Optimal environmental policy in the presence of multiple equilibria and reversible hysteresis*

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Abstract: We study optimal environmental policy in an economy-ecology model featuring multiple stable steady-state ecological equilibria. The policy instruments consist of public abatement and a tax on the polluting production input, which we assume to be the stock of capital. The isocline for the stock of pollution features two stable branches, a low-pollution (good) and a high-pollution (bad) one. Assuming that the ecology is initially located on the bad branch of the isocline, the ecological equilibrium is reversibly hysteretic and a suitably designed environmental policy can be used to steer the environment from the bad to the good equilibrium. We study both first-best and second-best social optima. We show that, compared to capital taxation, abatement constitutes a very cheap instrument of environmental policy.

Keywords: Ecological thresholds, nonlinear dynamics, environmental policy, abatement, capital taxes.

JEL Codes: D60, E62, H23, H63, Q20, Q28, Q50.

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1 Introduction

In this chapter we study optimal environmental policy using a dynamic model featuring interactions between the ecological system and the macro-economy. In line with the recent environmental literature, we assume that the ecological process is nonlinear such that (i) ecosystems do not respond smoothly to gradual changes in dirt flows and abrupt "catastrophic shifts" may be possible in the vicinity of threshold points, (ii) there may be multiple stable equilibria, and (iii) irreversibility and hysteresis are both possible (Scheffer *et al.*, 2001). The nonlinear ecological dynamics described by Scheffer (1998) and employed by us now carries the name Shallow-Lake Dynamics (SLD hereafter).[1]

To describe the macroeconomic system we use a standard Ramsey-Cass-Koopmans model of a closed economy. Households practice intertemporal consumption smoothing and accumulate capital that is rented out to perfectly competitive firms. Following Bovenberg and Heijdra (1998, 2002), we assume that the capital stock is the polluting production factor. Households enjoy living in a clean environment but act as free riders and thus fail to internalize the external effects caused by their capital accumulation decisions.

We assume that the initial steady-state confronting the policy maker has the following features. First, there is no pre-existing policy regarding the environment, i.e. public abatement activities are absent and there is no externality-correcting tax on capital in place. Second, the flow of dirt is such that there exist two stable ecological steady-state equilibria. Third, the ecological system has settled down at the "bad" equilibrium featuring a high stock of pollution. In this setting the policy maker is in principle able to engineer substantial welfare gains by choosing the appropriate mix of capital taxation and abatement activities.

The chapter is structured as follows. Section 2 presents the model, consisting of an ecological system featuring SLD and an economic system. Section 3 studies the first-best social optimum. The optimal environmental policy can be decentralized with the aid of time-varying abatement and capital taxation. Section 4 studies optimal environmental policy in a second-best setting. In particular we consider the repercussions of two types of constraints on the policy maker's choices, namely the unavailability of instruments and the insufficient flexibility of a given instrument. Finally, in Section 5 we offer a brief summary of the main results, whilst the Appendix presents some computational details.

[1]There is an emerging literature on the SLD approach as it is used in economics – see Heijdra and Heijnen (2013) for an extensive list of references.

2 The model[2]

We model the environment as a renewable resource stock, the quality of which depends negatively on the *flow* of dirt, $D(t)$, that is generated in the production process:

$$D(t) \equiv \kappa K(t) - \gamma G(t), \qquad \kappa > 0,\ \gamma > 0, \tag{1}$$

where $K(t)$ is the private capital stock (see below), and $G(t)$ represents abatement activities by the government. Capital is the polluting factor of production, just as in Bovenberg and Heijdra (1998, 2002). By definition the flow of dirt must be non-negative ($D(t) \geq 0$). Denoting the *stock* of pollution at time t by $P(t)$, we write the general form of the emission equation as:

$$\dot{P}(t) = -\Phi\left(P\left(t\right)\right) + D(t), \tag{2}$$

where $\dot{P}(t) \equiv dP(t)/dt$ and $\Phi\left(P\left(t\right)\right)$ is a nonlinear function whose definition and properties are stated in the following Lemma.

Lemma 1. *Let $\Phi\left(x\right)$ for $x \geq 0$ be given by:*

$$\Phi\left(x\right) \equiv \pi x - \frac{x^2}{x^2+1}, \qquad \frac{1}{2} < \pi < \frac{3\sqrt{3}}{8}.$$

The first- and second derivatives of $\Phi\left(x\right)$ are given by:

$$\Phi'\left(x\right) \equiv \pi - \frac{2x}{\left[x^2+1\right]^2}, \qquad \Phi''\left(x\right) \equiv \frac{2\left[3x^2-1\right]}{\left[x^2+1\right]^3}.$$

The following properties can be established: (i) $\Phi\left(x\right) = 0$ for $x = 0$ and $\Phi\left(x\right) > 0$ for $x > 0$; (ii) $\Phi\left(x\right)$ attains a local maximum at x_1 such that $\Phi'\left(x_1\right) = 0$ and $\Phi''\left(x_1\right) < 0$ and a local minimum at x_2 such that $\Phi'\left(x_2\right) = 0$ and $\Phi''\left(x_2\right) > 0$; (iii) $\Phi'\left(x\right) > 0$ for $0 < x < x_1$ and $x > x_2$; (iv) $\Phi'\left(x\right) < 0$ for $x_1 < x < x_2$.

The isocline for the stock of pollution is depicted in Figure 1. Given the range of values of π, the pollution isocline is S-shaped, with sharp turns at points C and B. The dirt levels associated with these threshold point are denoted by, respectively, D_L and D_U. The vertical arrows depict the dynamic forces operating on the stock of pollution off the isocline. The upward sloping branches of the isocline are locally stable: Lemma 1(iii) establishes that $\partial\dot{P}\left(t\right)/\partial P\left(t\right) = -\Phi'\left(P\left(t\right)\right) < 0$ there. In contrast, the downward sloping (dashed) branch is unstable because Lemma 1(iv) shows that $\partial\dot{P}\left(t\right)/\partial P\left(t\right) > 0$ for these points. For future reference we state the following Definition.

[2] Apart from the introduction of a tax on capital, the model used here is identical to the one discussed in more detail in Heijdra and Heijnen (2013).

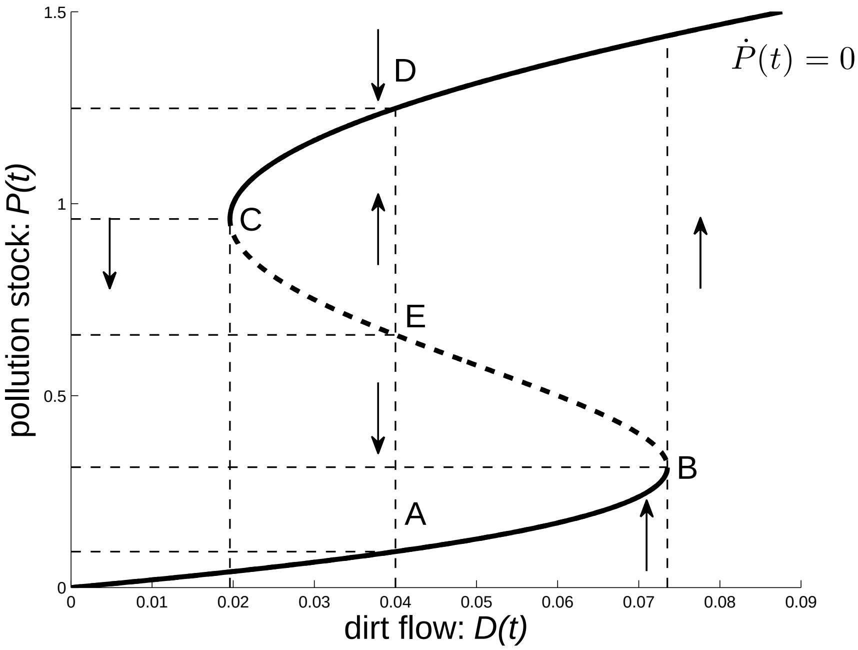


Figure 1: Ecological dynamics

Definition 1. *Define the clean branch of the pollution isocline as* $\Phi_C(x) \equiv \Phi(x)$ *for* $0 \leq x < x_1$ *and the dirty branch as* $\Phi_D(x) \equiv \Phi(x)$ *for* $x > x_2$.

Consider a time-invariant dirt flow $\hat{D}$. Depending on its magnitude, three regimes are possible:

- Unique stable and clean steady-state. For $0 \leq \hat{D} < D_L$ there exists a unique steady-state pollution level that is located on the lower branch of the pollution isocline.
- Multiple steady-state pollution levels. For $D_L \leq \hat{D} \leq D_U$ there exist three ecological steady-state equilibria, of which two are stable and one is unstable. For example, if $\hat{D} = 0.04$ the stable equilibria are at points A and D in Figure 1 whilst the instable one is at point E. Which particular steady state is attained depends on initial conditions, i.e. the ecological model features *reversible hysteresis*.
- Unique stable and polluted steady-state. For $\hat{D} > D_U$ there exists a unique steady-state pollution level that is located on the upper branch of the pollution isocline.

To capture the key features of the economic system we formulate a simple general equilibrium model of the macro-economy. This model describes a closed economy consisting of a government and representative households and firms who are blessed with perfect foresight.

The representative household lives forever, and features the following utility functional:

$$\Lambda(t) \equiv \int_t^\infty \left[\ln C(\tau) + \varepsilon_E \ln\left[\bar{E} - P(\tau)\right] \right] \cdot e^{-\rho(\tau - t)} d\tau, \tag{3}$$

where $C(\tau)$ denotes consumption of private commodities at time τ, $E(\tau) \equiv \bar{E} - P(\tau) > 0$ measures the quality of the environment, $\bar{E}$ is some pristine value attained in a non-polluting society, ε_E denotes the weight in overall utility attached to environmental amenities, and $\rho > 0$ stands for the pure rate of time preference. Since utility is separable in its two arguments, the quality of the environment does not directly affect household consumption. As the felicity function for private consumption is logarithmic, the model features a unitary intertemporal elasticity of substitution. Without leisure entering utility, labour supply is exogenously fixed.

Households face the following budget identity:

$$\dot{A}(\tau) = r(\tau)A(\tau) + w(\tau) - T(\tau) - C(\tau), \tag{4}$$

where $r(\tau)$ denotes the real rate of interest on financial assets, $w(\tau)$ represents the wage rate, $T(\tau)$ are net lump-sum taxes, and $A(\tau)$ stands for real financial assets owned in period τ.

The representative agent chooses paths for $C(\tau)$ and $A(\tau)$ which maximize (3) subject to (4) and a solvency requirement of the form $\lim_{\tau\to\infty} A(\tau) e^{-\int_t^\tau r(s)ds} = 0$. He takes as given the stock of financial assets in the planning period, $A(t)$. The optimal consumption level that the agent chooses at time t is given by:

$$C(t) = \rho[A(t) + H(t)], \tag{5}$$

where human wealth, $H(t)$, is defined as:

$$H(t) \equiv \int_t^\infty [w(\tau) - T(\tau)] \cdot e^{-\int_t^\tau r(s)ds} d\tau. \tag{6}$$

The optimal time profile for consumption is given by the Euler equation:

$$\frac{\dot{C}(\tau)}{C(\tau)} = r(\tau) - \rho, \qquad \tau \geq t. \tag{7}$$

The intuitive interpretation of these expressions is as follows. Equation (5) shows that the agent consumes a constant proportion of total wealth in the planning period, whilst equation (7) indicates that consumption growth over time is chosen to be equal to the anticipated gap between the interest rate and the rate of time preference. Finally, the expression in (6) implies that human wealth is given by the discounted value of after-tax wage payments using the market rate of interest for discounting purposes. Intuitively it represents the after-tax value of the agent's unitary time endowment.

The production sector of the economy is perfectly competitive. The production function is Cobb-Douglas, with constant returns to scale to the factors capital, $K(t)$, and labour, $L(t)$:

$$Y(t) \equiv F(K(t), L(t)) = \Omega_0 K(t)^{1-\varepsilon_L} L(t)^{\varepsilon_L}, \qquad \Omega_0 > 0,\ 0 < \varepsilon_L < 1, \tag{8}$$

where $Y(t)$ denotes gross output. The value of the firm, $V(t)$, is given by the present value of the after-tax cash flow using the market rate of interest for discounting purposes:

$$V(t) = \int_t^\infty \left[(1-\theta(\tau))\left[Y(\tau) - w(\tau)L(\tau)\right] - I(\tau)\right] \cdot e^{-\int_t^\tau r(s)ds} d\tau, \tag{9}$$

where $\theta(\tau)$ is the capital tax and $I(\tau)$ is gross investment. The capital stock evolves according to:

$$\dot{K}(\tau) = I(\tau) - \delta K(\tau), \tag{10}$$

where $\dot{K}(\tau) \equiv dK(\tau)/d\tau$ denotes the rate of change in the capital stock and δ is the depreciation rate ($\delta > 0$).

The representative firm chooses paths for $Y(\tau)$, $K(\tau)$, $L(\tau)$ and $I(\tau)$ which maximize the value of the firm (9) subject to the production function (8), and the capital accumulation identity (10). The capital stock in the planning period, $K(t)$, is taken as given. The first-order conditions yield the usual marginal productivity conditions:

$$\frac{\partial Y(\tau)}{\partial K(\tau)} = \frac{r(\tau)+\delta}{1-\theta(\tau)}, \tag{11}$$

$$\frac{\partial Y(\tau)}{\partial L(\tau)} = w(\tau). \tag{12}$$

Since we abstract from adjustment costs in investment, the value of equity corresponds to the replacement value of the capital stock, i.e. $V(t) = K(t)$.

For convenience, the key equations of the core model have been gathered in Table 1. Equation (T1.1) is the Euler equation (7), whilst equations (T1.5) and (T1.7)–(T1.8) just restate, respectively (8), (2), and (1). Labour supply is exogenous so $L(t) = 1$ – see (T1.6). The factor demand expressions in (11)–(11) have been rewritten by using the production function – see (T1.3) and (T1.4). Equation (T1.2) is obtained by combining (10) with the goods market clearing condition for a closed economy, i.e. $Y(\tau) = C(\tau) + I(\tau) + G(\tau)$. Finally, in the absence of government debt, claims on the capital stock are the only assets available, i.e. $A(t) = K(t)$.

The phase diagram for the economic system is depicted in Figure 2. The initial equilibrium, by assumption featuring no public abatement, is at point E_0. Steady-state consumption and the capital stock are given by, respectively, $\hat{C}$ and $\hat{K}$. The equilibrium is saddle-point sta-

Table 1: The model

$$\frac{\dot{C}(t)}{C(t)} = r(t) - \rho, \qquad \rho > 0 \tag{T1.1}$$

$$\dot{K}(t) = Y(t) - C(t) - G(t) - \delta K(t) \tag{T1.2}$$

$$[r(t) + \delta]\, K(t) = (1 - \varepsilon_L)\,[1 - \theta(t)]\, Y(t) \tag{T1.3}$$

$$w(t)L(t) = \varepsilon_L Y(t) \tag{T1.4}$$

$$Y(t) = \Omega_0 L(t)^{\varepsilon_L} K(t)^{1-\varepsilon_L}, \qquad \Omega_0 > 0,\ 0 < \varepsilon_L < 1 \tag{T1.5}$$

$$L(t) = 1 \tag{T1.6}$$

$$T(t) = G(t) - \theta(t)\,[Y(t) - w(t)L(t)] \tag{T1.7}$$

$$\dot{P}(t) = -\pi P(t) + \frac{P(t)^2}{P(t)^2 + 1} + D(t), \qquad \frac{1}{2} < \pi < \frac{3\sqrt{3}}{8} \tag{T1.8}$$

$$D(t) = \kappa K(t) - \gamma G(t), \qquad \kappa > 0,\ \gamma > 0 \tag{T1.9}$$

Endogenous: consumption, $C(t)$, capital stock, $K(t)$, output, $Y(t)$, interest rate, $r(t)$, wage rate, $w(t)$, employment, $L(t)$, pollution stock, $P(t)$, dirt flow, $D(t)$. **Exogenous**: capital tax $\theta(t)$ and government abatement, $G(t)$. **Parameters**: rate of time preference, ρ, depreciation rate of capital, δ, labour coefficient in the technology, ε_L, and scale factor in the technology, Ω_0. **Ecological parameters**: lake resilience, π, capital dirt coefficient, κ, and abatement clean-up coefficient, γ.

ble, with SP_0 representing the saddle path, and is dynamically efficient, i.e. $\hat{K}$ is strictly less than the golden-rule capital stock, $\hat{K}^{GR}$.

3 First-best social optimum

In the remainder of this chapter we consider optimal environmental policy. The initial situation facing the policy maker is as follows. First, both the economic and ecological systems are in a steady-state equilibrium and environmental abatement is zero. Second, the steady-state dirt flow resulting from the equilibrium capital stock is such that there exist three possible ecological steady-state equilibria. Third, for otherwise unspecified reasons, the ecological system has settled down at the "bad" equilibrium featuring a high stock of pollution. In Figure 2 the initial economic equilibrium is thus at point E_0. In Figure 1 the dirt flow equals $\kappa\hat{K}$ and the ecological equilibrium is located at point D. Given this initial condition, can the policy maker bring about substantial welfare gains by choosing the appropriate mix of capital taxation and abatement activities?

In this section we characterize the first-best social optimum, i.e. we study the allocation that would be selected by a benevolent social planner aiming to maximize lifetime utility of the representative agent. In the planning period $t = 0$, the planner chooses paths for $C(t)$,

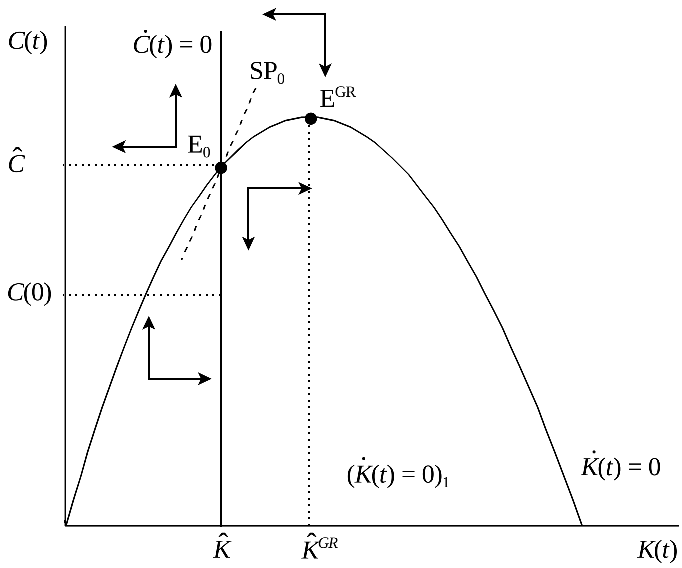


Figure 2: Consumption-capital dynamics

$P(t)$, and $K(t)$ (for $t \geq 0$) in order to maximize (3) subject to the resource constraint (T1.2), the emission equation (2), and the dirt flow definition (1). The initial conditions are:

$$K(0) = \hat{K}, \qquad P(0) = \hat{P}_B = \Phi_D^{-1}(\hat{D}_0), \tag{13}$$

where $\hat{P}_B$ is the steady-state pollution level consistent with the upper branch of the pollution isocline (see Definition 1) and with a dirt flow equal to $\hat{D}_0 = \kappa\hat{K}$ – see point D in Figure 1. Abatement, the dirt flow, and gross investment must remain non-negative:

$$G(t) \geq 0, \qquad [D(t) \equiv] \, \kappa K(t) - \gamma G(t) \geq 0 \qquad [I(t) \equiv] \, F\left(K(t), 1\right) - C(t) - G(t) \geq 0. \tag{14}$$

Dropping the time index, the current-value Hamiltonian can be written as:

$$\begin{aligned} \mathcal{H} \;\; \equiv \;\; & \ln C + \varepsilon_E \ln[\bar{E} - P] + \lambda_K \left[F(K, 1) - C - G - \delta K\right] \\ & + \lambda_P \left[-\Phi(P) + \kappa K - \gamma G\right] + \eta_D \left[\kappa K - \gamma G\right] + \eta_I \left[F(K, 1) - C - G\right]. \end{aligned}$$

The control variables for this optimization problem are C and G (and thus implicitly D and I), the state variables are K and P, the co-state variables are λ_K and λ_P, and η_D and η_I are the Lagrange multipliers for, respectively, the dirt and investment constraints. The first-order conditions are:

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial C} &= \frac{1}{C} - (\lambda_K + \eta_I) = 0, &(15)\\
\frac{\partial \mathcal{H}}{\partial G} &= -(\lambda_K + \eta_I) - \gamma(\lambda_P + \eta_D) \leq 0, \qquad G \geq 0, \qquad G\frac{\partial \mathcal{H}}{\partial G} = 0, &(16)\\
\frac{\partial \mathcal{H}}{\partial \eta_D} &= \kappa K - \gamma G \geq 0, \qquad \eta_D \geq 0, \qquad \eta_D \frac{\partial \mathcal{H}}{\partial \eta_D} = 0, &(17)\\
\frac{\partial \mathcal{H}}{\partial \eta_I} &= F(K,1) - C - G \geq 0, \qquad \eta_I \geq 0, \qquad \eta_I \frac{\partial \mathcal{H}}{\partial \eta_I} = 0, &(18)\\
\dot{\lambda}_K - \rho\lambda_K &= -\frac{\partial \mathcal{H}}{\partial K} = -\kappa(\lambda_P + \eta_D) - [F_K(K,1) - \delta]\lambda_K - \eta_I F_K(K,1), &(19)\\
\dot{\lambda}_P - \rho\lambda_P &= -\frac{\partial \mathcal{H}}{\partial P} = \frac{\varepsilon_E}{\bar{E} - P} + \lambda_P \Phi'(P). &(20)
\end{aligned}$$

The first-best social optimum is characterized by (2), (T1.2), (14), (15)–(20) and the transversality conditions:

$$\lim_{t\to\infty} e^{-\rho t}\lambda_K(t)K(t) = \lim_{t\to\infty} e^{-\rho t}\lambda_P(t)P(t) = 0. \tag{21}$$

3.1 Long-run optimum

We first study the long-run properties of the first-best equilibrium. In terms of notation, hatted variables denote steady-state values and the subscript "f" denotes first-best. In the steady state gross investment is strictly positive, i.e. $\hat{I}_f = \delta\hat{K}_f > 0$ and it follows from (18) that $\hat{\eta}_I = 0$. Depending on the structural parameters and the resulting magnitude of $\hat{G}_f$ two cases are possible.

Case 1: With long-run abatement Assume that $0 < \hat{G}_f < (\kappa/\gamma)\hat{K}_f$ so that $\hat{\eta}_D = 0$ and $\gamma\hat{\lambda}_P = -\hat{\lambda}_K < 0$. It follows that the steady-state first-best equilibrium is given by:

$$\begin{aligned}
F_K(\hat{K}_f, 1) &= \rho + \delta + \frac{\kappa}{\gamma}, &(22)\\
\rho + \Phi_C'(\hat{P}_f) &= \gamma\frac{\varepsilon_E \hat{C}_f}{\bar{E} - \hat{P}_f}, &(23)\\
F(\hat{K}_f, 1) &= \hat{C}_f + \hat{G}_f + \delta\hat{K}_f, &(24)\\
\Phi_C(\hat{P}_f) &= \kappa\hat{K}_f - \gamma\hat{G}_f, &(25)
\end{aligned}$$

where $\Phi_C(x)$ is the function representing the lower branch of the P-isocline – see Definition 1. The key thing to note is that a (Pigouvian) capital tax can be used to decentralize the first-best equilibrium. Equation (T1.3) shows that private saving behaviour will result in a steady-state capital stock such that $F_K(\hat{K}, 1) = (\rho + \delta)/(1 - \hat{\theta})$. By comparing this expression to (22) we find that $\hat{K} = \hat{K}_f$ if and only if the steady-state capital tax is set equal to:

$$\hat{\theta}_f = \frac{\kappa/\gamma}{\rho + \delta + \kappa/\gamma}. \tag{26}$$

The optimal Pigouvian capital tax is feasible (as it satisfies $0 < \hat{\theta}_f < 1$) and is increasing in the κ/γ. Intuitively, the more polluting is capital (κ up) and the less potent is abatement (γ down), the higher is the optimal environmental tax.

Case 2: Without long-run abatement Assume that $\hat{\lambda}_K > -\gamma\hat{\lambda}_P$ so that $\hat{G}_f = 0$. Since $\hat{K}_f > 0$ it follows that $\hat{D}_f > 0$ and thus $\hat{\eta}_D = 0$ also. The first-best steady-state equilibrium can now be written as:

$$F_K(\hat{K}_f, 1) = \rho + \delta - \kappa\hat{\lambda}_P\hat{C}_f, \tag{27}$$

$$\rho + \Phi_C'(\hat{P}_f) = -\frac{1}{\hat{\lambda}_P}\frac{\varepsilon_E}{\bar{E} - \hat{P}_f}, \tag{28}$$

$$F(\hat{K}_f, 1) = \hat{C}_f + \delta\hat{K}_f, \tag{29}$$

$$\Phi_C(\hat{P}_f) = \kappa\hat{K}_f. \tag{30}$$

Just as for the previous case, a capital tax is needed to decentralize the first-best optimum:

$$\hat{\theta}_f = \frac{-\kappa\hat{\lambda}_P\hat{C}_f}{\rho + \delta - \kappa\hat{\lambda}_P\hat{C}_f}. \tag{31}$$

Since $\hat{\lambda}_P < 0$ and $\hat{C}_f > 0$ it follows that the optimal Pigouvian capital tax is feasible, i.e. $0 < \hat{\theta}_f < 1$.

3.2 Optimal dynamic allocation

In order to avoid having to deal with a taxonomy of possible cases, we use a parameterized version of the model to illustrate its main properties. For reasons of comparison we use the same parameterization as in Heijdra and Heijnen (2013) – see Table 2. For these parameter values we find that $-\hat{\lambda}_K - \gamma\hat{\lambda}_P = -0.9198$, i.e. Case 2 is the relevant one and abatement is not needed in the long run, i.e. $\hat{G}_f = 0$. We furthermore compute $\hat{K}_f = 2.3177$, $\hat{C}_f = 0.7901$, $\hat{Y}_f = 0.9524$, $\hat{P}_f = 0.0766$, and $\hat{D}_f = 0.0340$. For ease of comparison, we report these values in column (b) in Table 3. The long-run Pigouvian capital tax is $\theta_f = 0.1066$ and consumption, output, and the capital stock are all lower than in the initial steady-state equilibrium the key features of which have been reported in column (a) of Table 3.

The dynamic properties of the first-best optimum are illustrated in Figure 3. Details of the computations are found in the Appendix. There are two critical dates characterizing the optimal solution, namely the earliest time at which the irreversibility constraint on in-

Table 2: Structural parameters and steady-state features

Economic system:					
$\rho = 0.04$	$\delta = 0.07$	$\varepsilon_L = 0.70$	$\Omega_0 = 0.7401$		
$\hat{r} = 0.04$	$\hat{K} = 2.7273$	$\hat{Y} = 1.000$	$\hat{C} = 0.8091$	$\hat{I} = 0.1909$	$G = 0$
Ecological system:					
$\pi = 0.52$	$\kappa = 0.0147$	$\gamma = 0.302$	$\varepsilon_E = 0.9$	$\bar{E} = 2$	
$D_L = 0.0196$	$D_U = 0.0735$	$\hat{D}_0 = 0.04$	$\hat{P}_B = 1.2482$	$\hat{P}_G = 0.0936$	$P_E = 0.6581$

vestment ceases to bind, $t_I = 1.27$, and the time at which the dirt constraint becomes slack, $t_D = 27.01$. Together these dates define the three regimes through which the optimal paths evolve.

3.2.1 Regime 1

For $0 \leq t \leq t_I$ both the dirt flow and gross investment are zero, i.e. $D_f(t) = 0$ and $I_f(t) = 0$. It follows that abatement is at its maximum feasible level given by $G_f(t) = (\kappa/\gamma) K_f(t)$, consumption is described by $C_f(t) = F\left(K_f(t), 1\right) - (\kappa/\gamma) K_f(t)$, whilst the capital stock satisfies $\dot{K}_f(t) = -\delta K_f(t)$. By combining these expressions and noting that $K(0) = \hat{K}$ we find:

$$
\begin{aligned}
K_f(t) &= \hat{K}e^{-\delta t}, \\
C_f(t) &= \hat{Y}e^{-\delta(1-\varepsilon_L)t} - \frac{\kappa}{\gamma}\hat{K}e^{-\delta t}, \\
G_f(t) &= \frac{\kappa}{\gamma}\hat{K}e^{-\delta t}.
\end{aligned}
$$

The transition paths for $K_f(t)$, $C_f(t)$, and $G_f(t)$ have been depicted in, respectively, panels (c), (d), and (a) of Figure 3. With the flow of dirt reduced to zero, the stock of pollution falls according to:

$$
\dot{P}_f(t) = -\Phi(P_f(t)).
$$

3.2.2 Regime 2

For $t_I < t \leq t_D$ the dirt flow is zero but gross investment is strictly positive, i.e. $D_f(t) = 0$ and $I_f(t) > 0$. Abatement remains at its maximum feasible level, $G_f(t) = (\kappa/\gamma) K_f(t)$. Since the capital stock is continuous for all t, it follows that the path of abatement is also continuous throughout this regime. Since the non-negativity constraint for gross investment ceases to be binding for $t > t_I$, the consumption path follows the Euler equation:

$$
\frac{\dot{C}_f(t)}{C_f(t)} = F\left(K_f(t), 1\right) - \left(\rho + \delta + \frac{\kappa}{\gamma}\right),
$$

Table 3: Quantitative effects of taxation and abatement[‡]

	BM	FBSO	SBSO			
			Taxation		Abatement	
			TV	TI	TV	TI
	(a)	(b)	(c)	(d)	(e)	(f)
$\hat{Y}$	1.0000	0.9524	0.9524	0.9524	1.0000	1.0000
$C(0)$		0.8677	1.0000	1.0000	0.6798	0.6933
$\hat{C}$	0.8091	0.7901	0.7901	0.7901	0.8091	0.8091
$\hat{K}$	2.7273	2.3177	2.3177	2.3177	2.7273	2.7273
$\hat{P}$	1.2482	0.0766	0.0766	0.0766	0.0936	0.0936
$\Lambda(0)$	−11.9092	−2.7722	−7.7638	−7.9525	−3.2471	−4.3087
$\theta(0)$		0.1234	0.1891	0.8500		
$\hat{\theta}$		0.1077	0.1077	0.1077		
$G(0)$		0.1326			0.1324	0.1166
$\hat{G}$		0.0000			0.0000	0.0000
t_E				39.5	28.2	30.0
$EV(0)$		44.1	17.1	16.2	40.5	34.5

[‡]BM: parameterized base model. FBSO: first-best social optimum. SBSO: second-best social optimum. Policy instrument lacking or not sufficiently flexible. TV: time-varying instrument. TI: time-invariant instrument. Notation: $x(0)$ and $\hat{x}$ denote, respectively, the impact- and long-run (steady-state) value of the variable $x(t)$.

whilst the stocks of capital and pollution evolve according to:

$$\begin{aligned}
\dot{K}_f(t) &= F\left(K_f(t),1\right) - C_f(t) - \left(\delta + \frac{\kappa}{\gamma}\right) K_f(t), \\
\dot{P}_f(t) &= -\Phi(P_f(t)).
\end{aligned}$$

Consumption is continuous at time t_I, i.e. $\lim_{t\nearrow t_I} C_f(t) = \lim_{t\searrow t_I} C_f(t) = C_f(t_I)$, so that $C_f(t_I) = \hat{Y}e^{-\delta(1-\varepsilon_L)t_I} - \frac{\kappa}{\gamma}\hat{K}e^{-\delta t_I}$ and $K_f(t_I) = e^{-\delta t_I}\hat{K}$ are the initial conditions for the system of differential equation in $C_f(t)$ and $K_f(t)$.

3.2.3 Regime 3

At time t_D abatement is permanently reduced to zero ($G_f(t) = 0$) and the dirt flow becomes positive (as $D_f(t) = \kappa K_f(t)$). The value of t_D is such that $-\gamma\lambda_P(t_D) C_f(t_D) = 1$. Again, like the stocks of capital and pollution, consumption is continuous at time t_D, i.e. $\lim_{t\nearrow t_D} C_f(t) = \lim_{t\searrow t_D} C_f(t) = C_f(t_D)$. The optimal path for $t > t_D$ is described by:

$$\begin{aligned}
\frac{\dot{C}_f(t)}{C_f(t)} &= F_K\left(K_f(t),1\right) - (\rho+\delta) + \kappa\lambda_P(t)C_f(t), \\
\dot{\lambda}_P(t) &= \frac{\varepsilon_E}{\bar{E} - P_f(t)} + \left[\rho + \Phi_C'(P_f(t))\right]\lambda_P(t), \\
\dot{K}_f(t) &= F\left(K_f(t),1\right) - C_f(t) - \delta K_f(t), \\
\dot{P}_f(t) &= -\Phi(P_f(t)) + \kappa K_f(t).
\end{aligned}$$

This system converges to the steady state given in (27)–(30).

In passing through the three regimes, the first-best social optimum is decentralized by means of a tax on capital, $\theta_f(t)$, which is implicitly defined by:

$$\frac{\dot{C}_f(t)}{C_f(t)} = (1-\theta_f(t))F_K\left(K_f(t),1\right) - (\rho+\delta).$$

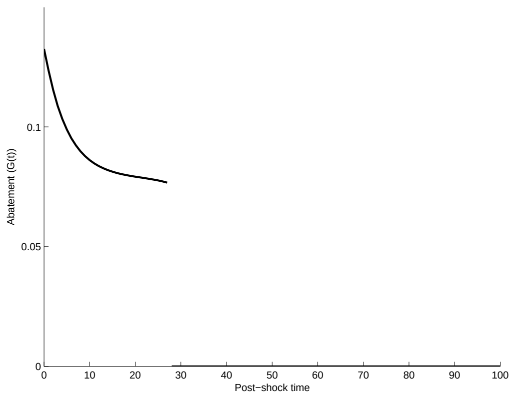
As is illustrated in panel (b) of Figure 3, the tax is quite high during the early phase of the environmental cleanup.

The welfare effect of the first-best optimal policy is considerable. Indeed, as our equivalent variation welfare measure $EV(0)$ in Table 3 reveals, the welfare gain due to the optimal environmental cleanup amounts to 44.1 percent of initial steady-state consumption.[3] Despite the fact consumption is lower that its initial level during much of the transition, the gradual improvement in environmental quality more than compensates for this.

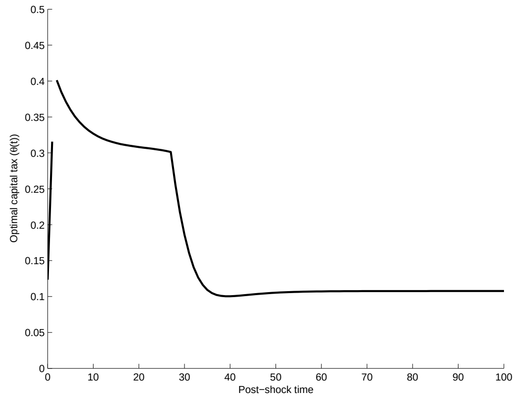
[3] See Heijdra and Heijnen (2013) for a further discussion of the equivalent variation measure used here.

Figure 3: The first-best optimal policy

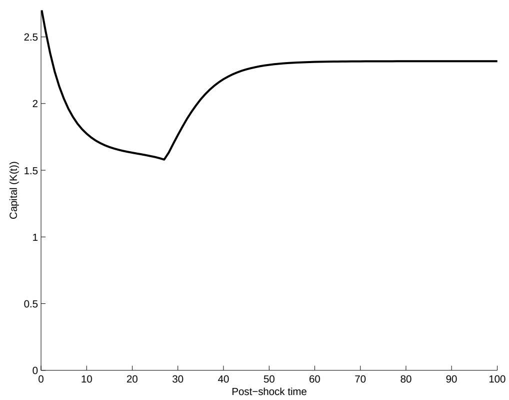
(a) government abatement $G(t)$



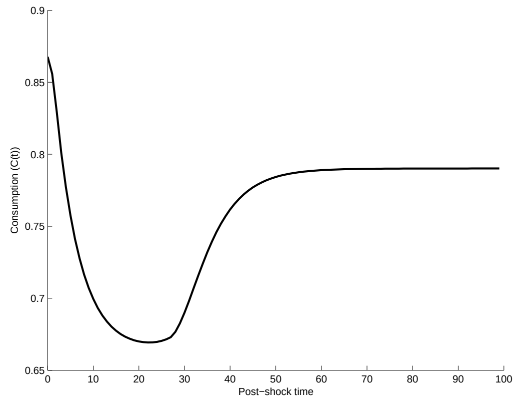
(b) capital tax $\theta(t)$



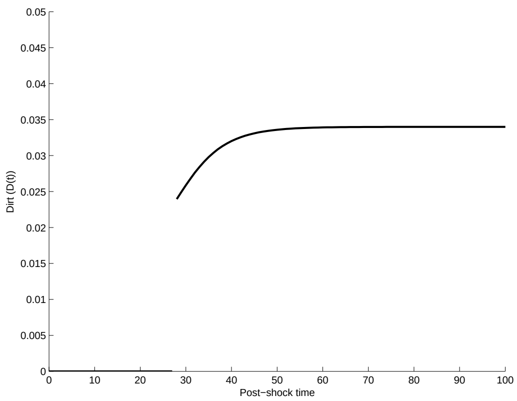
(c) capital stock $K(t)$



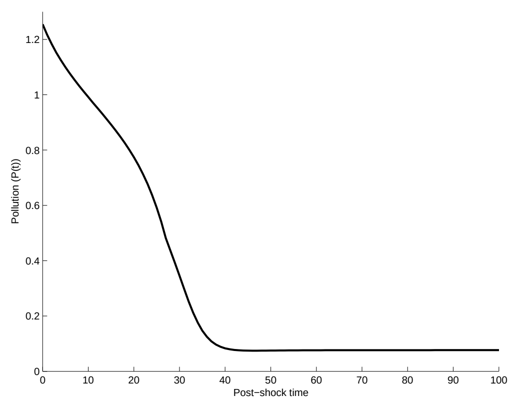
(d) consumption $C(t)$



(e) dirt flow $D(t)$



(f) pollution stock $P(t)$



Parameters: see Table 2. The initial ecological equilibrium is at point D in panel (b).

4 Second-best social optimum

In this section we study optimal environmental policy in a second-best setting. In particular we consider the repercussions of two types of constraints on the policy maker's choices, namely the unavailability of instruments and the insufficient flexibility of a given instrument. In subsections 4.1 and 4.2 we assume that the policy maker cannot use the abatement instrument and conducts constrained optimal environmental policy with either a time-varying capital tax (in subsection 4.1) or a time-invariant (step-wise) capital tax (in subsection 4.2).

In subsections 4.3 and 4.4 we study the alternative case in which the policy maker cannot use the tax instrument and is constrained to conduct optimal environmental policy with, respectively, a time-varying or time-invariant abatement program. The latter case coincides with the ad hoc policy studied in our earlier paper (Heijdra and Heijnen, 2013).

4.1 Time-varying taxation

The social planner chooses paths for $C(t)$, $P(t)$, and $K(t)$ (for $t \geq 0$) in order to maximize (3) subject to the resource constraint (T1.2), the emission equation (2), and the dirt flow definition (1). The initial conditions are as given in (13) above, and the non-negativity constraint on investment in (14) is still relevant. Compared to the first-best policy, however, the abatement instrument is not available, i.e. $G\left(t\right) = 0$ forms an additional constraint. As a result of this, the dirt flow constraint is slack, i.e. $D\left(t\right) > 0$ for all t. The second-best optimal plan can be decentralized with the aid of a time-varying tax on capital.

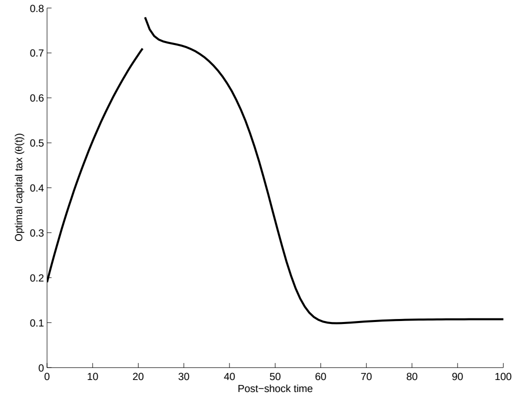
Of course, since abatement is not needed in the long-run first-best social optimum, the steady-state equilibrium under the second-best equilibrium considered here is still as given in (27)–(30) above, i.e. $\hat{K}_s^{TVT} = \hat{K}_f$, $\hat{C}_s^{TVT} = \hat{C}_f$, and $\hat{P}_s^{TVT} = \hat{P}_f$, where the subscript "$s$" denotes second-best and the superscript "TVT" indicates that the policy is decentralized with the aid of a time-varying tax. For convenience these quantitative results are reported in column (c) in Table 3.

Whereas the first- and second-best solutions are identical in the long run, the optimal transition paths differ substantially for these two cases. The dynamic properties of the second-best optimum are illustrated in Figure 4. There is one critical date characterizing the optimal solution, namely $t_I = 21.46$, and there exist two adjustment regimes. Since there is no abatement, the flow of dirt is proportional to the capital stock and environmental pollution evolves in both regimes according to:

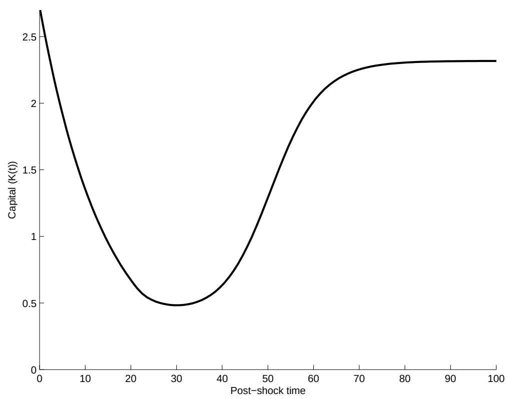
$$\dot{P}_s^{TVT}(t) = -\Phi(P_s^{TVT}\left(t\right)) + \kappa K_s^{TVT}(t).$$

Figure 4: Second-best optimal policy: Time-varying taxation

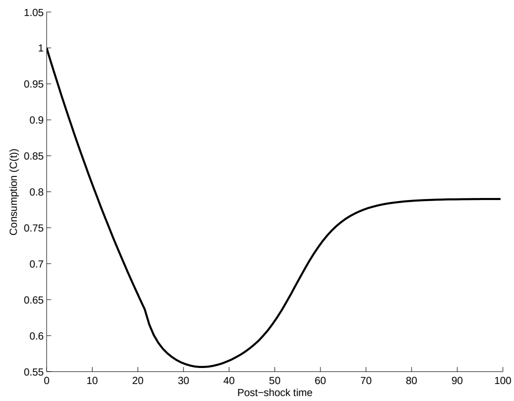
(a) capita tax $\theta(t)$



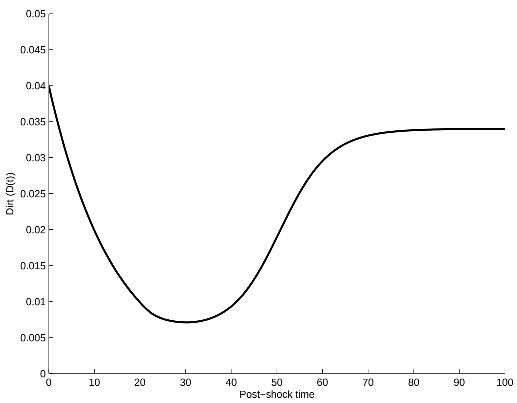
(b) capital stock $K(t)$



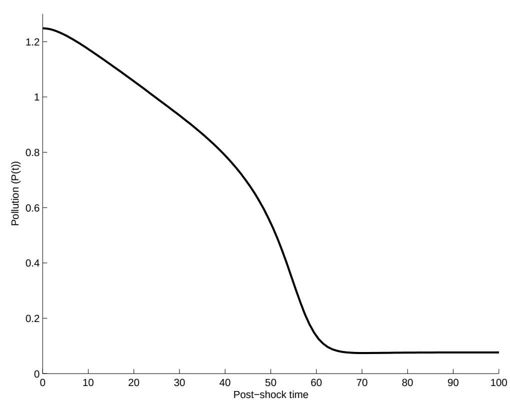
(c) consumption $C(t)$



(d) dirt flow $D(t)$



(e) pollution stock $P(t)$



4.1.1 Regime 1

For $0 \leq t \leq t_I$ gross investment is zero and the capital stock gradually falls. Since abatement is also absent, consumption is equal to output. To summarize we find for this regime that:

$$\begin{aligned} K_s^{TVT}(t) &= \hat{K}e^{-\delta t}, \\ C_s^{TVT}(t) &= \hat{Y}e^{-\delta(1-\varepsilon_L)t}. \end{aligned}$$

These paths have been depicted in panels (b) and (c) in Figure 4. Because consumption growth in the decentralized equilibrium follows the Euler equation (T1.1) and consumption growth during this social planning regime equals $-\delta(1-\varepsilon_L)$ we find that the second-best social optimum can be decentralized with a time-varying capital tax of the following form:

$$\theta_s^{TVT}(t) = 1 - \frac{\rho + \delta\varepsilon_L}{\rho + \delta} e^{-\delta\varepsilon_L t}.$$

The capital tax is increasing over time in order to ensure that the gap between the equilibrium interest rate and the rate of time preference stays constant despite the fact that the capital stock falls over time. See panel (a) in Figure 4.

4.1.2 Regime 2

For $t > t_I$ gross investment is strictly positive ($I_f(t) > 0$) and the consumption path is characterized by:

$$\begin{aligned} \frac{\dot{C}_s^{TVT}(t)}{C_s^{TVT}(t)} &= F_K(K_s^{TVT}(t), 1) - (\rho + \delta) + \kappa\lambda_P(t)\, C_s^{TVT}(t), \\ \dot{\lambda}_P(t) &= \frac{\varepsilon_E}{\bar{E} - P_s^{TVT}(t)} + \left[\rho + \Phi_C'(P_f(t))\right]\lambda_P(t) \end{aligned}$$

whilst the stock of capital evolves according to:

$$\dot{K}_s^{TVT}(t) = F(K_s^{TVT}(t), 1) - C_s^{TVT}(t) - \delta K_s^{TVT}(t).$$

Consumption is continuous at time t_I, i.e. $\lim_{t \nearrow t_I} C_s^{TVT}(t) = \lim_{t \searrow t_I} C_s^{TVT}(t) = C_s^{TVT}(t_I)$, so that $C_s^{TVT}(t_I) = \hat{Y}e^{-\delta(1-\varepsilon_L)t_I}$ and $K_s^{TVT}(t_I) = \hat{K}e^{-\delta t_I}$ are the initial conditions for the system of differential equation in $C_s^{TVT}(t)$ and $K_s^{TVT}(t)$. Since the optimal *growth rate* in consumption features a downward jump at $t = t_I$ and the capital stock is a predetermined variable, the optimal capital tax exhibits a discrete increase at that time – see panel (a) in Figure 4. In the long run the system converges to the steady-state equilibrium discussed above.

Even though steady-state allocations are the same in the first- and second-best social optimum, the "road traveled" to get from the initial (dirty) steady-state to the socially optimal (clean) equilibrium is much more expensive when the policy maker lacks the abatement instrument. Indeed, as is indicated in Table 3 our equivalent variation measure $EV(0)$ falls from 44.1% to 17.1% of current consumption when a time-varying capital tax is the sole environmental policy instrument available. The tax is thus a rather blunt instrument in the sense that it must be set at very high (and strongly distortionary) levels during much of the transition in order to sharply reduce the capital stock (and the associated dirt flow) such that the ecology is steered to the basin of attraction of the lower branch of the P-isocline in Figure 1. In contrast, in the first-best case abatement forms a very cheap instrument to get the pollution dynamics on the right track because it is financed by means of nondistortionary lump-sum taxes.

4.2 Time-invariant taxation

In this subsection we further restrict the policy makers instrumentarium by assuming that the capital tax can only take on two values.[4] In particular, we postulate that $\theta(t)$ is set according to:

$$\theta(t) = \begin{cases} \theta_h & \text{for } 0 \leq t \leq t_E \\ \theta_l & \text{for } t > t_E \end{cases} \tag{32}$$

where θ_h, θ_l, and t_E are chosen optimally by the social planner. Intuitively, in view of the results obtained from the time-varying taxation case (θ_h, t_E) must ensure that the ecology is out on the right track whereas θ_l corrects for the environmental externality in the long run.

Figure 5 depicts the optimal paths for the key variables whilst column (d) in Table 3 presents the quantitative results. Several things are worth noting. First, the long-run allocation is the same under time-varying and time-invariant taxation. Second, during transition the regime configuration is also the same although t_I (the time until which the investment constraint is binding) is highest under time-invariant taxes ($t_I = 32$). Third, the initial capital tax is quite high ($\theta_h = 0.85$) and must be maintained for quite a long time ($t_E = 39.5$) in order to move the ecology to the basin of attraction of the lower branch of the P-isocline in Figure 1. Fourth, the welfare cost of the instrument inflexibility is modest, i.e. the equivalent variation measure falls from 17.1% under time-varying taxation to 16.2% under time-invariant taxes.

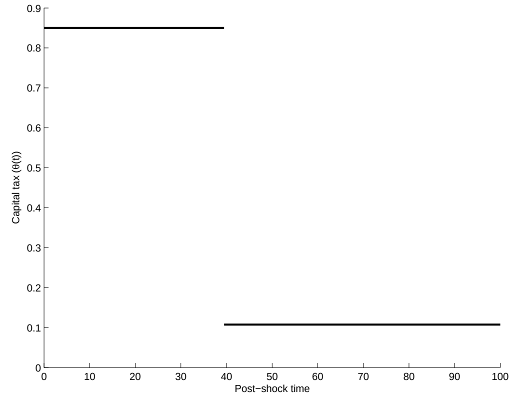
4.3 Time-varying abatement

In the absence of capital taxation, the policy maker must conduct environmental policy exclusively with the abatement instrument. In order to compute the second-best optimal policy, we follow the approach exposited by Judd (1999). In the determination of the best

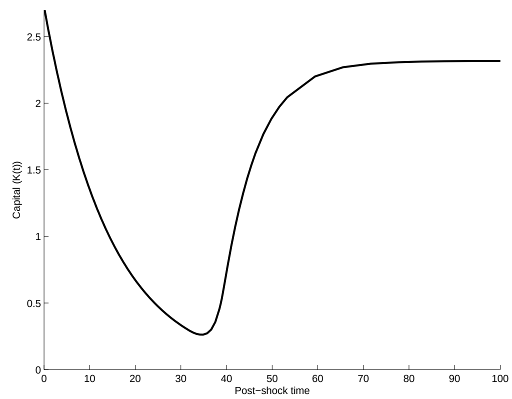
[4]See Moser *et al.* (2012) for a general analysis of multi-stage optimal control techniques in the presence of history dependence.

Figure 5: Second-best optimal policy: Time-invariant taxation

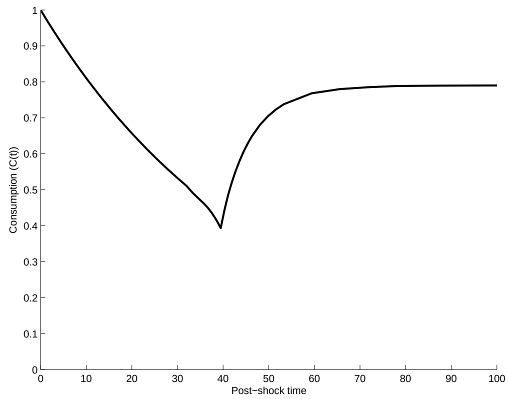
(a) capita tax $\theta(t)$



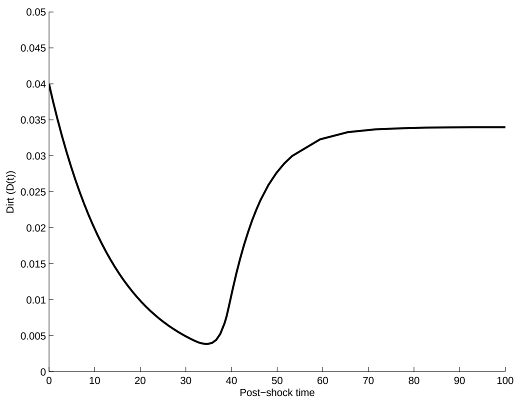
(b) capital stock $K(t)$



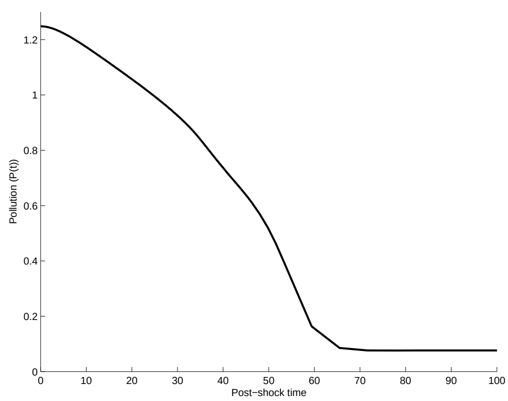
(c) consumption $C(t)$



(d) dirt flow $D(t)$



(e) pollution stock $P(t)$



feasible allocation the social planner faces not only the resource constraint (T1.2), the emission equation (2), and the dirt flow definition (1), but also the following private sector constraints:[5]

$$\lambda_H(t) = \frac{1}{C(t)}, \qquad \dot{\lambda}_H(t) = [\rho + \delta - F_K(K(t),1)]\,\lambda_H(t). \tag{33}$$

Substituting the dirt constraint into the emission equation and dropping the time index, the current-value Hamiltonian can now be written as:

$$\begin{aligned} \mathcal{H} \equiv\; & \ln C + \varepsilon_E \ln[\bar{E} - P] + \lambda_K [F(K,1) - C - G - \delta K] \\ & + \lambda_P[-\Phi(P) + \kappa K - \gamma G] + \eta_\lambda[\rho + \delta - F_K(K,1)]\,\lambda_H \\ & + \eta_C\left[\frac{1}{C} - \lambda_H\right] + \eta_D[\kappa K - \gamma G] + \eta_I[F(K,1) - C - G]. \end{aligned}$$

The control variables are C and G (and thus D and I), the state variables are K, P, and λ_H, the associated co-state variables are λ_K, λ_P, and η_λ, and the Lagrange multipliers are η_C, η_D and η_I. The most relevant first-order conditions are the expressions in (16)–(18), (20), (33) and:

$$\frac{\partial \mathcal{H}}{\partial C} = \frac{1}{C} - (\lambda_K + \eta_I) - \frac{\eta_C}{C^2} = 0, \tag{34}$$

$$\begin{aligned} \dot{\lambda}_K - \rho\lambda_K &= -\frac{\partial \mathcal{H}}{\partial K} = -\kappa(\lambda_P + \eta_D) - [F_K(K,1) - \delta]\,\lambda_K - \eta_I F_K(K,1) \\ &\quad + F_{KK}(K,1)\,\eta_\lambda \lambda_H, \end{aligned} \tag{35}$$

$$\dot{\eta}_\lambda - \rho\eta_\lambda = -\frac{\partial \mathcal{H}}{\partial \lambda_H} = [F_K(K,1) - \delta - \rho]\,\eta_\lambda + \eta_C. \tag{36}$$

Since the capital tax is unavailable, the long-run capital stock returns to its initial level:

$$\hat{K}_s^{TVA} = \hat{K},$$

where the superscript "TVA" stands for time-varying abatement. Whilst it is in principle possible for long-run abatement to be positive, the parameter values ensure that this case does not materialize (just as in the first-best social optimum). In summary we find that:

$$\begin{aligned} G_s^{TVA} &= 0, \\ \hat{C}_s^{TVA} &= F(\hat{K}_s^{TVA},1) - \delta \hat{K}_s^{TVA} = \hat{C}, \\ \hat{P}_s^{TVA} &= \Phi_I^{-1}(\kappa \hat{K}_s^{TVA}) = \hat{P}_G, \\ \hat{D}_s^{TVA} &= \hat{D}_0. \end{aligned}$$

In the second-best optimum, the ecology moves from point D to A in Figure 1. Of course, by

[5] Together these give rise to the Euler equation in the decentralized equilibrium, i.e. $\dot{C}(t)/C(t) = r(t) - \rho$, where $r(t) \equiv F_K(K(t),1) - \delta$.

construction, the second-best optimum can be decentralized with an abatement policy.

The dynamic properties of the second-best optimum are illustrated in Figure 6. There is one date characterizing the optimal solution, namely $t_D = 28.2$, and there exist two adjustment regimes. Throughout the two regimes consumption is constrained to follow its decentralized Euler equation:

$$\frac{\dot{C}_s^{TVA}(t)}{C_s^{TVA}(t)} = F_K\left(K_s^{TVA}, 1\right) - (\rho + \delta),$$

whilst gross investment remains non-negative ($I_s^{TVA}(t) \geq 0$). and the dynamic path for capital accumulation for $0 \leq t \leq t_D$ is given by (see panel (b)).

4.3.1 Regime 1

For $0 \leq t \leq t_D$ abatement is at its maximum feasible level and the dirt flow is reduced to zero ($D_s^{TVA}(t) = 0$). It follows from (1), (T1.2), and (2) that:

$$\begin{aligned}
G_s^{TVA}(t) &= \frac{\kappa}{\gamma} K_s^{TVA}(t), \\
\dot{K}_s^{TVA}(t) &= F\left(K_s^{TVA}(t), 1\right) - C_s^{TVA}(t) - \left(\delta + \frac{\kappa}{\gamma}\right) K_s^{TVA}(t), \\
\dot{P}_s^{TVT}(t) &= -\Phi(P_s^{TVT}(t)).
\end{aligned}$$

Together with the consumption Euler equation these conditions determine the paths depicted in Figure 6.

4.3.2 Regime 2

For $t > t_D$ abatement is reduced to zero and the dirt flow becomes positive. Together with the consumption Euler the paths for the main variables are given by:

$$\begin{aligned}
G_s^{TVA}(t) &= 0, \\
\dot{K}_s^{TVA}(t) &= F\left(K_s^{TVA}(t), 1\right) - C_s^{TVA}(t) - \delta K_s^{TVA}(t), \\
\dot{P}_s^{TVA}(t) &= -\Phi(P_s^{TVA}(t)) + \kappa K_s^{TVA}(t).
\end{aligned}$$

The optimization problem implies that consumption is continuous at time t_D, i.e. $\lim_{t \nearrow t_D} C_s^{TVA}(t) = \lim_{t \searrow t_D} C_s^{TVA}(t) = C_s^{TVT}(t_D)$. This system converges to the steady state discussed above. The quantitative effects of the optimal time-varying abatement policy are reported in column (e) in Table 3. At impact abatement is quite high ($G(0) = 0.13$) and consumption is reduced substantially by about 16 percent. During the early phase of transition the capital stock is crowded out though by a relatively small amount compared to the time-varying taxation case discussed above. The abatement policy is thus a cheap instrument to direct the ecology to the basin of attraction of the lower branch of the P-isocline in Figure 1. Indeed, as we report in column (e) of Table 3 the welfare gain under time-varying abatement is 40.5 percent of initial consumption which is quite close to the result under the first-best environmental policy.

4.4 Time-invariant abatement

In Heijdra and Heijnen (2013) we study the case in which the social planner uses an ad hoc abatement policy of the following form:

$$G(t) = \begin{cases} G & \text{for } 0 \le t \le t_E \\ 0 & \text{for } t > t_E \end{cases} \tag{37}$$

where G and t_E are chosen optimally by the policy maker. Intuitively, in view of the results obtained from the time-varying abatement case (G, t_E) must ensure that the ecology is put on the right track. The value of G must be chosen such that the non-negativity constraint on the dirt flow is violated *nowhere* along the adjustment path. The optimal policy is demonstrated to possess a "cold turkey" property: within the class of stepwise abatement function (37) the largest feasible G must be chosen for the briefest possible duration.

Figure 7 depicts the optimal paths for the key variables whilst column (f) in Table 3 presents the quantitative results. Several things are worth noting. First, the long-run allocation is the same under time-varying and time-invariant taxation. Second, abatement is set at $G = 0.1166$ which initially is lower than the values it takes under the time-varying policy. As a consequence, abatement must be continued for a slightly longer period ($t_E = 30$ instead of $t_E = t_D = 28.2$). Third, the welfare cost of the instrument inflexibility is relatively small, i.e. the equivalent variation measure falls from 40.3% under time-varying abatement to 34.5% under time-invariant abatement.

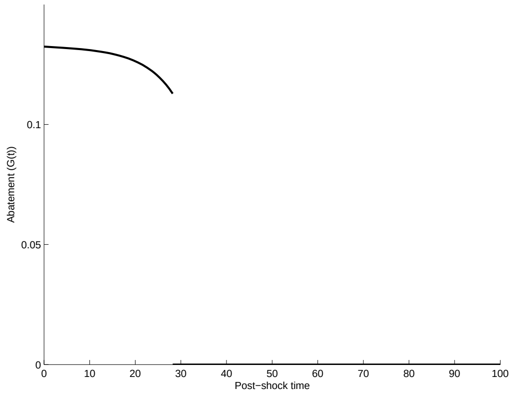
5 Conclusions

In this paper we have studied optimal environmental policy in the presence of an ecological process featuring multiple stable steady-state ecological equilibria and reversible hysteresis. Assuming that the ecological steady-state equilibrium is initially located on the high-pollution (low-welfare) branch of the pollution isocline, the policy maker is in principle able to engineer substantial welfare gains by choosing the appropriate mix of Pigouvian taxation and abatement activities.

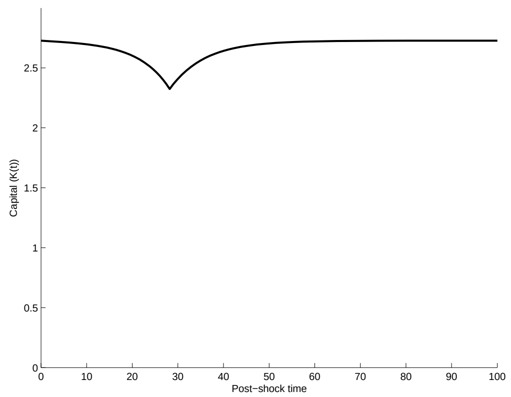
In the first-best social optimum the two available policy instruments each play a very distinct role. During the initial phase of the policy, abatement is used to choke off the flow of dirt as much as is feasible whereas the tax is employed to bring down the stock of the

Figure 6: Second-best optimal policy: Time-varying abatement

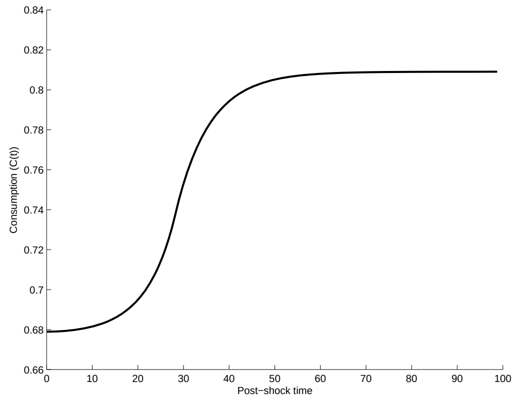
(a) government abatement $G(t)$



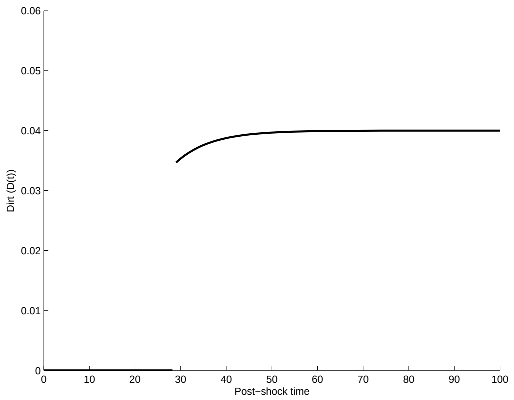
(b) capital stock $K(t)$



(c) consumption $C(t)$



(d) dirt flow $D(t)$



(e) pollution stock $P(t)$

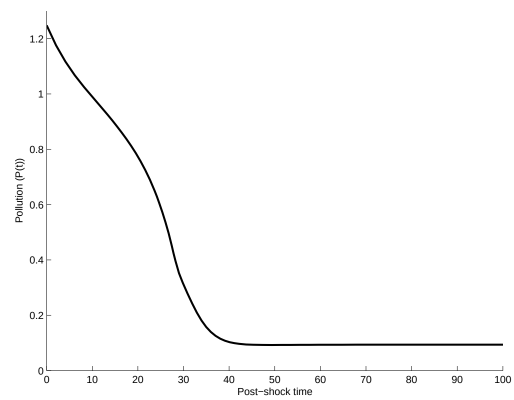
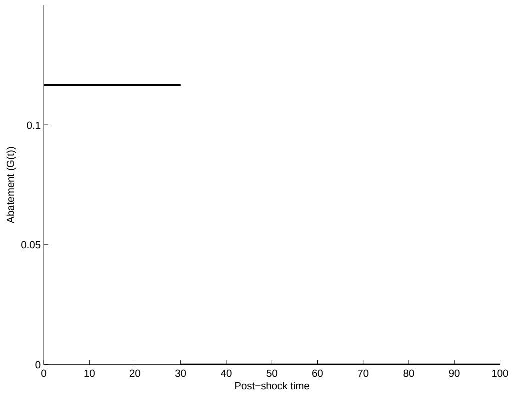
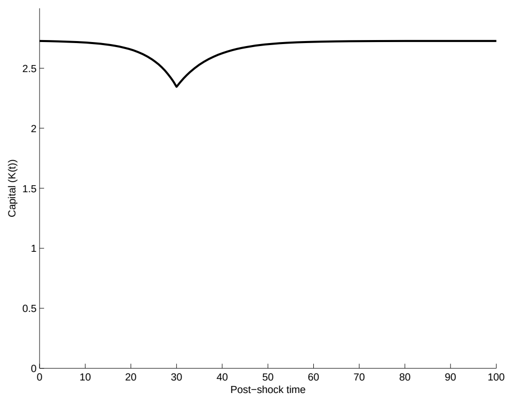


Figure 7: Second-best optimal policy: Time-invariant abatement

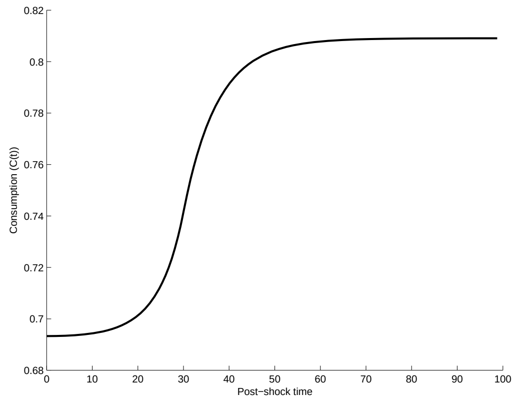
(a) government abatement $G(t)$



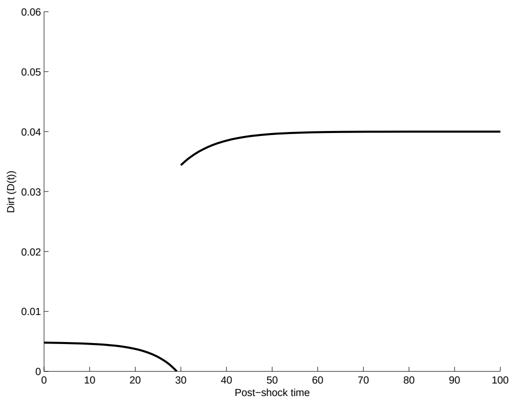
(b) capital stock $K(t)$



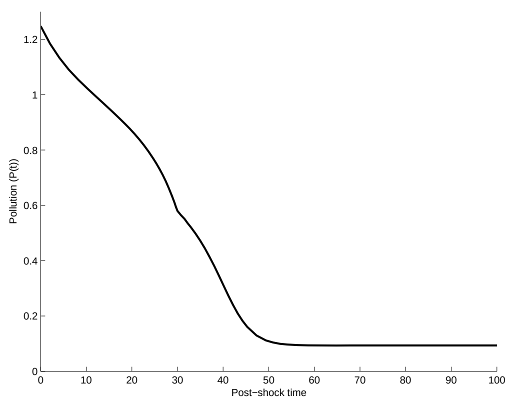
(c) consumption $C(t)$



(d) dirt flow $D(t)$



(e) pollution stock $P(t)$



Parameters: see Table 2. The initial ecological equilibrium is at point D in panel (b).

polluting capital input as quickly as possible. In the long run, however, abatement is no longer needed and the capital tax settles down at its externality-correcting Pigouvian level.

Interestingly, in a second-best setting it matters very much which additional constraint is faced by the policy maker. In the case where capital taxation is unavailable as an instrument for environmental policy, a suitably designed abatement policy can achieve a social outcome that is only marginally worse than the first-best result. Intuitively, lump-sum tax financed abatement is a cheap instrument to steer the ecology from the high- to the low-pollution equilibrium.

In contrast, if the abatement instrument is not available and the tax must be used to clean up the environment then the "road travelled" is a very expensive one. Intuitively, because of the distorting nature of the capital tax, using it to get out of the hysteretic equilibrium is a high-price option. Indeed, we show that in that case it is only marginally welfare improving to steer the ecology from the high- to the low-pollution equilibrium and to correct for the environmental externality.

Appendix: Computational Details

First-best (FB) We use a continuation method to compute the first-best. Let $X_s = (K, P)$ denote the state variables, $X_c = (\lambda_K, \lambda_P)$ the costates, and $X = (X_s, X_c)$. The controls and the Lagrange-multipliers are denoted by $U = (C, G, \eta_I, \eta_D)$. From Pontryagin's maximum principle, we get $U = U^*(X)$: the state- and costate-variables determine consumption, abatement and the multipliers for the investment- and dirt constraints. Recall that the other first-order conditions can be written as follows:

$$\dot{X}(t) = H(X(t), U^*(X(t))).$$

The optimal path is determined by constraints on $X_s(0)$ and $X(\infty)$. In particular $X_s(0) = (\hat{K}, \hat{P}_B)$ and $X(\infty) = X^*$, where X^* is a root of $H(\cdot)$. The end condition is replaced by the requirement that at time $T = 200$, the trajectory is orthogonal to the stable manifold of X^*. We approximate the first-order condition as follows. First, we discretize the time grid $t \in \{0, 1, \cdots, 200\}$ and at time t we replace the differential equation by a fourth-order Runge-Kutta approximation. This leads to a system of equations of which we have to find the root.

For the continuation method, we need a trivial solution. Note that $X(t) \equiv X^*$ is a solution for the initial condition $X_s(0) = X_s^*$. We slowly change this initial condition into the direction of the actual initial condition, using a simple predictor-correction algorithm. See Grass (2012) for details.

The time at which the investment constraint stops being binding is calculated in the following manner. Suppose that for $t \leq t^*$, we have $\eta_I(t) > 0$ (and $\eta_I(t) = 0$ for $t > t^*$). This means that the investment constraint is binding until $t_I \in [t^*, t^* + 1]$. Using cubic extrapolation, we determine the value of t_I. It turns out that $t_I = 1.27$. For the dirt constraint, we use a similar method and it turns out that $t_D = 27.01$.

Time-varying tax (TVT) In principle, in this case we should be able to use a similar algorithm as for the first-best. However, the continuation algorithm fails to terminate (the path "bends back" to the $X_s(0) = X_s^*$). We note that at some point the investment constraint becomes binding. Therefore, we postulate that the optimal path first goes through a regime where the investment constraint is binding. If the investment constraint is binding until $t = t_I$, then we can calculate the value of capital and pollution at $t = t_I$. We take these as the initial value for capital and pollution and solve for the optimal time-varying tax from that point onward. Then we choose t_I such that this is the point where the investment constraint stops being binding (i.e. $\lambda_K(t_I)F(K(t_I), 1) = 1$). It turns out that this is the case for $t_I = 21.46$.

N.B. In both FB and TVT the long-run tax rate is $\hat{\theta} = 0.1066$.

Time-invariant tax (TIT) In the long-run, we set the tax rate equal to $\hat{\theta}$, but we start with a higher tax rate to move the system towards a lower pollution level. It turns out that the initial tax rate θ_0 is high enough to make the investment constraint binding. This means that we have to determine t_I (the time at which the investment constraint stops being binding) and t_E (the time at which the tax rate shifts from θ_0 to $\hat{\theta}$). Since the consumption path cannot jump, we can only switch from θ_0 to $\hat{\theta}$ if we are on the stable saddle path leading to the clean equilibrium. Hence, the free variables are θ_0 and t_I. We somewhat crudely search for the lowest values that can force the system to the clean steady state by increasing θ_0 with step size 0.05 and t_I with step size 1. We end up with $\theta_0 = 0.85$ and $t_I = 32$. Since the EV under TIT is close to the EV under TIA (time-invariant abatement), we are confident that these values are close to the optimal tax of this form.

Time-varying abatement and time-invariant abatement (TVA and TIA) See Heijdra and Heijnen (2013). We have added a bit of accuracy for the case with TVA: full abatement until $t_E = 28.2$, increases the EV to 40.5%.

Calculate utility levels In the FB, we calculate utility level by calculating the Hamiltonian at time zero and dividing this value by ρ. In all other cases, we use the following method to calculate the utility of the representative consumer. Given paths for consumption and pollution, this amounts to evaluating an integral of the form

$$W = \int_0^\infty u(C(s), P(s))\mathrm{e}^{-\rho s} ds.$$

As inputs we have the levels of consumption and pollution at discrete points in time $t \in \{t_0, t_1, t_2, \ldots, t_n\}$, where t_n is sufficiently large for consumption and pollution to be close to the steady state values. Then, as is also noted by Heijnen and Wagener (2013), W is approximately equal to:

$$W \approx \frac{1}{2}\sum_{i=1}^{n}[u(C(t_i), P(t_i))\mathrm{e}^{-\rho t_i} + u(C(t_{i-1}), P(t_{i-1}))\mathrm{e}^{-\rho t_{i-1}}](t_i - t_{i-1}) + u(C(t_n), P(t_n))\frac{\mathrm{e}^{-\rho t_n}}{\rho}.$$

Since our grid is not very dense, this gives a rather rough approximation, limiting the accuracy with which we can calculate the optimal policy.

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