Foundations of Modern Macroeconomics
Second Edition

Chapter 16: Overlapping generations in continuous time
(sections 16.4.5 – 16.6)

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Outline

1. The open economy
    - The model
    - Lenders, borrowers, non-savers
    - What about hysteresis?
2. Endogenous labour supply
    - Optimal working hours decision
    - Extended BY model
    - Tax policy
3. The life-cycle and retirement

Bird's-eye view (1)

- Key points of the open-economy BY model:
  - The generational turnover mechanism ensures that the model is well defined even if the world rate of interest is unequal to the (exogenous) pure rate of time preference. Hence, we can distinguish creditor and debtor nations. (In open economy Ramsey model only the knife-edge case yields a well-defined equilibrium, though one exhibiting hysteresis–see Chapter 13.)
  - Just as in the open-economy Ramsey model, adjustment costs on investment are needed to limit the international mobility of physical capital–see Chapter 13.
  - In the SOE the dynamics of (q, K) and (A, H) (or (C, A)) decouple. Model can be solved recursively. See the Exercise & Solutions Manual for a worked example involving an oil price shock.

Bird's-eye view (2)

- Here we discuss a simplified model of the open economy due to Blanchard (1985). It abstracts from physical capital altogether. The model is:

$$\begin{aligned} \dot{C}(t) &= (r^* - \rho)C(t) - \mu(\rho + \mu)A_F(t) \\ \dot{A}_F(t) &= r^* A_F(t) + w(t) - C(t) \end{aligned}$$

  where A_F is net foreign assets.
- Technology is given by $Y(t) = Z(t)L(t)$ so that profit maximization leads to $w(t) = Z(t)$ (full employment also obtains and $L(t) = 1$).
- The system of differential equations is:

$$\begin{bmatrix} \dot{C}(t) \\ \dot{A}_F(t) \end{bmatrix} = \begin{bmatrix} r^* - \rho & -\mu(\rho + \mu) \\ -1 & r^* \end{bmatrix} \begin{bmatrix} C(t) \\ A_F(t) \end{bmatrix} + \begin{bmatrix} 0 \\ Z(t) \end{bmatrix}$$

Bird's-eye view (3)

- The determinant of the Jacobian is:

$$|\Delta| = r^*(r^* - \rho) - \mu(\rho + \mu) \qquad \text{(S1)}$$

  - $\dot{C} = 0$ implies: $C = \mu(\rho + \mu) A_F / (r^* - \rho)$.
  - $\dot{A}_F = 0$ implies: $C - r^* A_F = w$.
  - Using both results in (S1) we find:

$$|\Delta| = -\mu(\rho + \mu) \cdot \frac{w}{C} < 0$$

  - Provided the steady state exists, it is saddle-point stable!

Bird's-eye view (4)

- We can now look at several special cases of the model:
  - Creditor nation ($r^* > \rho$) versus debtor nation ($r^* < \rho$).
  - Non-saving nation ($r^* = \rho$).
  - Representative-agent knife-edge case ($r^* = \rho$ *and* $\mu = 0$).
- In each case we study the effects of (permanent or temporary) productivity shocks.

Creditor nation (1)

- The phase diagram is illustrated in **Figure 16.11**.
- The $\dot{C} = 0$ line is upward sloping:

$$C = \frac{\mu(\rho + \mu)}{r^* - \rho} A_F$$

- Consumption dynamics:

$$\frac{\partial \dot{C}}{\partial C} = r^* - \rho > 0$$

  See vertical arrows.

- The $\dot{A}_F = 0$ line is upward sloping (but flatter than $\dot{C} = 0$):

$$C = r^* A_F + Z$$

Creditor nation (2)

- Current account dynamics:

$$\frac{\partial \dot{A}_F}{\partial A_F} = r^* > 0$$

  See horizontal arrows.
- Model is saddle-point stable and features upward sloping saddle path.
- The effect of an unanticipated and permanent productivity shock are shown in **Figure 16.12**.
  - Impact effect: C jumps (human capital effect).
  - Transitional dynamics: gradual increase in C and A_F.
  - Long-run effect: both C and A_F increase.

Figure 16.11: A patient small open economy

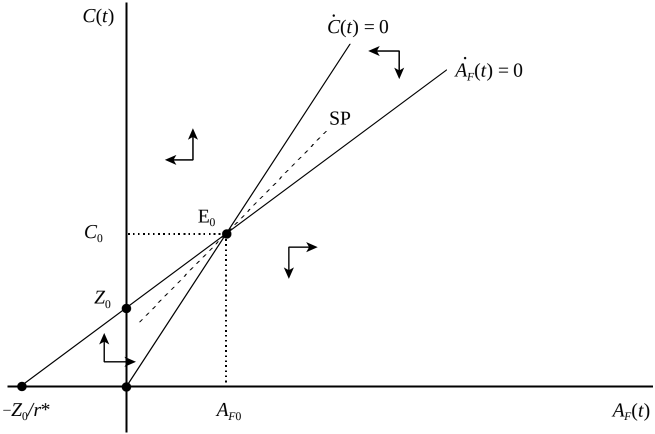
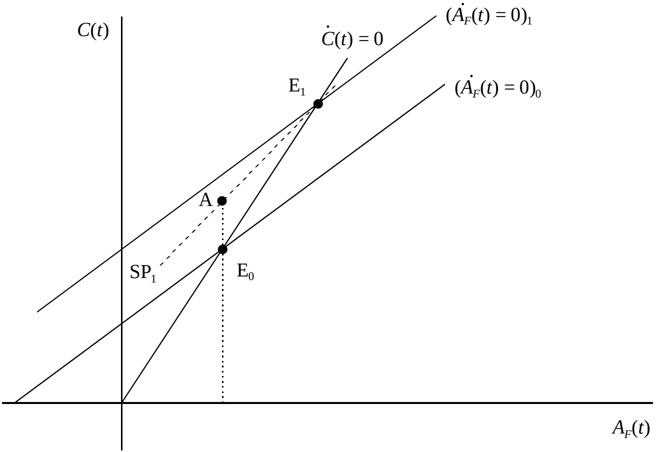


Figure 16.12: A productivity shock



Debtor nation (1)

- The phase diagram is illustrated in **Figure 16.13**.
- The $\dot{C} = 0$ line is downward sloping:

$$C = \frac{\mu(\rho + \mu)}{r^* - \rho} A_F$$

- Consumption dynamics:

$$\frac{\partial \dot{C}}{\partial C} = r^* - \rho < 0$$

  See vertical arrows.
- The $\dot{A}_F = 0$ line is upward sloping:

$$C = r^* A_F + Z$$

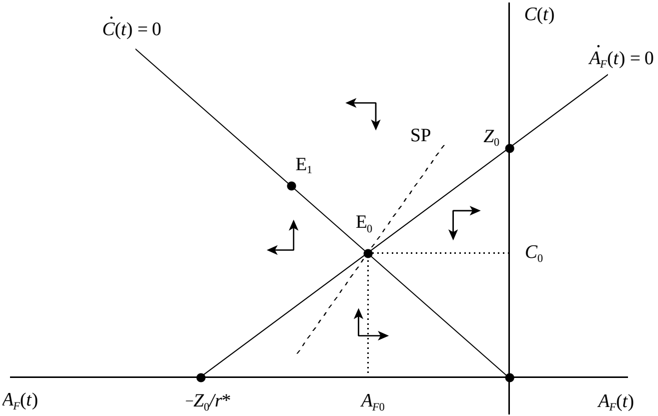
Debtor nation (2)

- Current account dynamics:

$$\frac{\partial \dot{A}_F}{\partial A_F} = r^* > 0$$

  See horizontal arrows.
- Model is saddle-point stable and features upward sloping saddle path.
- Effect of productivity shock left as an exercise.

Figure 16.13: An impatient small open economy



Non-saving nation (1)

- Special case with $r^* = \rho$: aggregate Euler equation is:

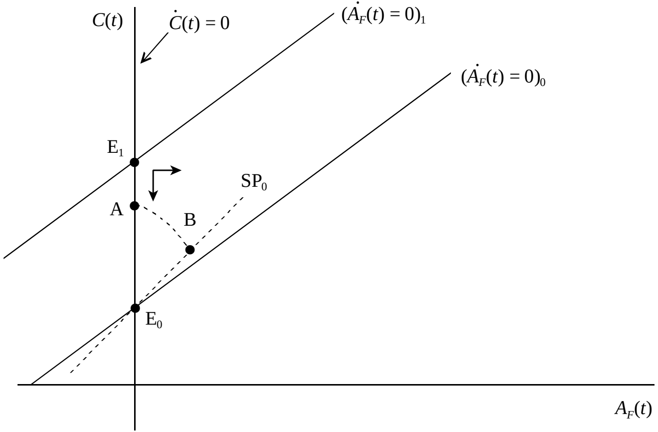
$$\dot{C}(t) = -\mu(\rho + \mu) A_F(t)$$

- $\dot{C} = 0$ line coincides with the vertical axis in **Figure 16.14**.
- Model is still saddle-point stable (as $|\Delta| = -\mu(\rho + \mu) < 0$) and the SP line is the saddle path.

Non-saving nation (2)

- Effects of temporary productivity shock:
  - Impact: upward jump in consumption (human wealth effect).
  - Transition during high productivity: gradual decline in C and increase in A_F (saving to smooth consumption). Human wealth of newborns declines during transition.
  - Transition after high productivity: gradual decline in both C and A_F.
  - Long run: no effect on C and A_F.
- Temporary shock only has temporary effects (no hysteresis).

Figure 16.14: A temporary productivity shock in a non-saving nation



Representative agent model (1)

- Special case with $r^* = \rho$ and $\mu = 0$: aggregate (and individual) Euler equation is:

$$\dot{C}(t) = 0$$

- Model features one unstable root ($\lambda_2 = r^* = \rho > 0$) and one zero root ($\lambda_1 = 0$): hysteresis.
- The consumption level is fully determined by the requirement of national solvency:

$$A_F(0) = \int_0^{\infty} \left[C(t) - Z(t)\right] e^{-\rho t} dt$$

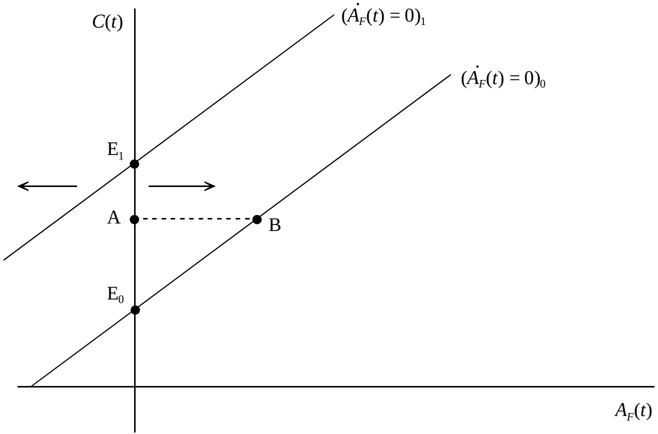
- Recall that $C(0) = \rho \cdot \left[A_F(0) + H(0)\right]$ so that:

$$C(0) = \rho \cdot \left[A_F(0) + \int_0^{\infty} Z(t) e^{-\rho t} dt\right]$$

Representative agent model (2)

- In **Figure 16.15** we show the effects of a temporary productivity shock under the assumption that the country holds no foreign assets initially ($A_F(0) = 0$).
  - Impact: upward jump in C (human wealth effect).
  - During transition, dynamics of E_1 dictates adjustment: net saving takes place (from A to B).
  - Long-run effect: consumption and net foreign assets permanently higher.
- Temporary shock has permanent effects (hysteresis).

Figure 16.15: A temporary productivity shock in the RA model



Back to the closed economy: Further extensions of the BY model

- *Endogenous labour supply*: introduce endogenous leisure choice into the household model. Issues:
  - How do various taxes affect the labour supply decision?
  - How do these taxes affect the aggregate economy and the intergenerational distribution of resources?
- *Retirement*: can we mimic life-cycle phenomena in consumption, assets, and labour supply in the perpetual youth model?

Individual agents (1)

- *Motivation*: To make the model suitable for tax policy analysis it is important to have an endogenous labour supply decision.
- Individual households: Lifetime utility is now:

$$E(\Lambda(v,t)) \equiv \int_t^\infty \ln\left[C(v,\tau)^{\varepsilon_C}\left[1-L(v,\tau)\right]^{1-\varepsilon_C}\right] e^{(\rho+\mu)(t-\tau)}d\tau$$

Note: Unit intertemporal and intratemporal substitution elasticities just as in the RBC model.

Individual agents (2)

- Budget identity:

$$\begin{aligned}\dot{A}(v,\tau) &= [r(\tau)+\mu]\,A(v,\tau)+w(\tau)(1-t_L)L(v,\tau)+TR(\tau)\\ &\quad -(1+t_C)C(v,\tau)\\ &= [r(\tau)+\mu]\,A(v,\tau)+w(\tau)(1-t_L)+TR(\tau)\\ &\quad -X(v,\tau) \qquad \text{(S2)}\end{aligned}$$

  where $TR(\tau)$ is government transfers and $X(v,\tau)$ represents *full consumption*:

$$X(v,\tau)\equiv(1+t_C)C(v,\tau)+w(\tau)(1-t_L)\left[1-L(v,\tau)\right] \quad \text{(S3)}$$

Individual agents (3)

- Step (1) (static) For given level of full consumption choose consumption and leisure such that felicity is maximized. Optimality condition requires equality between the MRS between consumption and leisure and the after-tax wage rate:

$$\frac{(1-\varepsilon_C)/\left[1-L(v,\tau)\right]}{\varepsilon_C/C(v,\tau)} = w(\tau)\frac{1-t_L}{1+t_C} \tag{S4}$$

Note: The tax on consumption (e.g. BTW) directly distorts the labour supply choice! Using (S4) in (S3) we get the "conditionally optimal" solutions:

$$(1+t_C)C(v,\tau) = \varepsilon_C X(v,\tau) \tag{S5}$$

$$w(\tau)(1-t_L)\left[1-L(v,\tau)\right] = (1-\varepsilon_C)X(v,\tau) \tag{S6}$$

Note: Due to CD assumption gross spending on consumption and leisure are constant proportion of full consumption.

Individual agents (4)

- By substituting the "conditionally optimal" choices for consumption and leisure back into the felicity function we can rewrite lifetime utility in terms of full consumption and a true cost-of-living index:

$$E\Lambda(v,t) \equiv \int_t^\infty \ln\left(\frac{X(v,\tau)}{P_\Omega(\tau)}\right)\cdot e^{(\rho+\mu)(t-\tau)}d\tau \qquad \text{(S7)}$$

$$P_\Omega(\tau) \equiv \left(\frac{1+t_C}{\varepsilon_C}\right)^{\varepsilon_C}\left(\frac{w(\tau)(1-t_L)}{1-\varepsilon_C}\right)^{1-\varepsilon_C}$$

Individual agents (5)

- Step (2) (dynamic) The household now chooses the optimal time path for full consumption which maximizes (S7) subject to the budget identity (S2) (plus a NPG condition). Outcome of this problem:

$$
\begin{aligned}
X(v,t) &= (\rho+\mu)\left[A(v,t)+H(t)\right] \\
\frac{\dot{X}(v,\tau)}{X(v,\tau)} &= r(\tau)-\rho, \quad \text{for } \tau \in [t,\infty) \\
H(t) &\equiv \int_t^{\infty} \left[w(\tau)(1-t_L)+TR(\tau)\right] e^{-R^A(t,\tau)} d\tau
\end{aligned}
$$

  Note: Full consumption proportional to total wealth, the Euler equation is now in terms of full consumption, and human wealth includes the government transfers.
- Aggregate households: aggregation just as before.
- Rest of the model unchanged: **Table 16.2**.

Table 16.2: The extended Blanchard-Yaari model

$$\frac{\dot{C}(t)}{C(t)} = r(t) - \rho - \varepsilon_C \mu(\rho + \mu) \cdot \frac{K(t)}{(1+t_C)C(t)} \quad \text{(T2.1)}$$

$$\dot{K}(t) = Y(t) - C(t) - \delta K(t) \quad \text{(T2.2)}$$

$$TR(t) = t_L w(t) L(t) + t_C C(t) \quad \text{(T2.3)}$$

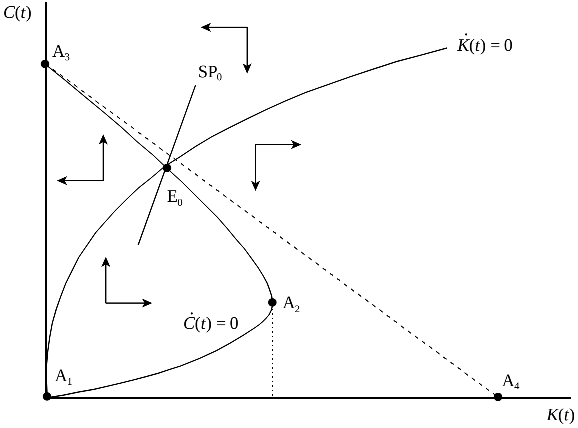
$$r(t) + \delta = (1 - \varepsilon_L) \frac{Y(t)}{K(t)} \quad \text{(T2.4)}$$

$$w(t) = \varepsilon_L \frac{Y(t)}{L(t)} \quad \text{(T2.5)}$$

$$w(t)\,[1 - L(t)] = \frac{1 - \varepsilon_C}{\varepsilon_C} \frac{1 + t_C}{1 - t_L} C(t), \quad 0 < \varepsilon_C \leq 1. \quad \text{(T2.6)}$$

$$Y(t) = K(t)^{1-\varepsilon_L} L(t)^{\varepsilon_L}, \quad 0 < \varepsilon_L < 1 \quad \text{(T2.7)}$$

Figure 16.16: Phase diagram for the extended Blanchard-Yaari model



Phase diagram (1)

- In the text we derive the phase diagram for the extended BY model. The resulting diagram is found in **Figure 16.16**. *Features*:
- The $\dot{K} = 0$ line is the same as in Chapter 15 [apart from the supply-side effects of the various tax rates: $\dot{K} > 0$ (< 0) for points below (above) the line – see the horizontal arrows.
- The $\dot{C} = 0$ line combines two mechanisms:
  - **FS**: factor scarcity effect which explains the slope of the $\dot{C} = 0$ line for the representative agent model of Chapter 15.
  - **GT**: generational turnover effect which explains the slope of the $\dot{C} = 0$ line for the standard BY model with exogenous labour supply.

Phase diagram (2)

- The $\dot{C} = 0$ line . . . .
  - In the lower branch of the $\dot{C} = 0$ line, $L \approx 1$ and the GT effect dominates the FS effect. For points above the line $\dot{C} > 0$:

$$\underbrace{\frac{\dot{C}}{C}}_{\uparrow} = \underbrace{r(C,K)}_{\downarrow} - \rho - \frac{\mu \varepsilon_C(\rho+\mu)}{1+t_C} \underbrace{\frac{K}{C}}_{\downarrow\downarrow}$$

  - In the upper branch of the $\dot{C} = 0$ line, $L \approx 0$ and the FS effect dominates the GT effect. For points above the line $\dot{C} < 0$:

$$\underbrace{\frac{\dot{C}}{C}}_{\downarrow} = \underbrace{r(C,K)}_{\downarrow\downarrow} - \rho - \frac{\mu \varepsilon_C(\rho+\mu)}{1+t_C} \underbrace{\frac{K}{C}}_{\downarrow}$$

Increasing the consumption tax

- In the text we show how the model can be used to analyze the macroeconomic effects of a change in the consumption tax.
  - Impact, transitional, and long-run effects can be obtained from log-linearized version of the model (small tax changes).
  - The steady state effect on the capital stock depends on the relative strength of the FS and GT effects. *Intuition*:
    - Dominant GT effect: $t_C \uparrow$ causes $K(\infty) \uparrow$. Redistribution from old to young. ($C(v, 0)$ high the older one is: old pay more on consumption tax.) r virtually unchanged (weak FS effect). $C(0) \downarrow\downarrow$ and $Y(0) \downarrow$ so $\dot{K}(0) \uparrow$. Capital accumulation takes place.
    - Dominant FS effect: $t_C \uparrow$ causes $K(\infty) \downarrow$. Great ratios: (K/L) virtually unchanged in long run. As $L(\infty) \downarrow$ so must $K(\infty)$.
  - Simple as the extended BY model is, similar results are obtained in much more detailed *computable general equilibrium* models (e.g. Auerbach & Kotlikoff, 1989).

Figure 16.17: Factor markets

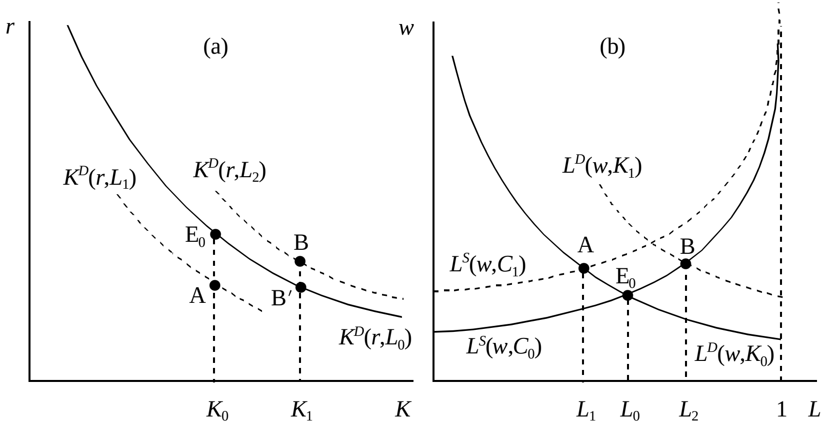


Figure 16.18: Consumption taxation with a dominant GT effect

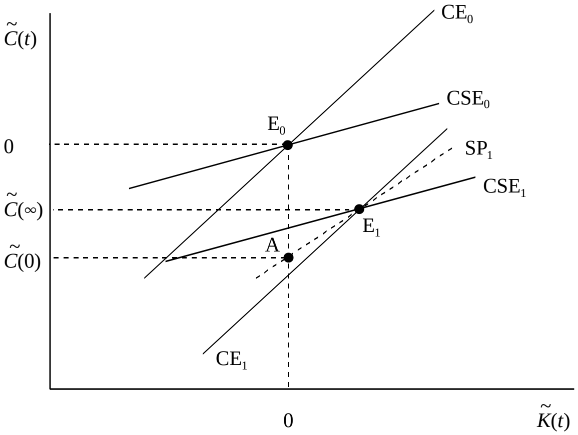


Figure 16.19: Consumption taxation with a dominant FS effect

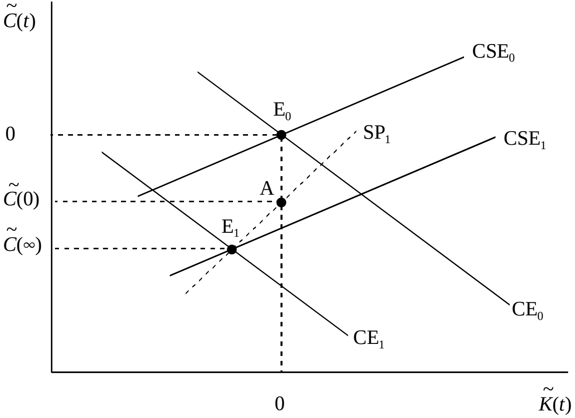


Table 16.3: The loglinearized extended model

$$\dot{\tilde{C}}(t) = r\tilde{r}(t) + (r-\rho)\left[\tilde{C}(t) + \tilde{t}_C - \tilde{K}(t)\right] \quad \text{(T3.1)}$$

$$\dot{\tilde{K}}(t) = (\delta/\omega_I)\left[\tilde{Y}(t) - \omega_C\tilde{C}(t) - \omega_I\tilde{K}(t)\right] \quad \text{(T3.2)}$$

$$\widetilde{TR}(t) = (1+t_C)\omega_C\left[\tilde{t}_C + \frac{t_C}{1+t_C}\cdot\tilde{C}(t)\right] + \varepsilon_L t_L\tilde{Y}(t) \quad \text{(T3.3)}$$

$$r\cdot\tilde{r}(t) = (r+\delta)\left[\tilde{Y}(t) - \tilde{K}(t)\right] \quad \text{(T3.4)}$$

$$\tilde{w}(t) = \tilde{Y}(t) - \tilde{L}(t) \quad \text{(T3.5)}$$

$$\tilde{L}(t) = \omega_{LL}\left[\tilde{w}(t) - \tilde{t}_C - \tilde{C}(t)\right] \quad \text{(T3.6)}$$

$$\tilde{Y}(t) = \varepsilon_L\tilde{L}(t) + (1-\varepsilon_L)\tilde{K}(t) \quad \text{(T3.7)}$$

Table 16.4: The birth rate and the GT effect

μ	$1/\mu$	ρ	*GT effect*	*FS effect*
0.005	200	0.0156	0.000312	0.0457
0.01	100	0.0151	0.000762	0.0457
0.02	50	0.0138	0.002054	0.0457
0.04	25	0.0098	0.006051	0.0457
0.07229	13.83	0	0.015868	0.0457

Retirement (1)

- Wealth effect ensures that leisure rises above unity, $1 - L(u) > 1$ (where u is the worker's age)
- But this means negative labour supply, $L(u) < 0$
- Ignored up to now, but not realistic
- Develop simple model of life-cycle labour supply with:
  - hump-shaped age pattern in labour productivity
  - endogenous retirement age, u^*
  - retirement assumed a permanent decision (absorbing state)
- Lifetime utility function:

$$E(\Lambda(u)) \equiv \int_u^\infty \left[\varepsilon_C \ln C(s) + (1 - \varepsilon_C) \ln [1 - L(s)]\right] e^{(\rho+\mu)(u-s)} ds$$

Retirement (2)

- Budget identity:

$$\dot{A}(s) = (r+\mu)A(s) + w(s)L(s) - C(s) - T$$

- Inequality constraint:

$$L(s) \geq 0$$

- **Figure 16.20** shows labour supply choice over the life cycle.
- **Figure 16.21** shows key aspects of the agent's individual consumption, labour supply, and savings decisions over the life cycle.

Figure 16.20: Life-cycle labour supply and retirement

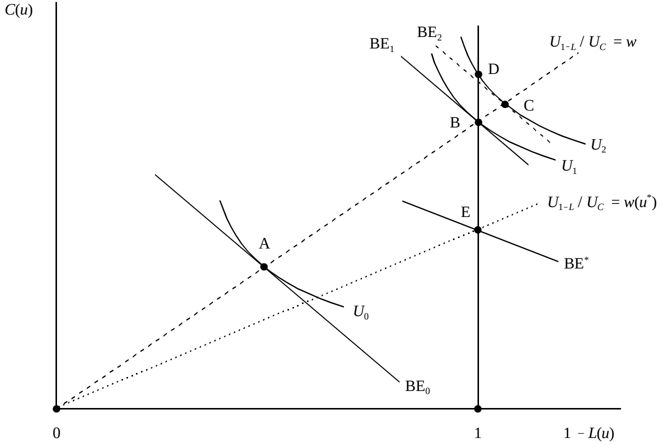


Figure 16.21(a): Age-dependent wage

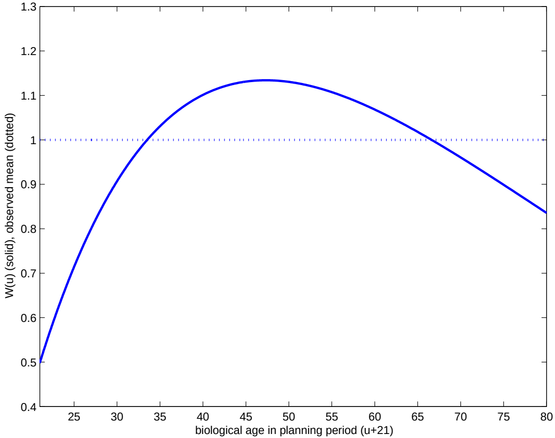


Figure 16.21(d): Age-dependent labour supply

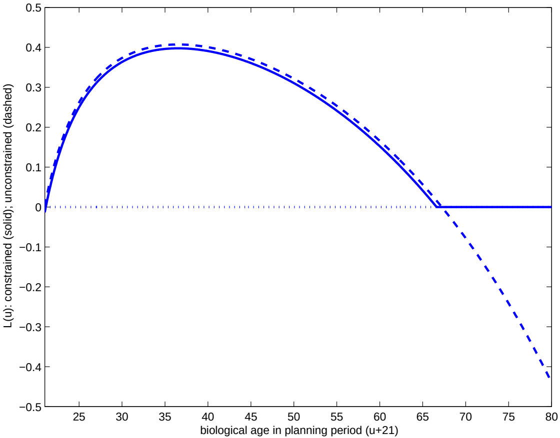


Figure 16.21(e): Age-dependent consumption

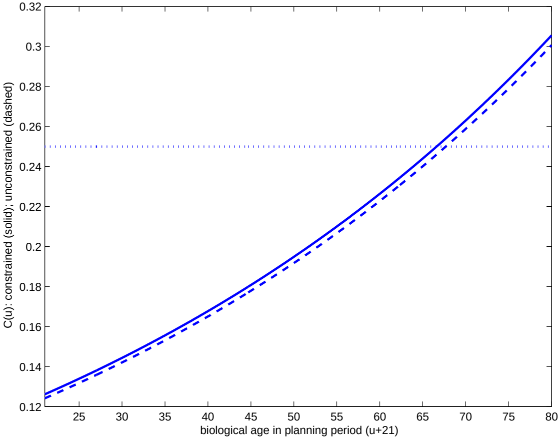
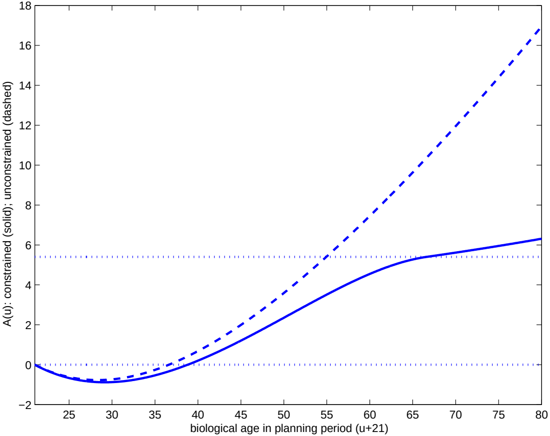


Figure 16.21(f): Age-dependent assets



Punchlines (1)

- Insights of Yaari:
  - Positive death rate leads to more severe discounting of future felicity.
  - With actuarially fair life insurance the household can fully insure against the unpleasant aspects of being mortal.
- Blanchard shows that Yaari's consumption model can be embedded in general equilibrium framework.
  - Fully tractable GE model with heterogeneous agents.
  - Both efficiency and intergenerational distribution matter.
  - Ricardian equivalence not valid.

Punchlines (2)

- BY model can be easily extended.
  - Endogenous labour supply (tax distortions).
  - Life-cycle issues.
  - Endogenous growth.
  - Saving for rainy day (dynamic inefficiency).
  - Open economy (non-degenerate dynamics).
- Approach fully deserves its current "work horse" status.