{KR}.

Phase diagram of the Blanchard-Yaari model (3)

- Continued.
  - The $\dot{C} = 0$ line is upward sloping. Can be understood by comparing points E_0, B, and C in **Figure 16.7**. In E_0 and B r is the same but K/C is higher in B. To restore $\dot{C} = 0$ we must have a move to point C, where K is lower than in B (r higher) and K/C is lower.
  - For points above (below) the $\dot{C} = 0$ line, the generational turnover effect is too low (too strong), and aggregate consumption growth is positive (negative). See the vertical arrows in Figure 16.7.
- The BY model without a government features a unique equilibrium at E_0 which is saddle point stable.

Table 16.1: The Blanchard-Yaari model

$$\dot{C}(t) = [r(t) - \rho]\, C(t) - \mu(\rho + \mu)\, [K(t) + B(t)] \qquad \text{(T1.1)}$$

$$\dot{K}(t) = Y(t) - C(t) - G(t) - \delta K(t) \qquad \text{(T1.2)}$$

$$\dot{B}(t) = r(t)B(t) + G(t) - T(t) \qquad \text{(T1.3)}$$

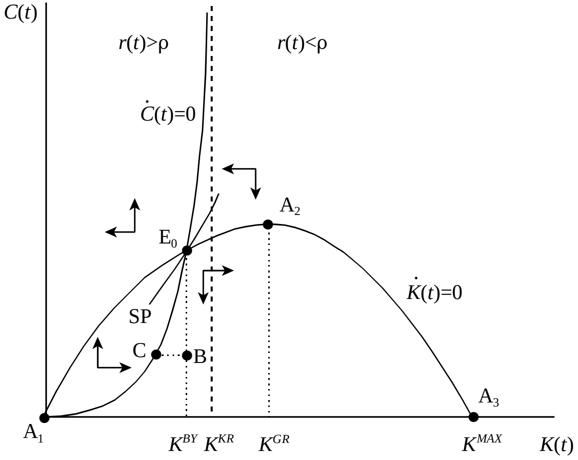
$$r(t) + \delta = F_K(K(t), L(t)) \qquad \text{(T1.4)}$$

$$w(t) = F_L(K(t), L(t)) \qquad \text{(T1.5)}$$

$$L(t) = 1 \qquad \text{(T1.6)}$$

$$Y(t) = F(K(t), L(t)) \qquad \text{(T1.7)}$$

Figure 16.7: Phase diagram of the Blanchard-Yaari model



Some basic model properties

- *Fiscal policy*: increase in government consumption financed by means of lump-sum taxes. Issues:
  - Crowding out of private by public consumption?
  - Intergenerational redistribution of resources? How does this work?
- *Non-neutrality of debt*.
  - Does government debt matter?
  - Do deficit-financed policies differ from balanced-budget policies?

Fiscal policy (1)

- Unanticipated and permanent increase in G financed by increase in T (recall T is the same for all agents, regardless of their vintage).
- Abstract from government debt: $\dot{B} = B = 0$ and GBC is static, $G = T$.
- The shock is analyzed in **Figure 16.8**.
  - The $\dot{K} = 0$ line shifts down by the amount of the shock.
  - The $\dot{C} = 0$ line is unchanged (no supply effect of tax).
  - Steady state shifts from E_0 to E_1: $C(\infty) \downarrow$ and $K(\infty) \downarrow$ (the latter does not occur in Ramsey model).

Fiscal policy (2)

- Continued.
  - Transitional dynamics: jump from E_0 to A (at impact) followed by gradual move along saddle path from A to E_1 thereafter. (Recall: no t.d. in Ramsey model.)
  - Crowding out results:

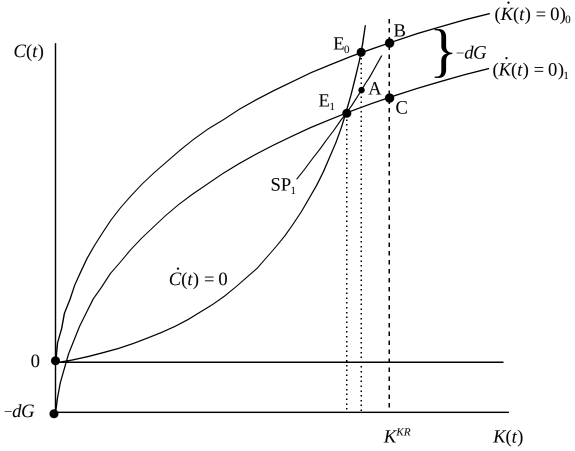
$$
\begin{aligned}
-1 &< \frac{dC(0)}{dG} < 0 \\
\frac{dC(\infty)}{dG} &< -1
\end{aligned}
$$

Less than one-for-one at impact but more than one-for-one in the long run!

Fiscal policy (3)

- Economic intuition: the $T \uparrow$ causes an intergenerational redistribution of resources away from future towards present generations.
  - At impact $C(v,0) \downarrow$ because $H(0) \downarrow$ (due to $T \uparrow$).
  - Households discount net labour income stream, $w - T$, by annuity rate $r + \mu$ (higher than market interest rate, r).
  - Hence, the drop in $C(v,0)$, $C(0)$, and $H(0)$ is not large enough, so that private investment in crowded out: $\dot{K}(0) \downarrow$.
  - Over time $K(t) \searrow$, so that $[w(t) - T] \searrow$, $r(t) \nearrow$, and $H(t) \searrow$.
  - Future newborns poorer than newborns in initial steady state (the former have less capital to work with).

Figure 16.8: Fiscal policy in the Blanchard-Yaari model



Non-neutrality of debt (1)

- The fact that T causes intergenerational redistribution in the fiscal policy case hints at the non-neutrality of debt.
- Ricardian non-equivalence can be proven by looking at a simple accounting exercises: substitute the GBC into the HBC.
- The aggregate wealth constraint facing household features the following definition for total wealth:

$$\begin{aligned} A(t) + H(t) &\equiv K(t) + B(t) + H(t) \\ &= K(t) + B(t) + \int_t^{\infty} [w(\tau) - T(\tau)]\, e^{-R^A(t,\tau)} d\tau \\ &= K(t) + \int_t^{\infty} [w(\tau) - G(\tau)]\, e^{-R^A(t,\tau)} d\tau + \Omega(t) \end{aligned}$$

Non-neutrality of debt (2)

- Here $\Omega(t)$ is defined as:

$$\Omega(t) \equiv B(t) - \int_t^{\infty} [T(\tau) - G(\tau)] \underbrace{e^{-R^A(t,\tau)}}_{(a)} d\tau \qquad \text{(S22)}$$

Note: Ricardian equivalence holds iff $\Omega(t) \equiv 0$!

- Recall that the GBC can be written as:

$$0 = B(t) - \int_t^{\infty} [T(\tau) - G(\tau)] \underbrace{e^{-R(t,\tau)}}_{(b)} d\tau \qquad \text{(S23)}$$

- In (S22) primary surpluses are discounted with the annuity rate (see (a)) whereas the market rate is used in (S23) (see (b)).
  - Hence, $\Omega(t)$ only vanishes iff the birth rate is zero, so that $R^A(t,\tau) = R(t,\tau)$, i.e. in the Ramsey model.
  - If $\mu > 0$ then $\Omega(t) \neq 0$ and Ricardian equivalence fails: the path of $T(\tau)$ and the initial debt level do not drop out of the aggregate HBC.