Foundations of Modern Macroeconomics Second Edition

Chapter 14: Endogenous economic growth (sections 14.1 – 14.3)

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13 November 2012

Outline

1. Introduction
2. Physical capital fundamentalism
   - Factor substitutability
   - A private sector AK model
   - A public sector AK model
3. Human capital and growth
   - The model
   - Steady-state growth
   - Transitional dynamics

Aims of this chapter (1)

- Study the main theories of endogenous growth.
- *Key notion*: Can we devise a growth theory in which the steady-state growth rate is *endogenous*, i.e. depends not just on exogenous things like the population growth rate and the rate of Harrod-neutral technological change?
- Can we open up the black box of technological change?
- Following the influential work of Paul Romer in the mid 1980s a very active research field has developed.

Aims of this chapter (2)

- We will give a selective overview of this huge body of literature. Three groups can be distinguished:
  - "Capital fundamentalist" models. Physical capital forms the engine of growth.
  - Human capital formation. The knowledge inside human heads is crucial to growth. People accumulate knowledge with good purpose.
  - Endogenous technology. Profit-seeking firms engage in research & development to make new products or services, or devise new production processes.

Recall the Inada conditions

- In exogenous growth models there exist diminishing returns to capital
- As $k(t)$ rises over time, the average product of capital falls:

$$\frac{d\left[f(k(t))/k(t)\right]}{dk(t)} = -\frac{\left[f(k(t)) - k(t)f'(k(t))\right]}{k(t)^2} < 0 \qquad \text{(S1)}$$

- Property (S1) necessary but not sufficient for existence of steady-state capital-labour ratio
- Inada conditions are strong enough:

$$\lim_{k(t)\to 0} \frac{f(k(t))}{k(t)} = \lim_{k(t)\to 0} \frac{f'(k(t))}{1} = \infty \qquad \text{(S2)}$$

$$\lim_{k(t)\to \infty} \frac{f(k(t))}{k(t)} = \lim_{k(t)\to \infty} \frac{f'(k(t))}{1} = 0 \qquad \text{(S3)}$$

Inada conditions violated

- *Key idea*: There are perfectly reasonable production functions which do not satisfy the Inada conditions.
- Take, for example, the CES production function:

$$F(K(t), L(t)) \equiv Z \cdot \left[\alpha K(t)^{1/\xi} + (1-\alpha)L(t)^{1/\xi}\right]^{\xi} \quad \Leftrightarrow$$

$$f(k(t)) \equiv Z \cdot \left[1 - \alpha + \alpha k(t)^{1/\xi}\right]^{\xi}$$

where ξ is a coefficient involving the substitution elasticity between capital and labour, σ_{KL}:

$$\xi \equiv \frac{\sigma_{KL}}{\sigma_{KL} - 1}$$

- The average product of capital (APK) equals:

$$\frac{f(k(t))}{k(t)} = Z \cdot \left[(1-\alpha)k(t)^{-1/\xi} + \alpha\right]^{\xi} \qquad \text{(S4)}$$

CES production function (1)

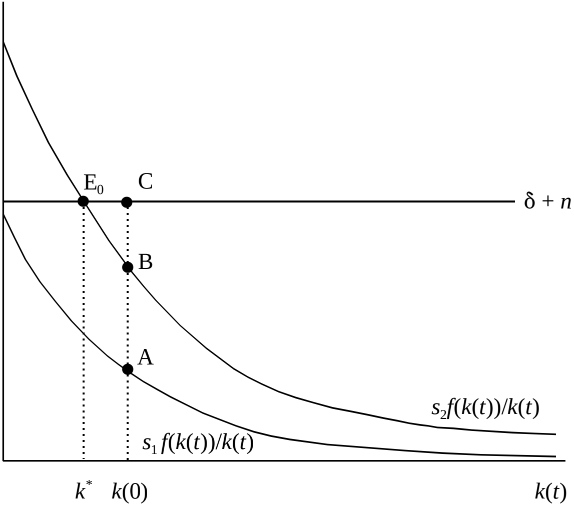
- Two cases must be distinguished:
  - **Case A**: Difficult substitution between K and L ($0 < \sigma_{KL} < 1$ and $\xi < 0$).
  - **Case B**: Easy substitution between K and L ($\sigma_{KL} > 1$ and $\xi > 1$).
- Case A. With difficult substitution we obtain from (S4):

$$0 < \lim_{k(t)\to 0} \frac{f(k(t))}{k(t)} = Z \cdot \alpha^{\xi} < \infty$$

$$\lim_{k(t)\to\infty} \frac{f(k(t))}{k(t)} = Z \cdot \lim_{k(t)\to\infty} \frac{f'(k(t))}{1} = 0$$

The APK is finite for $k(t) \to 0$. Hence, it may not be even high enough to sustain a non-trivial steady state (see Figure 14.1).

Figure 14.1: Difficult substitution between labour and capital



CES production function (2)

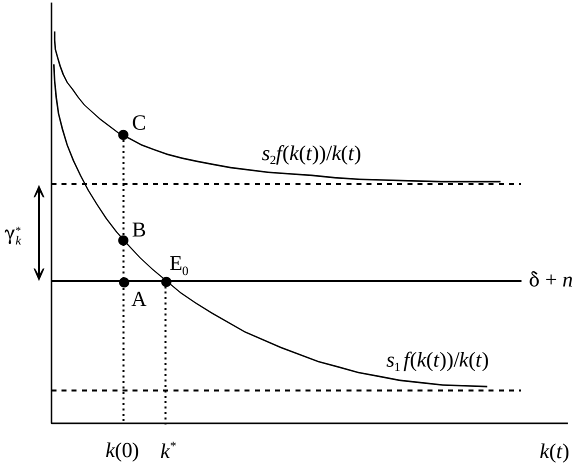
- **Case B**. With easy substitution we derive from (S4):

$$\lim_{k(t)\to 0} \frac{f(k(t))}{k(t)} = Z \cdot \lim_{k(t)\to 0} \frac{f'(k(t))}{1} = \infty$$

$$\lim_{k(t)\to \infty} \frac{f(k(t))}{k(t)} = Z \cdot \alpha^{\xi} > 0$$

There is a lower bound on the APK as $k(t) \to \infty$. Hence, there may be perpetual growth in $k(t)$ (see **Figure 14.2**).

Figure 14.2: Easy substitution between labour and capital



CES production function (3)

- The asymptotic growth rate is:

$$\gamma^* = sZ\alpha^{\sigma_{KL}/(\sigma_{KL}-1)} - (\delta + n) > 0$$

- We call γ^* an *endogenous* growth rate because the savings rate, s, affects it!
- Even though there are diminishing returns to capital, K and L substitute easily. Hence, labour does not become an effective constraint. Scarce labour is substituted by capital indefinitely.
- But, the share of labour goes to zero – contra SF3 and SF5.

Macroeconomic technology linear in capital

- An even more radical model is the so-called *AK model*.
- macroeconomic technology is:

$$Y(t) = Z \cdot K(t) \tag{S5}$$

- The MPK is constant and labour is eliminated from the model altogether!
- Eq. (S5) is not as silly as one might think.
- Two ways to rationalize this macroeconomic relationship.
  - There exist external productivity effects between individual firms
  - Public infrastructure causes external productivity effects on individual firms

Inter-firm technological externalities (1)

- *Key idea*: individual firms experience diminishing returns to labour and capital. But external effects between firms render the marginal product of aggregate capital constant.
- Technology available to firm i:

$$Y_i(t) = F(K_i(t), L_i(t)) \equiv Z(t) K_i(t)^{\alpha} L_i(t)^{1-\alpha} \qquad \text{(S6)}$$

with $0 < \alpha < 1$. Here Y_i, K_i, and L_i, stand for, respectively, output, capital input, and labour input of firm i $(= 1, \cdots, N_0)$, and N_0 is the fixed number of firms. $Z(t)$ is the general level of factor productivity which is taken as given by the individual firm.

Inter-firm technological externalities (2)

- Firm i's objective function:

$$V_i(0) = \int_0^\infty \Big[F(K_i(t), L_i(t)) - w(t)L_i(t) - (1 - s_I)\, I_i(t)\Big] e^{-R(t)} dt$$

where $R(t) \equiv \int_0^t r(\tau)d\tau$ is the cumulative discount factor, and s_I is the investment subsidy.

- Marginal productivity conditions for labour and capital:

$$w(t) = F_L(K_i(t), L_i(t)) = (1-\alpha)\, Z(t)\, k_i(t)^\alpha \quad \text{(S7)}$$
$$R^K(t) = F_K(K_i(t), L_i(t)) = \alpha Z(t)\, k_i(t)^{\alpha-1} \quad \text{(S8)}$$

- Rental rate of capital:

$$R^K(t) \equiv (r(t) + \delta)(1 - s_I)$$

Inter-firm technological externalities (3)

- Symmetric solution: the rental rate on each factor is the same for all firms, i.e. they all choose the same capital intensity and $k_i(t) = k(t)$ for all $i = 1, \cdots, N_0$. Aggregation over firms simple.
- Inter-firm externality takes the following form:

$$Z(t) = z_0 K(t)^{1-\alpha}, \qquad z_0 > 0 \tag{S9}$$

where $K(t) \equiv \sum_i K_i(t)$ is the aggregate capital stock.

- With a fixed macro labour supply, using (S9) in (S6)–(S8) results in:

$$Y(t) = Z_0 K(t) \tag{S10}$$

$$w(t) L_0 = (1-\alpha) Y(t) \tag{S11}$$

$$R^K(t) = \alpha Z_0 \tag{S12}$$

where $Y(t) \equiv \sum_i Y_i(t)$ is aggregate output ($Z_0 \equiv z_0 L_0^{1-\alpha}$).

Inter-firm technological externalities (4)

- The national income share of labour is positive and there are constant returns to capital at the macroeconomic level. This result follows from the fact that the exponents for K_i in (S6) and for K in (S9) **precisely** add up to unity.
- Household side: Ramsey model with fixed labour supply L_0.
- Infinitely-lived representative household.

$$\Lambda(0) = \int_0^\infty \frac{C(t)^{1-1/\sigma} - 1}{1 - 1/\sigma} e^{-\rho t} dt$$

$$\dot{A}(t) = r(t)A(t) + w(t) L_0 - (1 + t_C) C(t) - T(t)$$

- Euler equation:

$$\frac{\dot{C}(t)}{C(t)} = \sigma \cdot (r(t) - \rho)$$

Inter-firm technological externalities (5)

- Closed economy
- No government debt. Government consumption $G(t) = gY(t)$ Government budget equation:

$$T(t) + t_C C(t) = G(t) + s_I I(t)$$

- The only financial asset which can be accumulated consists of company shares. Replacement value of capital equals $1 - s_I$, so $A(t) = [1 - s_I(t)] K(t)$.
- The key equations of the basic *AK* growth model have been summarized in Table 14.1.

Table 14.1: An AK growth model with inter-firm external effects

$$\frac{\dot{C}(t)}{C(t)} = \sigma \cdot (r(t) - \rho) \quad \text{(T1.1)}$$

$$\dot{K}(t) = [(1-g) \cdot Z_0 - \delta] \cdot K(t) - C(t) \quad \text{(T1.2)}$$

$$r(t) = \frac{\alpha Z_0}{1 - s_I} - \delta \quad \text{(T1.3)}$$

Main properties of the model with inter-firm technological externalities

- The growth rate in the economy is:

$$\gamma^* = \frac{\dot{C}(t)}{C(t)} = \frac{\dot{Y}(t)}{Y(t)} = \frac{\dot{K}(t)}{K(t)} = \frac{\dot{I}(t)}{I(t)} = \sigma \left[\frac{\alpha Z_0}{1 - s_I} - \delta - \rho \right]$$

- Endogenous growth: the policy maker can affect it by setting s_I. Indeed, $d\gamma^*/ds_I > 0$. In **Figure 14.3**, $\theta \equiv C/K$ falls. The lump-sum tax increase makes people poorer.
- Consumption tax t_C and government consumption g do not affect growth rate.
- Growth path is not Pareto-efficient. Firms fail to take inter-firm externality into account.
- No transitional dynamics (for case with $\dot{s}_I(t) = 0$).

Why is there no transitional dynamics in this model? (1)

- Define $\theta(t) \equiv C(t)/K(t)$ and note that:

$$\frac{\dot{\theta}(t)}{\theta(t)} = \frac{\dot{C}(t)}{C(t)} - \frac{\dot{K}(t)}{K(t)}$$

- Use (T1.1)-(T1.3) to find:

$$\frac{\dot{\theta}(t)}{\theta(t)} = \theta(t) - \theta^* \qquad \text{(A)}$$

  where θ^* is defined as:

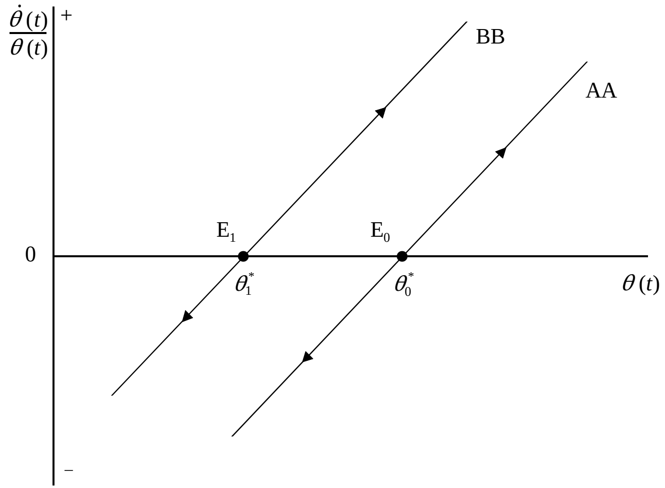
$$\theta^* \equiv (1-g) Z_0 - \delta + \sigma(\rho+\delta) - \frac{\alpha\sigma Z_0}{1-s_I} > 0.$$

- Equation (A) is an unstable differential equation for which the only economically feasible solution is the steady-state, i.e. $\theta(t) = \theta^*$. See **Figure 14.3**.

Why is there no transitional dynamics in this model? (2)

- Hence the capital stock, investment, and output, must feature the same growth rate as consumption.
- The level of the different variables can be determined by using the initial condition regarding the capital stock and noting that $C(0) = \theta^* K(0)$.
- In the absence of shocks in the interval $(0, t)$, we thus find that $K(t) = K(0)\, e^{\gamma^* t}$, $C(t) = \theta^* K(t)$, $Y(t) = Z_0 K(t)$, etcetera.
- The *growth rate* of the economy can be permanently affected by the investment subsidy! In **Figure 14.3** an increase in s_I shifts the equilibrium from θ_0^* to θ_1^*. The lump-sum tax increase needed to finance the higher subsidy makes people poorer.

Figure 14.3: Consumption-capital ratio



Public infrastructural capital (1)

- Barro suggest a model in which productive government spending affects productivity (and growth). Technology is still as in (S6) but there is an output tax, t_Y
- The productivity conditions for individual firms are:

$$\begin{aligned} w(t) &= (1-\alpha)(1-t_Y) Z(t) k_i(t)^{\alpha} \\ r(t)+\delta_K &= \alpha(1-t_Y) Z(t) k_i(t)^{\alpha-1} \end{aligned}$$

- In the *spirit* of Barro's model we assume:

$$Z(t) = z_0 K_G(t)^{1-\alpha} \tag{S13}$$

where $K_G(t)$ is the *stock* of public capital, consisting of infrastructural objects like roads, airports, bridges, and the like.

Public infrastructural capital (2)

- Aggregating over all firms i gives:

$$
\begin{aligned}
Y(t) &= Z_0 K(t)^{\alpha} K_G(t)^{1-\alpha} \\
w(t) L_0 &= (1-\alpha)(1-t_Y) Y(t) \\
r(t) + \delta_K &= \alpha (1-t_Y) Z_0 \left(\frac{K_G(t)}{K(t)} \right)^{1-\alpha}
\end{aligned}
$$

  - Diminishing returns to private capital also at the macro level, but...
  - If the government maintains constant K_G/K ratio then model is like *AK* model.

Public infrastructural capital (3)

- Accumulation equation for public capital:

$$\dot{K}_G(t) = I_G(t) - \delta_G K_G(t)$$

  where $I_G(t)$ is the **flow** of public investment (exogenous), and δ_G is the depreciation rate of public capital.

- The government budget constraint is:

$$t_Y Y(t) = I_G(t) + gY(t)$$

- The key equations of the model have been summarized in **Table 14.2**.

Table 14.2: An AK growth model with public capital

$$\frac{\dot{C}(t)}{C(t)} = \sigma \cdot [r(t) - \rho] \tag{T2.1}$$

$$\frac{\dot{K}(t)}{K(t)} = (1 - t_Y) Z_0 \left(\frac{K(t)}{K_G(t)} \right)^{\alpha - 1} - \frac{C(t)}{K(t)} - \delta_K \tag{T2.2}$$

$$\frac{\dot{K}_G(t)}{K_G(t)} = (t_Y - g) Z_0 \left(\frac{K(t)}{K_G(t)} \right)^{\alpha} - \delta_G \tag{T2.3}$$

$$r(t) = \alpha (1 - t_Y) Z_0 \left(\frac{K(t)}{K_G(t)} \right)^{\alpha - 1} - \delta_K \tag{T2.4}$$

Public infrastructural capital (4)

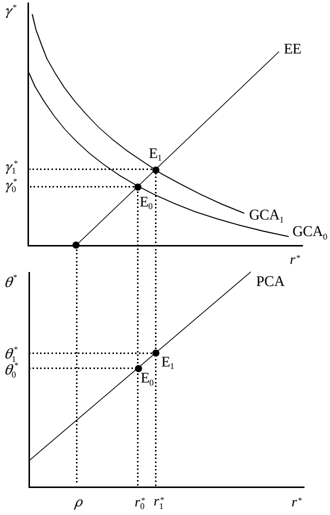
- The steady-state growth rate is γ^*:

$$\gamma^* = \sigma \left[r^* - \rho \right] \quad \text{(A)}$$
$$\gamma^* = (1 - t_Y) Z_0 (\kappa^*)^{\alpha - 1} - \theta^* - \delta_K \quad \text{(B)}$$
$$\gamma^* = (t_Y - g) Z_0 (\kappa^*)^{\alpha} - \delta_G \quad \text{(C)}$$
$$r^* = \alpha (1 - t_Y) Z_0 (\kappa^*)^{\alpha - 1} - \delta_K \quad \text{(D)}$$

- In **Figure 14.4** we illustrate the nature of the solution *for a given tax rate* t_Y.
  - EE is the Euler Equation (A).
  - GCA is the Government Capital Accumulation locus (solve (D) for κ^* and substitute in (C).
  - PCA is the Private Capital Accumulation locus (solve (D) for κ^* and substitute in (B).

Figure 14.4: Steady-state growth

Effect of a decrease in g



Public infrastructural capital (5)

- Implicit expression for γ^*:

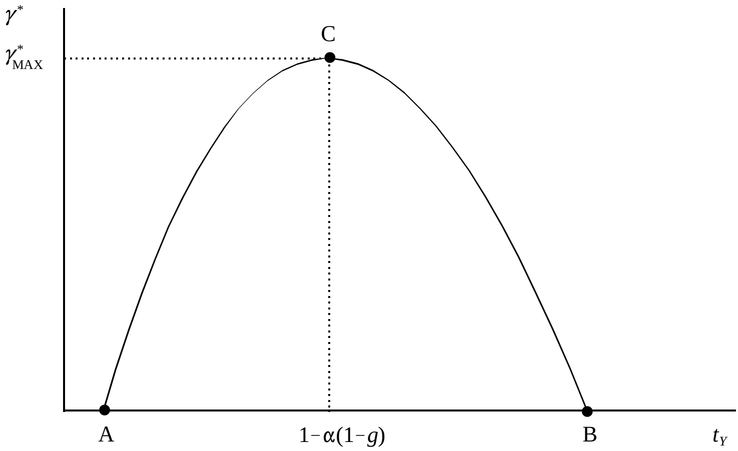
$$\gamma^* + \delta_G = (\alpha^\alpha Z_0)^{1/(1-\alpha)} (t_Y - g) \left(\frac{\sigma (1 - t_Y)}{\gamma^* + \sigma (\rho + \delta_K)} \right)^{\alpha/(1-\alpha)}$$

- Slope of the growth line:

$$\frac{t_Y - g}{\gamma^* + \delta_G} \cdot \frac{d\gamma^*}{dt_Y} = \frac{1 - \frac{\alpha}{1-\alpha} \frac{t_Y - g}{1 - t_Y}}{1 + \frac{\alpha}{1-\alpha} \frac{\gamma^* + \delta_G}{\gamma^* + \sigma(\rho + \delta_K)}}.$$

- In **Figure 14.5** we plot the steady-state growth rate as a function of the output tax, t_Y.
- This *AK* model features non-trivial transitional dynamics.

Figure 14.5: Productive government spending and growth



Transitional dynamics in the infrastructural model (1)

- Consider the model in Table 14.2.
- Define: $\theta(t) \equiv C(t)/K(t)$ and $\kappa(t) \equiv K(t)/K_G(t)$.
- Rewrite the system:

$$\frac{d\ln\theta(t)}{dt} = \sigma\left[r(t)-\rho\right] - (1-t_Y)\,Z_0\kappa(t)^{\alpha-1} + \theta(t) + \delta_K$$

$$\frac{d\ln\kappa(t)}{dt} = (1-t_Y)\,Z_0\kappa(t)^{\alpha-1} - (t_Y - g)\,Z_0\kappa(t)^{\alpha} - \theta(t) + \delta_G - \delta_K$$

$$r(t) = \alpha(1-t_Y)\,Z_0\kappa(t)^{\alpha-1} - \delta_K$$

Transitional dynamics in the infrastructural model (2)

- In order to study the dynamic properties of the model, we loglinearize it around the steady-state point (θ^*, κ^*) to obtain:

$$\begin{bmatrix} \frac{d\ln\theta(t)}{dt} \\ \frac{d\ln\kappa(t)}{dt} \end{bmatrix} = \Delta \cdot \begin{bmatrix} \ln\theta(t) - \ln\theta^* \\ \ln\kappa(t) - \ln\kappa^* \end{bmatrix}$$

- Δ is the Jacobian matrix:

$$\Delta \equiv \begin{bmatrix} \theta^* & \frac{(1-\alpha)(1-\alpha\sigma)(r^*+\delta_K)}{\alpha} \\ -\theta^* & -\frac{(1-\alpha)(r^*+\delta_K)+\alpha^2(\gamma^*+\delta_G)}{\alpha} \end{bmatrix}$$

Transitional dynamics in the infrastructural model (3)

- The determinant of Δ is given by:

$$|\Delta| \equiv -\theta^{*}\left[\,(1-\alpha)\,\sigma\,(r^{*}+\delta_{K})+\alpha\,(\gamma^{*}+\delta_{G})\,\right]<0$$

- There is one negative (stable root) $-\lambda_1 < 0$ and one positive (unstable) root, $\lambda_2 > 0$, and the model is saddle-path stable.
- $\theta\,(t)$ is a jumping variable (because $C\,(t)$ is) whilst $\kappa\,(t)$ is predetermined (because $K\,(t)$ and $K_G\,(t)$ are).
- Given initial values $K\,(0)$ and $K_G\,(0)$ (and thus for $\kappa\,(0) \equiv K\,(0)\,/K_G\,(0)$), the model converges along the saddle path toward the steady-state equilibrium.
- The transition speed is equal to the absolute value of the stable root, λ_1.

Human capital accumulation as the engine of growth

- *Key idea*: (Uzawa) all technical knowledge is embodied in labour. Educational sector uses labour to augment the state of knowledge in the economy.
- Uzawa (1965) assumed:

$$\frac{\dot{Z}_L}{Z_L} = \Psi\left(\frac{L_E}{L}\right)$$

where Z_L is labour-augmenting technical progress and L_E is labour used in the educational sector ($\Psi' > 0 > \Psi''$).

- Basic idea was taken over by Lucas (1988). He interprets Z_L as human capital ("skills") and calls it H. Rational agents accumulate human capital by dedicating some of their time on education (hence, the name of the model: "learning or doing").

Lucas-Uzawa model (1)

- Human capital accumulation function:

$$\frac{\dot{H}(t)}{H(t)} = Z_E \frac{L_E(t)}{L(t)} - \delta_H \tag{S14}$$

where Z_E is a positive constant.

- Lifetime utility of the representative household:

$$\Lambda(0) = \int_0^\infty \frac{C(t)^{1-1/\sigma} - 1}{1 - 1/\sigma} e^{-\rho t} dt \tag{S15}$$

- Time constraint of the household:

$$L_E(t) + L_P(t) = L_0 \tag{S16}$$

where L_P is time spent working ("doing") rather than going to school

Lucas-Uzawa model (2)

- Aggregate production function:

$$\begin{aligned} Y(t) &= F(K(t), N_P(t)) = Z_Y N_P(t)^{1-\alpha} K(t)^{\alpha} \\ N_P(t) &\equiv H(t) L_P(t) \end{aligned}$$

- Factors receive their respective marginal products:

$$\begin{aligned} R^K(t) &= F_K(K(t), N_P(t)) = \alpha Z_Y k(t)^{\alpha-1} \\ w(t) &= H(t) \cdot F_N(K(t), N_P(t)) \\ &= (1-\alpha) Z_Y H(t) \cdot k(t)^{\alpha} \qquad \text{(S17)} \end{aligned}$$

with $k(t) \equiv K(t)/N_P(t)$

- Equation (S17) shows that the wage rises with the skill level. Household has an incentive to accumulate human capital.

Lucas-Uzawa model (3)

- The household takes $k(t)$ and thus F_N and F_K as given. These are **macro** variables.
- The household chooses sequences for consumption and the stocks of physical and human capital in order to maximize lifetime utility (S15) subject to:
  - the time constraint (S16)
  - the accumulation identity for physical capital, $\dot{K}(t) = I(t) - \delta_K K(t)$
  - the budget identity:

$$I(t) + C(t) + T(t) = w(t)L_P(t) + R^K(t)K(t) + s_E w(t) L_E(t)$$

  where $T(t)$ is a lump-sum tax and s_E is a time-invariant education subsidy received from the government ($\dot{s}_E = 0$).

Lucas-Uzawa model (4)

- Current-value Hamiltonian:

$$\begin{aligned}\mathcal{H}_C(t) &= \frac{C(t)^{1-1/\sigma}-1}{1-1/\sigma} + \mu_H(t)\cdot\left[Z_E\frac{L_E(t)}{L_0}-\delta_H\right]\cdot H(t) \\ &+\mu_K(t)\cdot\left[\left(R^K(t)-\delta_K\right)K(t)+H(t)F_N(k(t),1)(L_0-L_E(t))\right. \\ &\left.+s_E H(t)F_N(k(t),1)L_E(t)-C(t)-T(t)\right]\end{aligned}$$

where $\mu_K(t)$ and $\mu_H(t)$ are the co-state variables.

- First-order necessary conditions:

$$\begin{aligned}C(t)^{-1/\sigma} &= \mu_K(t) \\ \mu_H(t)\frac{Z_E}{L_0} &= \mu_K(t)(1-s_E)F_N(k(t),1) \\ \frac{\dot{\mu}_K(t)}{\mu_K(t)} &= \rho+\delta_K-F_K(k(t),1) \\ \frac{\dot{\mu}_H(t)}{\mu_H(t)} &= \rho+\delta_H-Z_E\frac{L_E(t)}{L_0}-\frac{\mu_K(t)}{\mu_H(t)}\left[L_0-(1-s_E)L_E(t)\right]F_N(k(t),1) \\ 0 &= \lim_{t\to\infty}e^{-\rho t}\mu_K(t)K(t)=\lim_{t\to\infty}e^{-\rho t}\mu_H(t)H(t)\end{aligned}$$

Lucas-Uzawa model (5)

- Fundamental principle of valuation: the rate of return on different assets with the same riskiness must be equalized.
- For each asset the rate of return can be computed as the sum of dividends plus capital gains divided by the price of the asset:

$$\rho = \frac{\dot{\mu}_K(t) + D_K(t)}{\mu_K(t)} = \frac{\dot{\mu}_H(t) + D_H(t)}{\mu_H(t)}$$

where $D_K(t)$ and $D_H(t)$ are "dividend payments" on physical and human capital, respectively:

$$\begin{aligned} D_K(t) &\equiv \mu_K(t)\left[F_K(k(t),1) - \delta_K\right] \\ D_H(t) &\equiv \mu_H(t)\left[\frac{Z_E}{1-s_E} - \delta_H\right] \end{aligned}$$

- The key expressions of the Lucas-Uzawa model are gathered in **Table 14.3** ($p(t) \equiv \mu_H(t)/\mu_K(t)$ is the relative shadow price of human capital).

Table 14.3: The Lucas-Uzawa model of growth and human capital accumulation

$$\frac{\dot{p}(t)}{p(t)} = r(t) + \delta_H - \frac{Z_E}{1 - s_E} \tag{T3.1}$$

$$\frac{\dot{C}(t)}{C(t)} = \sigma \left[r(t) - \rho \right] \tag{T3.2}$$

$$\frac{\dot{K}(t)}{K(t)} = (1 - g) Z_Y k(t)^{\alpha - 1} - \frac{C(t)}{K(t)} - \delta_K \tag{T3.3}$$

$$\frac{\dot{H}(t)}{H(t)} = Z_E l_E(t) - \delta_H \tag{T3.4}$$

$$p(t) = (1 - s_E)(1 - \alpha) \frac{Z_Y L_0}{Z_E} k(t)^{\alpha} \tag{T3.5}$$

$$k(t) \equiv \frac{K(t)}{\left[1 - l_E(t)\right] L_0 H(t)} \tag{T3.6}$$

$$r(t) \equiv \alpha Z_Y k(t)^{\alpha - 1} - \delta_K \tag{T3.7}$$

Steady-state growth in the Lucas-Uzawa model (1)

- Balanced growth path easy to analyze.
- Define $\theta(t) \equiv C(t)/K(t)$ and $\kappa(t) \equiv K(t)/H(t)$.
- Along the balanced growth path, consumption and the stocks of physical and human capital all grow at the same exponential growth rate, γ^*, so that $\theta(t) = \theta^*$ and $\kappa(t) = \kappa^*$. Also $p(t) = p^*$, $l_E(t) = l_E^*$, $k(t) = k^*$, and $r(t) = r^*$. The steady state can be solved recursively.
- **Step 1**: Equation (T3.1) fixes the steady-state interest rate:

$$r^* = \frac{Z_E}{1 - s_E} - \delta_H$$

Steady-state growth in the Lucas-Uzawa model (2)

- **Step 2**: Given r^*, (T3.2) and (T3.7) determine, respectively, γ^* and k^*:

$$\gamma^* = \sigma \cdot [r^* - \rho] = \sigma \cdot \left[\frac{Z_E}{1 - s_E} - \delta_H - \rho \right] \quad \text{(S18)}$$

$$k^* = \left(\frac{\alpha Z_Y}{r^* + \delta_K} \right)^{1/(1-\alpha)}$$

- **Step 3**: Given γ^* and k^* we find from (T3.3)-(T3.5):

$$\theta^* = \frac{1-g}{\alpha} (r^* + \delta_K) - \gamma^* - \delta_K$$

$$l_E^* = \frac{\gamma^* + \delta_H}{Z_E}$$

$$p^* = (1 - s_E)(1 - \alpha) \frac{Z_Y L_0}{Z_E} (k^*)^\alpha$$

Steady-state growth in the Lucas-Uzawa model (3)

- **Step 4**: Given k^* and l_E^* we obtain from (T3.6):

$$\kappa^* = k^* \cdot [1 - l_E^*] \cdot L_0$$

- **Step 5**: It remains to be checked that the (common) growth rate given in (S18) is actually feasible ($l_E^* < 1$). The feasibility requirement thus places an upper limit on the allowable intertemporal substitution elasticity:

$$\sigma < \frac{Z_E - \delta_H}{Z_E / (1 - s_E) - (\rho + \delta_H)}$$

Transitional dynamics in the Lucas-Uzawa model (1)

- In esssence only three fundamental dynamic variables in Table 14.3: $p(t)$, $\theta(t) \equiv C(t)/K(t)$, and $\kappa(t) \equiv K(t)/H(t)$.
- Two quasi-reduced-form relationships.
- **Relationship 1**. It follows from (T3.5) that $k(t)$ is an increasing function of both $p(t)$ and s_E:

$$k(t) = \left(\frac{Z_E p(t)}{(1-\alpha) Z_Y L_0 (1-s_E)} \right)^{1/\alpha} \equiv \Psi(\underset{+}{p(t)}, \underset{+}{s_E}) \quad \text{(S19)}$$

- **Relationship 2**. We find from (T3.6) that $l_E(t)$ depends negatively on $\kappa(t)$ and positively on $k(t)$ (and thus, via (S19), on $p(t)$ and s_E):

$$l_E(t) = 1 - \frac{\kappa(t)}{L_0 \Psi(p(t), s_E)} \quad \text{(S20)}$$

Transitional dynamics in the Lucas-Uzawa model (2)

- Hence, it follows from (S19)–(S20) that $k(t)$ and $l_E(t)$ are uniquely determined by the fundamental state variables, $p(t)$ and $\kappa(t)$.
- In order to study the dynamic properties of the model, we log-linearize it around the steady-state point (θ^*, κ^*) to obtain:

$$\begin{bmatrix} \dfrac{d\ln p(t)}{dt} \\ \dfrac{d\ln \theta(t)}{dt} \\ \dfrac{d\ln \kappa(t)}{dt} \end{bmatrix} = \Delta \cdot \begin{bmatrix} \ln p(t) - \ln p^* \\ \ln \theta(t) - \ln \theta^* \\ \ln \kappa(t) - \ln \kappa^* \end{bmatrix}$$

Transitional dynamics in the Lucas-Uzawa model (3)

- Δ is the Jacobian matrix:

$$\Delta \equiv \begin{bmatrix} -\dfrac{(1-\alpha)(r^*+\delta_K)}{\alpha} & 0 & 0 \\ -\dfrac{(1-\alpha)\sigma(r^*+\delta_K)+Z_E\left(1-l_E^*\right)}{\alpha} & 0 & Z_E\left(1-l_E^*\right) \\ -\dfrac{(1-\alpha)(1-g)(r^*+\delta_K)+\alpha Z_E\left(1-l_E^*\right)}{\alpha^2} & -\theta^* & Z_E\left(1-l_E^*\right) \end{bmatrix}$$

Transitional dynamics in the Lucas-Uzawa model (4)

- The determinant of Δ is given by:

$$|\Delta| \equiv -\frac{(1-\alpha)(r^* + \delta_K) Z_E (1 - l_E^*) \theta^*}{\alpha} < 0$$

so it follows that the product of the characteristic roots of Δ is negative, i.e. there is an odd number of negative roots.

- In the text we prove saddle-point stability: one stable root ($-\lambda_1 < 0$) and two unstable roots ($\lambda_2 > 0$ and $\lambda_3 > 0$).
- The model features two jumping variables ($p(t)$ and $\theta(t)$) and one predetermined (sticky) variable ($\kappa(t)$). The adjustment speed in the economy is given by λ_1. Given initial values for $K(0)$ and $H(0)$ (and thus for $\kappa(0) \equiv K(0)/H(0)$), the model converges along the saddle path toward the steady-state equilibrium.