# Foundations of Modern Macroeconomics Second Edition

## Chapter 17: Overlapping generations in discrete time (sections 17.3 – 17.4)

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# Outline

1. Human capital accumulation
   - Azariadis-Drazen model
   - Eckstein-Zilcha model
2. Public investment
3. Endogenous fertility

# Extensions to the basic Diamond-Samuelson model

- Human capital accumulation.
    - Automatic knowledge transfer and endogenous growth.
    - A family externality and the benefit of a mandatory education system.
- Public investment.
    - Macroeconomic effects.
    - Some modified golden rules.
- Endogenous fertility.
    - What determines the population growth rate?
    - Is the Ricardian Equivalence Theorem still valid?

# Human capital accumulation

- Human capital creation may be an important engine of growth in the economy.
- We study an OLG version of the Lucas-Uzawa model (also studied in Chapter 14) proposed by Azariadis-Drazen.

# Households: utility

- Work full time during second period of life.
- Divide time between training and working during first period.
- Lifetime utility:

$$\Lambda_t^{Y,i} \equiv \Lambda^Y (C_t^{Y,i}, C_{t+1}^{O,i})$$

  - No direct utility attached to leisure and to training (knowledge not value *per se*).

# Households: constraints

- Budget identities:

$$
\begin{aligned}
C_t^{Y,i} + S_t^i &= w_t H_t^i N_t^i \\
C_{t+1}^{O,i} &= (1 + r_{t+1}) S_t^i + w_{t+1} H_{t+1}^i
\end{aligned}
$$

  - $w_t$ is the wage rate per efficiency unit of labour.
  - $H_t^i$ is the level of human capital of worker $i$ at time $t$.
  - $N_t^i$ is the amount of time spent working (rather than training) during youth.
  - $C_t^{Y,i}$, $C_{t+1}^{O,i}$, and $S_t^i$ have their usual meaning.
- Time constraint during youth:

$$
E_t^i = 1 - N_t^i \geq 0
$$

  - Time endowment is unity.
  - $E_t^i$ is time spent on training during youth.

# Households: constraints

- Training technology:

$$H_{t+1}^{i} = G(E_t^i) \cdot H_t^i$$

  - Positive but non-increasing returns to training ($G' > 0 \geq G''$).
  - No knowledge depreciation ($G(0) = 1$).
- Household optimization in two steps:
  - Choose training level to maximize lifetime income.
  - Choose consumption and savings (subject to lifetime income).

# Step 1: Training decision (1)

- Household chooses $E_t^i$ such that lifetime income is maximized:

$$I_t^i(E_t^i) \equiv H_t^i \cdot \left[ w_t(1 - E_t^i) + \frac{w_{t+1} G(E_t^i)}{1 + r_{t+1}} \right]$$

- First-order (Kuhn-Tucker) condition:

$$\begin{aligned} \frac{dI_t^i}{dE_t^i} &= H_t^i \cdot \left[ -w_t + \frac{w_{t+1} G'(E_t^i)}{1 + r_{t+1}} \right] \leq 0 \\ E_t^i &\geq 0, \qquad E_t^i \cdot \frac{dI_t^i}{dE_t^i} = 0 \end{aligned}$$

# Step 1: Training decision (2)

- Two possible solutions.
  - No-training solution:

$$G'(0) < \frac{w_t(1+r_{t+1})}{w_{t+1}} \qquad \Rightarrow \qquad E_t^i = 0$$

    Corner solution because training technology not productive enough!
  - Training solution:

$$E_t^i > 0 \qquad \Rightarrow \qquad 1 + r_{t+1} = \frac{w_{t+1}}{w_t} \cdot G'(E_t^i)$$

    Invest in human capital until its yield equals the yield on financial assets.

# Step 2: Consumption-saving decision

- Household chooses $C_t^{Y,i}$, $C_{t+1}^{O,i}$ and $S_t^i$ in order to maximize lifetime utility subject to the lifetime budget constraint:

$$C_t^{Y,i} + \frac{C_{t+1}^{O,i}}{1 + r_{t+1}} = I_t^i$$

where $I_t^i$ is now maximized lifetime income (see Step 1).

- Key expression is the savings function:

$$S_t^i = S\Big(r_{t+1}, (1 - E_t^i)w_t H_t^i, w_{t+1} H_{t+1}^i\Big)$$

# Further elements of the model

- Initial condition for household $i$:

$$H_t^i = H_t$$

  Household "inherits" average level of human capital in the economy (*osmotic* human capital transfer across generations).
- Model is symmetric so index $i$ can be dropped.
- Constant population ($L_t = L_{t-1} = 1$).
- Total labour supply in efficiency units is $N_t = (1 - E_t)H_t + H_t$.

# Table 17.2: Growth, human capital, and overlapping generations

$$
\begin{aligned}
N_{t+1}k_{t+1} &= S(r_{t+1}, (1-E_t)w_t H_t, w_{t+1}H_{t+1}) && \text{(T2.1)} \\
r_{t+1} + \delta &= f'(k_{t+1}) && \text{(T2.2)} \\
w_t &= f(k_t) - k_t f'(k_t) && \text{(T2.3)} \\
N_t &= (2 - E_t) H_t && \text{(T2.4)} \\
1 + r_{t+1} &= \frac{w_{t+1}}{w_t} G'(E_t) && \text{(T2.5)} \\
H_{t+1} &= G(E_t) H_t && \text{(T2.6)}
\end{aligned}
$$

# Summary of the Azariadis-Drazen model (1)

- Model displays endogenous growth in the steady state. This is illustrated in **Figure 17.7** for the unit-elastic case with technology and log-linear utility:

$$y_t = k_t^{1-\varepsilon_L}$$

$$\Lambda_t^Y = \ln C_t^Y + (1/(1+\rho)) \ln C_{t+1}^O.$$

  - PB is portfolio balance line: $(k, E)$ combinations for which yields on physical and human capital equalize.
  - SI is the savings-investment line: $(k, E)$ combinations for which savings equals investment.
  - equilibrium at $E_0$.
  - Growth rate is $\gamma \equiv G(E) - 1$ is depicted in bottom panel.

# Summary of the Azariadis-Drazen model (2)

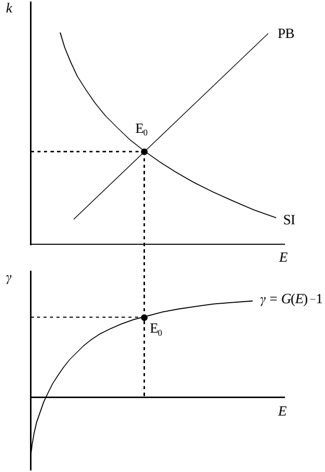
- Engine of growth in the model is the training technology:

$$H^i_{t+1} = G(E^i_t)H^i_t$$

.

  - *Level* of training explains *growth rate* in human capital.
  - Knowledge/technical skills are disembodied (live on after agent has died).
  - Edogenous growth vanishes if knowledge/technical skills are embodied.

# Figure 17.7: Endogenous growth due to human capital formation



# Choosing your offspring's education (1)

- Eckstein-Zilcha: why do we have compulsory education systems?
- *Key idea*: Parents may under-invest in the human capital of their children (*intra*-family external effect).
- We discuss a simple version of the E-Z model to demonstrate underinvestment result.
- Utility function:

$$\Lambda_t^Y \equiv \Lambda^Y(C_t^Y, C_{t+1}^O, Z_t, O_{t+1})$$

  - $C_t^Y$ is consumption when young.
  - $C_{t+1}^O$ is consumption when old.
  - $Z_t$ is leisure during youth.
  - $O_{t+1} \equiv (1+n)H_{t+1}$ is total human capital of the agent's offspring ($H_{t+1}$ is human capital per child, $1+n$ is the number of children).

# Choosing your offspring's education (2)

- In-house training technology (no schools).

$$H_{t+1} = G(E_t) \cdot H_t^{\beta}$$

  - $E_t$ is educational effort per child.
  - $G(\cdot)$ is the training curve (satisfying $0 < G(0) \leq 1$, $G(1) > 1$, $G' > 0 \geq G''$).
  - Positive but diminishing marginal product of human capital: $0 < \beta \leq 1$.
  - Note difference with A-D model: now parent chooses human capital of children (costs and benefits accrue to different agents).

# Choosing your offspring's education (3)

- Time endowment: households has two units of time available:
  - One unit is supplied inelastically to the labour market.
  - One is divided over leisure and training: $Z_t + (1+n)E_t = 1$.
- Household's lifetime budget constraint:

$$C_t^Y + \frac{C_{t+1}^O}{1+r_{t+1}} = w_t \cdot H_t$$

  - $w_t = F_N\left(K_t, N_t\right)$ where $N_t \equiv L_t H_t$ (efficiency units of labour).
  - $r_t + \delta = F_K\left(K_t, N_t\right)$.

# Choosing your offspring's education (4)

- Household chooses $C_t^Y$, $C_{t+1}^O$, $Z_t$, $E_t$, and $H_{t+1}$ to maximize lifetime utility subject to (a) the training technology, (b) the time constraint, and (c) the consolidated budget constraint. Key first-order conditions:

$$\begin{aligned}
\frac{\partial \Lambda^Y / \partial C_t^Y}{\partial \Lambda^Y / \partial C_{t+1}^O} &= 1 + r_{t+1} \\
\frac{\partial \Lambda^Y}{\partial O_t} G'(E_t) H_t^{\beta} - \frac{\partial \Lambda^Y}{\partial Z_t} &< 0 \implies E_t = 0 \\
\frac{\partial \Lambda^Y}{\partial O_t} G'(E_t) H_t^{\beta} - \frac{\partial \Lambda^Y}{\partial Z_t} &= 0 \impliedby E_t > 0 \quad \text{(S1)}
\end{aligned}$$

  - Corner solution if the net marginal benefit of training is negative.
  - For interior solution training provided until net marginal benefit of training is zero (all gains exhausted).

# Choosing your offspring's education (5)

- Assume that interior solution (S1) obtains. Note that (S1) only contains costs and benefits of the parent! First hint at underinvestment problem. Not all benefits are taken into account.
- Formal analysis of underinvestment problem: Social Welfare Function approach.
- The social welfare function is:

$$SW_0 \equiv \sum_{t=0}^{\infty} \lambda_t \Lambda_t^Y = \sum_{t=0}^{\infty} \lambda_t \Lambda^Y(C_t^Y, C_{t+1}^O, Z_t, O_{t+1})$$

  - $SW_0$ is social welfare in the planning period ($t = 0$).
  - $\{\lambda_t\}_{t=0}^{\infty}$ is a positive monotonically decreasing sequence of weights attached to the different generations (which satisfies $\sum_{t=0}^{\infty} \lambda_t < \infty$).

# Choosing your offspring's education (6)

- Resource constraint:

$$C_t^Y + \frac{C_t^O}{1+n} + (1+n)k_{t+1} = F(k_t, H_t) + (1-\delta)k_t$$

where $k_t \equiv K_t/L_t$.

- Social planner chooses sequences for consumption ($\{C_t^Y\}_{t=0}^{\infty}$ and $\{C_{t+1}^O\}_{t=0}^{\infty}$), the stocks of human and physical capital ($\{K_{t+1}\}_{t=0}^{\infty}$ and $\{H_{t+1}\}_{t=0}^{\infty}$), and the educational effort ($\{E_t\}_{t=0}^{\infty}$) in order to maximize $SW_0$ subject to (a) the training technology, (b) the time constraint, and (c) the resource constraint.

# Choosing your offspring's education (7)

- Most interesting (for our purposes) first-order condition:

$$\begin{aligned}\frac{\partial \Lambda^Y(\hat{x}_t)}{\partial Z_t} &= G'(\hat{E}_t)\hat{H}_t^{\beta} \cdot \left[\frac{\partial \Lambda^Y(\hat{x}_t)}{\partial O_t}\right. \\ &\quad + \frac{\partial \Lambda^Y(\hat{x}_t)}{\partial C_{t+1}^O} F_N(\hat{k}_{t+1}, \hat{H}_{t+1}) \\ &\quad \left. + \frac{\beta(1+n)\hat{H}_{t+2}}{G'(\hat{E}_{t+1})\hat{H}_{t+1}^{1+\beta}} \frac{\partial \Lambda^Y(\hat{x}_t)}{\partial C_{t+1}^O} \cdot \frac{\partial \Lambda^Y(\hat{x}_{t+1})/\partial Z_{t+1}}{\partial \Lambda^Y(\hat{x}_{t+1})/\partial C_{t+1}^Y}\right]\end{aligned}$$

  - Marginal social costs (LHS) must be equated to marginal social benefits (RHS).
  - Marginal social benefits consist of three terms:
    - "Own" term, affecting decision maker directly (line 1).
    - "Induced" term, affecting earning power of children (line 2).
    - "Induced" term, affecting incentive of children to educate *their* children provide (line 3).

# Choosing your offspring's education (8)

- Second and third effects are ignored by parents which leads to under-investment in human capital. Policy options:
  - Complex set of incentives (taxes/subsidies) to correct the parent's behaviour.
  - Compulsory education.

# Public investment

- Empirical work by Aschauer prompts a number of questions:
  - What are the macroeconomic effects of public investment?
  - How much public capital should a country possess?
- Study these questions with a modified D-S model.
  - Exogenous labour supply / lump-sum taxes.
  - Public capital is a stock variable.
  - Public capital affects factor productivity (e.g. bridges, roads, airports, etc.).

# Public investment

- Accumulation identity:

$$G_{t+1} - G_t = I_t^G - \delta_G G_t$$

  - $G_{t+1}$ is public capital stock.
  - $I_t^G$ is public investment.
  - $\delta_G$ is depreciation rate on public capital.

- Technology:

$$Y_t = F(K_t, L_t, g_t)$$

  where $g_t \equiv G_t / L_t$.
  - CRTS in $(K_t, L_t)$.
  - Positive but diminishing marginal product of public capital, $F_g > 0$, $F_{gg} < 0$.

# Public investment

- Competitive production yields rental expressions:

$$\begin{aligned} r_t &= r(k_t, g_t) \equiv f_k(k_t, g_t) - \delta, \\ w_t &= w(k_t, g_t) \equiv f(k_t, g_t) - k_t f_k(k_t, g_t), \end{aligned}$$

  where $f(k_t, g_t) \equiv F(K_t/L_t, 1, g_t)$ is the intensive-form production function. By assumption, $g_t$ affects $r_t$ and $w_t$ positively. (Example: $Y_t = K_t^{1-\varepsilon_L} L_t^{\varepsilon_L} g_t^{\eta}$ with $0 < \eta < \varepsilon_L$.)

- Household utility:

$$\Lambda_t^Y = \ln C_t^Y + \frac{1}{1+\rho} \ln C_{t+1}^O$$

- Household lifetime budget constraint:

$$\hat{w}_t \equiv w_t - T_t^Y - \frac{T_{t+1}^O}{1+r_{t+1}} = C_t^Y + \frac{C_{t+1}^O}{1+r_{t+1}}$$

# Public investment

- Savings function:

$$S_t = (1 - c)\left(w_t - T_t^Y\right) + c\frac{T_{t+1}^O}{1 + r_{t+1}}$$

where $c \equiv \frac{1+\rho}{2+\rho}$.

# Model summary (1)

- Model is:

$$(1+n)g_{t+1} = i_t^G + (1-\delta_G)g_t \qquad \text{(S2)}$$

$$i_t^G = T_t^Y + \frac{T_t^O}{1+n} \qquad \text{(S3)}$$

$$(1+n)k_{t+1} = (1-c)\left[W(k_t,g_t) - T_t^Y\right] + \frac{cT_{t+1}^O}{1+r(k_{t+1},g_{t+1})} \qquad \text{(S4)}$$

  - Eq. (S2): Accumulation identity for public capital per worker ($i_t^G \equiv I_t^G/L_t$).
  - Eq. (S3): Government budget constraint.
  - Eq. (S4): Link between savings and private capital formation.
- Immediately obvious that financing method critically affects the model: who pays for $i_t^G$ affects (S4) and thus the private capital stock and the rest of the economy.

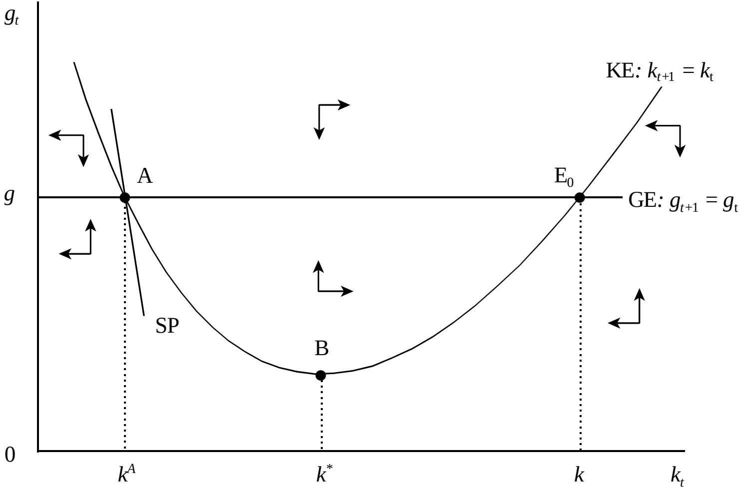
# Model summary (2)

- In Figure 17.8 we consider the case in which the old are untaxed ($T_t^O = 0$ for all $t$).
  - GE line: $(g, k)$ combinations for which $g_{t+1} = g_t$.
    - Horizontal.
    - $g_t$ rises (falls) for points above (below) the GE line–see the vertical arrows.
  - KE line: $(g, k)$ combinations for which $k_{t+1} = k_t$.
    - Slope is negative (positive) for low (high) private capital stock. (Intuition: slope determined by $1 + n - (1 - c) W_k$; $W_k$ high (low) if $k$ is low (high).)
    - $k_t$ increases (decreases) for points above (below) the KE line–see the horizontal arrows.
  - Two steady-state equilibria:
    - $\mathrm{E}_0$: stable node (stable monotonic or cyclical adjustment).
    - A: saddle point (unstable because both $g_t$ and $k_t$ are predetermined variables).

# Model summary (3)

- Focus on equilibrium $\mathrm{E}_0$: what happens if $i_t^G$ is increased?
  - Both GE and KE shift up.
  - Effect on $g$ unambiguously positive.
  - Effect on $k$ depends on relative scarcity of public capital ($w \uparrow$ but $T_t^Y \uparrow$ so net effect ambiguous): $k$ rises (falls) if $i^G/y < \eta\varepsilon_L$ ($> \eta\varepsilon_L$), i.e. if public capital is initially relatively scarce (abundant).

# Figure 17.8: Public and private capital



# How much public capital should a country have? (1)

- SWF approach in an OLG setting.
- Social welfare function:

$$SW_0 \equiv \left(\frac{1+n}{1+\rho_G}\right)^{-1} \Lambda^Y(C_{-1}^Y, C_0^O) + \sum_{t=0}^{\infty} \left(\frac{1+n}{1+\rho_G}\right)^t \Lambda^Y(C_t^Y, C_{t+1}^O)$$

  - Benthamite format ("...greatest happiness of the greatest number...").
  - $\rho_G$ is the planner's discount rate ($\rho_G > n$). May or may not equal $\rho$.
  - Special treatment of current old generation to avoid dynamic inconsistency of the social optimum (see Intermezzo).
- Resource constraint:

$$C_t^Y + \frac{C_t^O}{1+n} + (1+n)\left[k_{t+1} + g_{t+1}\right] = f(k_t, g_t) + (1-\delta)k_t + (1-\delta_G)g_t$$

# How much public capital should a country have? (2)

- Social planner chooses $\{C_t^Y\}_{t=0}^{\infty}$, $\{C_t^O\}_{t=0}^{\infty})$, $\{g_{t+1}\}_{t=0}^{\infty}$, and $\{k_{t+1}\}_{t=0}^{\infty})$, in order to maximize $SW_0$ subject to the resource constraint, taking $k_0$ and $g_0$ as given.
- Key first-order conditions for the social optimum:

$$\frac{\partial \Lambda^Y(\hat{x}_t)/\partial C_t^Y}{\partial \Lambda^Y(\hat{x}_t)/\partial C_{t+1}^O} = 1 + \hat{r}_{t+1} \tag{S5}$$

$$\hat{r}_{t+1} = f_k(\hat{k}_{t+1}, \hat{g}_{t+1}) - \delta = f_g(\hat{k}_{t+1}, \hat{g}_{t+1}) - \delta_G \tag{S6}$$

$$\frac{\partial \Lambda^Y(\hat{x}_t)/\partial C_t^Y}{\partial \Lambda^Y(\hat{x}_{t-1})/\partial C_t^O} = 1 + \rho_G \tag{S7}$$

  - Eq. (S5): Socially optimal Euler equation; MRS between present and future consumption equated to gross interest factor.
  - Eq. (S6): Yields on private and public capital should be equalized (efficient investment).

# How much public capital should a country have? (3)

- Continued.
  - Eq. (S7): $\rho_G$ determines optimal *intra*temporal division of consumption. With additively separable preferences we get:

$$\frac{U'(\hat{C}_t^Y)}{U'(\hat{C}_t^O)} = \frac{1+\rho_G}{1+\rho}$$

    - If $\rho_G > \rho$ then planner ensures that $U'(\hat{C}_t^Y) > U'(\hat{C}_t^O)$ i.e. that $\hat{C}_t^Y < \hat{C}_t^O$ (favour the old).
    - If $\rho_G = \rho$ then planner ensures that $U'(\hat{C}_t^Y) = U'(\hat{C}_t^O)$ i.e. that $\hat{C}_t^Y = \hat{C}_t^O$ (egalitarian solution).
    - If $\rho_G < \rho$ then planner ensures that $U'(\hat{C}_t^Y) < U'(\hat{C}_t^O)$ i.e. that $\hat{C}_t^Y > \hat{C}_t^O$ (favour the young).

# Some final remarks on public capital

- In the steady state, $\hat{r}_t = \rho_G$ so (b) simplifies to:

$$[\hat{r} \equiv] \quad f_k(k,g) - \delta = \rho_G = f_g(k,g) - \delta_G$$

  Hence, modified golden rules for private and public capital accumulation feature the social planner's discount rate.
- First-best social optimum can be decentralized if and only if the right policy instruments are available:
  - $i^G$ (and thus $g$) is set correctly.
  - *Age-specific* lump-sum taxes are available (even stronger requirement than in the representative-agent model).
- If one or more of the policy variables is not available, the problem becomes a second-best optimization problem (Ramsey taxation, modified Samuelson rule).

# Endogenizing the birth rate

- Simplified version of the Lapan-Enders model
- Large number of dynastic families
- Youth:
  - fully dependent on parent
  - no economic decisions
- Adulthood:
  - inherits wealth from parent
  - supplies one unit of labour
  - decides on consumption
  - decides on number of kids (born at beginning of the period)
  - decides on bequest to each child
- There are $L_t$ adults

# Choices of adult $i$

- Lifetime utility of adult $i$:

$$\Lambda_t^i \equiv U(c_t^i, n_t^i) + \alpha \Lambda_{t+1}^i, \qquad 0 < \alpha < 1$$

  - $\alpha$ is the altruism parameter
  - $c_t^i$ is consumption
  - $n_t^i$ is the number of children
  - $\Lambda_{t+1}^i$ is the *maximized* lifetime utility per child
- Budget constraint of adult $i$:

$$(1 + r_t) \cdot a_t^i + w_t = c_t^i + tax_t^i + n_t^i \cdot \left[\bar{c} + a_{t+1}^i\right]$$

  - $r_t$ is the real interest rate
  - $a_t^i$ is the bequest received at the beginning of adulthood
  - $w_t$ is the wage rate
  - $tax_t^i$ is the lump-sum tax
  - $\bar{c}$ is the cost of raising a child
  - $a_{t+1}^i$ is the bequest granted to each child at the end of life

# Dynastic choices

- Provided bequests remain operative ($a^i_{t+\tau} > 0$ for $\tau = 1, 2, \cdots$) we have an "as if" infinitely-lived agent with lifetime utility function:

$$\Lambda^i_t \equiv \sum_{\tau=0}^{\infty} \alpha^\tau U(c^i_{t+\tau}, n^i_{t+\tau})$$

- Choice variables: $c^i_{t+\tau}$, $n^i_{t+\tau}$, and $a^i_{t+\tau}$.
- First-order condition for consumption:

$$\frac{U_c(c^i_{t+\tau+1}, n^i_{t+\tau+1})}{U_c(c^i_{t+\tau}, n^i_{t+\tau})} = \frac{n^i_{t+\tau}}{\alpha\,[1 + r_{t+\tau+1}]} \qquad \text{(S8)}$$

  - Eq. (S8): Euler equation depends on the capital interest rate, the biological interest rate, and the altruism parameter ("impatience")

# Dynastic choices

- First-order conditions for kids:

$$\frac{U_n(c^i_{t+\tau}, n^i_{t+\tau})}{U_c(c^i_{t+\tau}, n^i_{t+\tau})} = \bar{c} + a^i_{t+\tau+1} \tag{S9}$$

  - Eq. (S9): marginal benefit of a kid equals marginal cost

- Two financial assets; capital and government bonds (perfect substitutes):

$$a^i_{t+\tau} = k^i_{t+\tau} + b^i_{t+\tau}$$

# Aggregate outcomes

- future population:

$$L_{t+1} \equiv \sum_{i=1}^{L_t} n_t^i = \bar{n}_t L_t$$

- debt per adult:

$$\bar{n}_t b_{t+1} = (1 + r_t)\, b_t + g_t - tax_t$$

- features of production:

$$\begin{aligned} y_t &= f(k_t) \\ w_t &= f(k_t) - k_t f'(k_t) \\ r_{t+1} + \delta &= f'(k_{t+1}) \end{aligned}$$

# Ricardian Equivalence Theorem

- Ricardian Equivalence Theorem valid iff all conditions are satisfied:
  (a) The chain of bequests is unbroken, i.e. $a^i_{t+\tau} > 0$ for all $\tau$ and $i$. This ensures that each dynasty is effectively infinitely lived;
  (b) Fertility is not a choice variable but is exogenously given, i.e. $n^i_{t+\tau} = n_0$, where $n_0$ is exogenous (and assumed to be constant for notational convenience);
  (c) The government does not engage in redistribution between dynasties, i.e. $tax^i_{t+\tau} = tax_{t+\tau}$ for all $i$ and $\tau$, so that the government solvency condition implies (at the individual and per capita level) that:

$$b^i_t = b_t = \sum_{\tau=0}^{\infty} n_0^{\tau} R_{t-1,\tau} \left[ tax_{t+\tau} - g_{t+\tau} \right], \quad R_{t-1,\tau} \equiv \prod_{s=0}^{\tau} \frac{1}{1+r_{t+s}}$$

  With a symmetric fiscal treatment of dynasties, per capita and individual debt coincide.

# Ricardian non-equivalence: debt matters

- With endogenous fertility:

$$\frac{U_c(c^i_{t+1}, n^i_{t+1})}{U_c(c^i_t, n^i_t)} = \frac{n^i_t}{\alpha\left[1 + r_{t+1}\right]} \tag{S10}$$

$$\frac{U_n(c^i_t, n^i_t)}{U_c(c^i_t, n^i_t)} = \bar{c} + k^i_{t+1} + b_{t+1} \tag{S11}$$

$$\left(1 + r_{t+\tau}\right) k^i_t + w_t = c^i_t + g_t + \left(n^i_t - \bar{n}_t\right) b_{t+1} + n^i_t \left[\bar{c} + k^i_{t+1}\right] \tag{S12}$$

- Debt non-neutral because:
  - Eq. (S10): it affects the relative price of children
  - Eq. (S12): "fiscal externality" economy-wide average fertility, $\bar{n}_t$, reduces the tax burden of individual agents (who treat $\bar{n}_t$ parametrically). Free riding on child production by others thus explains that children will be underproduced (and fertility will be too low) in the presence of public debt.

- Symmetric steady-state equilibrium (unit-elastic):

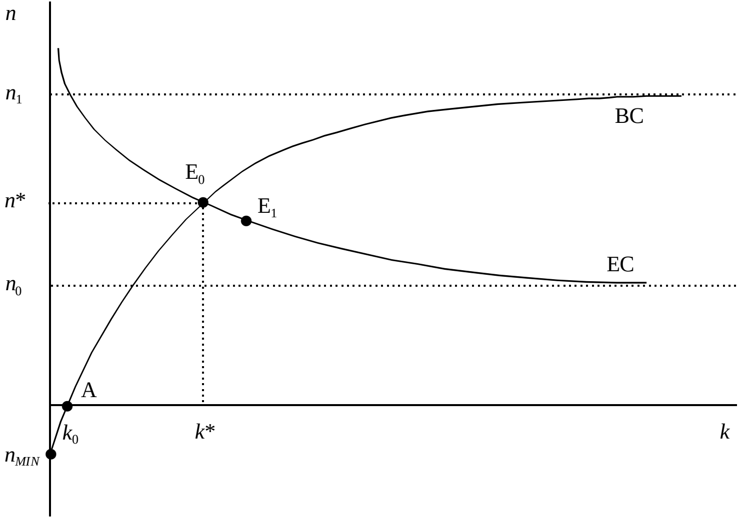
$$n = \alpha \cdot \left[1 - \delta + (1 - \varepsilon_L)\, k^{-\varepsilon_L}\right] \qquad \text{(S13)}$$

$$c = \frac{\varepsilon_C}{1 - \varepsilon_C} n \cdot [\bar{c} + k + b] \qquad \text{(S14)}$$

$$c = k^{1-\varepsilon_L} + (1 - \delta)\, k - n \cdot [\bar{c} + k] - g \qquad \text{(S15)}$$

  - See **Figure 17.9** for equilibrium.
  - EC: efficiency condition Eq. (S13)
  - BC: budget constraint Eqs. (S14)–(S15)
  - increase in $\bar{c}$ or $b$ rotates BC clockwise around point A: $dn/d\bar{c} < 0$, $dn/db < 0$, $dk/d\bar{c} > 0$, and $dk/db > 0$

# Figure 17.9: Steady-state fertility rate and capital intensity



# Punchlines

- Studied workhorse model of macroeconomics and public finance.
  - Life-cycle saving.
  - Dynamic inefficiency quite possible.
  - Wide set of applications.
- Pensions.
  - Fully funded: neutral (saving by the government).
  - PAYG: not neutral (welfare and crowding-out effects).
  - Transitional problems.
  - Population ageing may lead to extra saving under PAYG system.

# Punchlines

- Human capital.
  - Osmotic transfer and growth.
  - Mandatory education.
- Public capital.
  - Macroeconomic effects.
  - Some more golden rules.
- Endogenous fertility and economic incentives.
  - Dynastic model with operative bequests
  - Ricardian Equivalence unlikely
  - Economic effects on fertility rate