A tragedy of annuitization? Mathematical appendix*

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Abstract: This document contains all derivations for "A tragedy of annuitization? Longevity insurance in general equilibrium" by B. J. Heijdra, J. O. Mierau, and L. S. M. Reijnders. It will be made available via the first author's website (www.heijdra.org) once the paper has been accepted for publication in a professional journal.

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A.1 Proof of Proposition 1

Proposition 1. *[Golden rules] Define steady-state welfare of a young agent (P1.1), the economy-wide resource constraint (P1.2), and the macroeconomic production function (P1.3) as follows:*

$$\mathbb{E}\Lambda^y \equiv U(C^y) + \frac{1-\pi}{1+\rho}U(C^o), \tag{P1.1}$$

$$f(k) - (\delta+n)\,k = C^y + \frac{1-\pi}{1+n}C^o + g, \tag{P1.2}$$

$$f(k) = \Omega_0 k^{\alpha+\eta}. \tag{P1.3}$$

The social planner chooses non-negative values for C^y, C^o, k, and g in order to maximize $\mathbb{E}\Lambda^y$ subject to the constraints (P1.2)–(P1.3). In addition to satisfying the constraints the first-best social optimum has the following features:

$$\frac{U'(\tilde{C}^y)}{U'(\tilde{C}^o)} = \frac{1+n}{1+\rho}, \tag{S1}$$

$$f'(\tilde{k}) = n+\delta, \tag{S2}$$

$$\tilde{g} = 0. \tag{S3}$$

Proof. The Lagrangian for the optimization problem is:

$$\mathcal{L} \equiv U(C^y) + \frac{1-\pi}{1+\rho}U(C^o) + \lambda\left[f(k) - (\delta+n)\,k - C^y - \frac{1-\pi}{1+n}C^o - g\right].$$

The first-order conditions for the first-best social optimum are:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial C^y} &= U'(C^y) - \lambda = 0,\\
\frac{\partial \mathcal{L}}{\partial C^o} &= \frac{1-\pi}{1+\rho}U'(C^o) - \lambda\frac{1-\pi}{1+n} = 0,\\
\frac{\partial \mathcal{L}}{\partial g} &= -\lambda \le 0, \quad g \ge 0, \quad g\frac{\partial \mathcal{L}}{\partial g} = 0,\\
\frac{\partial \mathcal{L}}{\partial k} &= \lambda\left[f'(k) - (\delta+n)\right] = 0,\\
\frac{\partial \mathcal{L}}{\partial \lambda} &= f(k) - (\delta+n)\,k - C^y - \frac{1-\pi}{1+n}C^o - g = 0.
\end{aligned}$$

Since the marginal utility of consumption is positive everywhere (by the property of non-satiation), it follows that λ is strictly positive. Complementary slackness then implies that $\tilde{g} = 0$. Rewriting the remaining first-order conditions yields expressions (S1)–(S3). □

A.2 Proof of Proposition 2

In this section we provide elaborate proofs for the items contained in Proposition 2 in the paper. We study each scenario in turn. In the paper itself the perturbation parameters identify scenarios:

- WE model: $z_1 = z_2 = z_3 = z_3^- = 0$
- TO model: $z_1 = 1$, $z_2 = z_3 = z_3^- = 0$
- TY model: $z_2 = 1$, $z_1 = z_3 = z_3^- = 0$
- PA model: $z_3 = z_3^- = 1$, $z_1 = z_2 = 0$

A.2.1 WE model

We write the marginal propensity to consume in terms of the capital intensity as:

$$\Phi(k) \equiv \frac{1}{1 + \left(\frac{1-\pi}{1+\rho}\right)^{\sigma} (1 - \delta + \alpha\Omega_0 k^{\alpha+\eta-1})^{\sigma-1}}, \quad k > 0, \tag{A.1}$$

which corresponds to equation (21) in the text with $z_3 = 0$ substituted and the superfluous argument suppressed. The parameters satisfy $0 < \alpha < 1$, $0 \leq \eta < 1 - \alpha$, and $0 < \sigma \leq \bar{\sigma}$, where $\bar{\sigma}$ is given by:

$$\bar{\sigma} \equiv \frac{2 - \alpha - \eta}{1 - \alpha - \eta}. \tag{A.2}$$

We define:

$$\bar{\Phi}(\sigma) \equiv \frac{1}{1 + \left(\frac{1-\pi}{1+\rho}\right)^{\sigma} (1 - \delta)^{\sigma-1}}, \tag{A.3}$$

such that $0 < \bar{\Phi}(\sigma) < 1$ for $0 < \sigma \leq \bar{\sigma}$.

The following lemma states some crucial properties of $\Phi(k)$.

Lemma A.1. *[Properties of the $\Phi(k)$ function] Let $\Phi(k)$ be defined as in* (A.1)*. This function has the following properties:*

(i) $\lim_{k\to 0} \Phi(k) = \begin{cases} 1 & \text{for } 0 < \sigma < 1 \\ \bar{\Phi}(1) & \text{for } \sigma = 1 \\ 0 & \text{for } 1 < \sigma \leq \bar{\sigma} \end{cases}$;

(ii) $\lim_{k\to\infty}\Phi(k)=\bar{\Phi}(\sigma)$;

(iii) For $0<\sigma\leq 1$*:*

$$\lim_{k\to 0}\frac{1-\Phi(k)}{k}=+\infty;$$

(iv) For $1\leq\sigma\leq\bar{\sigma}$*:*

$$\lim_{k\to 0}\frac{\Phi(k)}{k}=+\infty;$$

(v) $\lim_{k\to 0}\Phi'(k)=\begin{cases}-\infty & \textit{for } 0<\sigma<1\\ 0 & \textit{for } \sigma=1\\ +\infty & \textit{for } 1<\sigma\leq\bar{\sigma}\end{cases}.$

Proof. For future reference we provide the following two equivalent representations of $\Phi(k)$:

$$\Phi(k)=\frac{1}{1+\phi_0(\phi_1+k^{\alpha+\eta-1})^{\sigma-1}}, \tag{A.4}$$

$$=\frac{1}{1+\phi_0 k^{(1-\sigma)(1-\alpha-\eta)}(\phi_1 k^{1-\alpha-\eta}+1)^{\sigma-1}}, \tag{A.5}$$

where ϕ_0 and ϕ_1 are given by:

$$\phi_0\equiv\left(\frac{1-\pi}{1+\rho}\right)^{\sigma}(\alpha\Omega_0)^{\sigma-1}>0,\quad \phi_1\equiv\frac{1-\delta}{\alpha\Omega_0}>0. \tag{A.6}$$

Part (i) can be proved by using expression (A.5) and noting that:

$$\lim_{k\to 0}\left(\phi_1 k^{1-\alpha-\eta}+1\right)^{\sigma-1}=1,\quad 0<\sigma\leq\bar{\sigma}, \tag{A.7}$$

$$\lim_{k\to 0}k^{(1-\sigma)(1-\alpha-\eta)}=\begin{cases}0 & \text{for } 0<\sigma<1\\ 1 & \text{for } \sigma=1\\ +\infty & \text{for } 1<\sigma\leq\bar{\sigma}\end{cases}. \tag{A.8}$$

Part (ii) follows directly from (A.4) as $\lim_{k\to\infty}k^{\alpha+\eta-1}=0$.

To prove part (iii) we write:

$$\frac{1-\Phi(k)}{k}=\frac{\phi_0 k^{(1-\sigma)(1-\alpha-\eta)-1}\left(\phi_1 k^{1-\alpha-\eta}+1\right)^{\sigma-1}}{1+\phi_0 k^{(1-\sigma)(1-\alpha-\eta)}\left(\phi_1 k^{1-\alpha-\eta}+1\right)^{\sigma-1}}. \tag{A.9}$$

The result in (iii) follows by using (A.7) and the fact that $\lim_{k\to 0}k^{(1-\sigma)(1-\alpha-\eta)-1}=+\infty$ (as $0\leq(1-\sigma)(1-\alpha-\eta)\leq 1$ for $0<\sigma\leq 1$).

To prove part (iv) we write:

$$\frac{\Phi(k)}{k} = \frac{1}{k + \phi_0 k^{1-(\sigma-1)(1-\alpha-\eta)} \left(\phi_1 k^{1-\alpha-\eta} + 1\right)^{\sigma-1}}. \tag{A.10}$$

The result in (iv) follows by using (A.7) and the fact that $\lim_{k\to 0} k^{1-(\sigma-1)(1-\alpha-\eta)} = 0$ (as $0 \le (\sigma - 1)(1 - \alpha - \eta) < 1$ for $1 \le \sigma \le \bar{\sigma}$).

To prove part (v) we use (A.4) to find $\Phi'(k)$:

$$\Phi'(k) \equiv -(1-\sigma)(1-\alpha-\eta) \frac{\Phi(k)\left[1-\Phi(k)\right]}{k} \frac{1}{\phi_1 k^{1-\alpha-\eta} + 1}. \tag{A.11}$$

Using parts (iii) and (iv) establishes part (v). □

If a steady-state equilibrium of the WE model satisfying $k_{t+1} = k_t = \hat{k}$ exists, then it is a solution to the following equation:

$$\Psi(\hat{k}) - \Gamma(\hat{k}) = 0, \tag{A.12}$$

where $\Psi(k)$ and $\Gamma(k)$ are defined as:

$$\Psi(k) \equiv \begin{cases} \dfrac{k}{1-\Phi(k)} & \text{for } k > 0 \\ 0 & \text{for } k = 0 \end{cases}, \tag{A.13}$$

$$\Gamma(k) \equiv \frac{(1-\alpha)\Omega_0}{1+n} k^{\alpha+\eta}, \quad k \ge 0. \tag{A.14}$$

Note that (A.14) is obtained from (20) in the paper by setting $z_2 = z_3^- = 0$ and suppressing the superfluous arguments. The next two lemmas cover the properties of $\Gamma(k)$ and $\Psi(k)$.

Lemma A.2. *[Properties of the $\Gamma(k)$ function] Let $\Gamma(k)$ be defined as in* (A.14). *This function has the following properties:*

(i) $\Gamma(0) = 0$;

(ii) $\Gamma'(k) = (\alpha+\eta) \dfrac{(1-\alpha)\Omega_0}{1+n} k^{\alpha+\eta-1} > 0$ *for* $k > 0$;

(iii) $\Gamma''(k) = -(1-\alpha-\eta)(\alpha+\eta) \dfrac{(1-\alpha)\Omega_0}{1+n} k^{\alpha+\eta-2} < 0$ *for* $k > 0$;

(iv) $\lim_{k\to 0} \Gamma'(k) = +\infty, \qquad \lim_{k\to\infty} \Gamma'(k) = 0.$

Proof. Obvious by differentiation. □

Lemma A.3. *[Properties of the $\Psi(k)$ function] Let $\Psi(k)$ be defined as in* (A.13). *This function has the following properties:*

(i) $\lim_{k\to 0}\Psi(k)=0$;

(ii) $\Psi'(k)>0$ *for* $k>0$;

(iii) $\lim_{k\to 0}\Psi'(k)=\begin{cases}+\infty & \text{for } 0<\sigma<1\\ \dfrac{1}{1-\bar{\Phi}(1)} & \text{for } \sigma=1\\ 1 & \text{for } 1<\sigma\leq\bar{\sigma}\end{cases}$;

(iv) $\lim_{k\to 0}\dfrac{\Gamma'(k)}{\Psi'(k)}=+\infty$;

(v) $\lim_{k\to\infty}\Psi'(k)=\dfrac{1}{1-\bar{\Phi}(\sigma)}>0$.

Proof. Part (i) follows readily from Lemma A.1(i) for $1<\sigma\leq\bar{\sigma}$ and from Lemma A.1(iii) for $0<\sigma\leq 1$.

To prove part (ii) we compute:

$$\Psi'(k)=\frac{1-(1-\sigma)(1-\alpha-\eta)\Phi(k)\frac{1}{\phi_1 k^{1-\alpha-\eta}+1}}{1-\Phi(k)}>0, \tag{A.15}$$

where we have used (A.11). The sign follows from the fact that $0<\alpha+\eta<1$, $0<\Phi(k)<1$, $1/\left(\phi_1 k^{1-\alpha-\eta}+1\right)<1$, and $\sigma>0$.

Part (iii) follows from (A.15) by using Lemma A.1(i).

To prove part (iv) we use (A.15) and Lemma A.2(ii) to write:

$$\frac{\Gamma'(k)}{\Psi'(k)}=\frac{(1-\alpha)(\alpha+\eta)\Omega_0}{1+n}\frac{[1-\Phi(k)]\,k^{\alpha+\eta-1}}{1-(1-\sigma)(1-\alpha-\eta)\Phi(k)\frac{1}{\phi_1 k^{1-\alpha-\eta}+1}}. \tag{A.16}$$

For $1\leq\sigma\leq\bar{\sigma}$ the result follows immediately from Lemma A.1 and the fact that $\lim_{k\to 0}k^{\alpha+\eta-1}=+\infty$. For $0<\sigma<1$ we write:

$$\lim_{k\to 0}\frac{1-\Phi(k)}{k}k^{\alpha+\eta}=\lim_{k\to 0}\frac{\phi_0 k^{-\sigma(1-\alpha-\eta)}\left(\phi_1 k^{1-\alpha-\eta}+1\right)^{\sigma-1}}{1+\phi_0 k^{(1-\sigma)(1-\alpha-\eta)}\left(\phi_1 k^{1-\alpha-\eta}+1\right)^{\sigma-1}}=+\infty, \tag{A.17}$$

where we have used the results in (A.7) and (A.8) above and the fact that $\lim_{k\to 0}k^{-\sigma(1-\alpha-\eta)}=+\infty$.

Part (v) follows from (A.15) and Lemma A.1(ii). □

We now proceed to the proof of Proposition 2 for the WE model.

Proposition 2. *[Existence and stability of the WE model] Consider the WE model and adopt Assumption 1. The following properties can be established:*

(i) The model has two steady-state solutions; the trivial one features $k_{t+1} = k_t = 0$, and the economically relevant one satisfies $k_{t+1} = k_t = \hat{k}^{WE}$, where $\hat{k}^{WE}$ is the solution to:

$$\frac{\hat{k}^{WE}}{1 - \Phi(\hat{k}^{WE})} = \frac{(1-\alpha)\,\Omega_0}{1+n}(\hat{k}^{WE})^{\alpha+\eta}.$$

(ii) The trivial steady-state solution is unstable whilst the non-trivial solution is stable:

$$0 < \frac{dk_{t+1}}{dk_t} < 1, \qquad \textit{for } k_{t+1} = k_t = \hat{k}^{WE}.$$

For any positive initial value the capital intensity converges monotonically to $\hat{k}^{WE}$.

Proof. The steady-state equation (A.12) has two roots. By Lemmas A.2(i) and A.3(i) one root is at $\hat{k} = 0$. To investigate the stability of that root we write:

$$\left.\frac{dk_{t+1}}{dk_t}\right|_{\hat{k}=0} = \lim_{k\to 0}\frac{\Gamma'(k)}{\Psi'(k)} = +\infty, \tag{A.18}$$

where we have used Lemma A.3(iv). It follows that the trivial solution is unstable and that $\Gamma(k)$ lies above $\Psi(k)$ for positive values of k close to the origin. We know that $\Gamma(k)$ is concave and satisfies $\lim_{k\to\infty}\Gamma'(k) = 0$ while $\Psi(k)$ is strictly increasing with $\lim_{k\to\infty}\Psi'(k) > 0$. Hence there is a unique positive and finite nontrivial root, $\hat{k}^{WE}$. At $k = \hat{k}^{WE}$, $\Psi(k)$ cuts $\Gamma(k)$ from below such that $0 < \Gamma'(\hat{k}^{WE})/\Psi'(\hat{k}^{WE}) < 1$, thus proving stability of the nontrivial equilibrium. □

A.2.2 TO model

If a steady state equilibrium of the TO model satisfying $k_{t+1} = k_t = \hat{k}$ exists, then it is a solution to the following equation:

$$\Psi(\hat{k},1) - \Gamma(\hat{k}) = 0, \tag{A.19}$$

where $\Gamma(k)$ has been defined in (A.14) and $\Psi(k, z_1)$ is given by:

$$\Psi\left(k,z_1\right) \equiv \begin{cases} \dfrac{1+z_1\frac{\pi}{1-\pi}\Phi(k)}{1-\Phi(k)}k & \text{for } k>0 \\ 0 & \text{for } k=0 \end{cases}, \tag{A.20}$$

with z_1 a perturbation parameter ($0 \leq z_1 \leq 1$). Note that (A.20) is obtained from (19) in the text by setting $z_3 = 0$ and dropping the superfluous arguments. The following lemma covers the properties of $\Psi(k, z_1)$.

Lemma A.4. *[Properties of the $\Psi(k, z_1)$ function] Let $\Psi\left(k, z_1\right)$ be defined as in* (A.20)*. This function has the following properties:*

(i) $\lim_{k\to 0} \Psi\left(k, z_1\right) = 0$;

(ii) $\Psi_{z_1}\left(k, z_1\right) \equiv \partial\Psi(k, z_1)/\partial z_1 > 0$ *and* $\Psi_k\left(k, z_1\right) \equiv \partial\Psi(k, z_1)/\partial k > 0$ *for* $k > 0$;

(iii) $\lim_{k\to 0} \Psi_k\left(k, z_1\right) = \begin{cases} +\infty & \textit{for } 0<\sigma<1 \\ \dfrac{1-\pi+\pi z_1\bar{\Phi}(1)}{(1-\pi)\left[1-\bar{\Phi}\left(1\right)\right]} & \textit{for } \sigma=1 \\ 1 & \textit{for } 1<\sigma\leq\bar{\sigma} \end{cases}$;

(iv) $\lim_{k\to 0} \dfrac{\Gamma'\left(k\right)}{\Psi_k\left(k,1\right)} = +\infty$;

(v) $\lim_{k\to\infty} \left(1-\pi\right)\Psi_k\left(k,1\right) = -\pi + \dfrac{1}{1-\bar{\Phi}\left(\sigma\right)} > 0.$

Proof. The proof of part (i) follows from Lemma A.1(i) and (iii) for $0 < \sigma < 1$. In case $\sigma = 1$, $\Psi\left(k, z_1\right)$ is proportional to k and therefore tends to zero as k goes to zero. For $1 < \sigma \leq \bar{\sigma}$ we apply Lemma A.1(i).

To prove part (ii) we compute:

$$\Psi_{z_1}(k,z_1) = \frac{\pi}{1-\pi}\frac{k\Phi(k)}{1-\Phi(k)} > 0, \tag{A.21}$$

$$\Psi_k(k,z_1) = \frac{-z_1\pi}{1-\pi} + \frac{1-(1-z_1)\pi}{1-\pi}\left[\frac{1-(1-\sigma)(1-\alpha-\eta)\Phi(k)\frac{1}{\phi_1 k^{1-\alpha-\eta}+1}}{1-\Phi(k)}\right] > 0, \tag{A.22}$$

where we have used (A.11) to arrive at the second expression. The inequality in (A.22) follows from the fact that the term in square brackets is greater than or equal to one for all values of σ, as $0 < \alpha+\eta < 1$, $0 < \Phi(k) < 1$, $1/\left(\phi_1 k^{1-\alpha-\eta}+1\right) < 1$, and $\sigma > 0$.

Part (iii) follows readily from (A.22) by using Lemma A.1(i).

To prove part (iv) we use (A.22) and Lemma A.2(ii) to write:

$$\frac{\Gamma'(k)}{\Psi_k(k,1)} = \frac{(1-\alpha)(\alpha+\eta)\Omega_0}{1+n}\frac{(1-\pi)[1-\Phi(k)]k^{\alpha+\eta-1}}{-\pi[1-\Phi(k)]+\left[1-(1-\sigma)(1-\alpha-\eta)\Phi(k)\frac{1}{\phi_1 k^{1-\alpha-\eta}+1}\right]}.$$

For $1 \leq \sigma \leq \bar{\sigma}$ the result follows immediately from Lemma A.1 and the fact that $\lim_{k\to 0} k^{\alpha+\eta-1} = +\infty$, while for $0 < \sigma < 1$ we make use of (A.17).

Part (v) follows from (A.22) and Lemma A.1(ii). □

We now proceed to the proof of Proposition 2 for the TO model.

Proposition 2. *[Existence and stability of the TO model] Consider the TO model and adopt Assumption 1. The following properties can be established:*

(i) The model has two steady-state solutions, the trivial one features $k_{t+1} = k_t = 0$, and the economically relevant one satisfies $k_{t+1} = k_t = \hat{k}^{TO}$, where $\hat{k}^{TO}$ is the solution to:

$$\frac{1+\frac{\pi}{1-\pi}\Phi(\hat{k}^{TO})}{1-\Phi(\hat{k}^{TO})}\hat{k}^{TO} = \frac{(1-\alpha)\Omega_0}{1+n}(\hat{k}^{TO})^{\alpha+\eta}.$$

(ii) The trivial steady-state solution is unstable whilst the non-trivial solution is stable:

$$0 < \frac{dk_{t+1}}{dk_t} < 1, \qquad \text{for } k_{t+1} = k_t = \hat{k}^{TO}.$$

For any positive initial value the capital intensity converges monotonically to $\hat{k}^{TO}$.

(iii) The steady-state capital intensity satisfies the following inequality:

$$0 < \hat{k}^{TO} < \hat{k}^{WE}.$$

Proof. The steady-state equation (A.19) has two roots. By Lemmas A.2(i) and A.4(i) one root is at $\hat{k} = 0$. To investigate the stability of that root we write:

$$\left.\frac{dk_{t+1}}{dk_t}\right|_{\hat{k}=0} = \lim_{k\to 0} \frac{\Gamma'(k)}{\Psi_k(k,1)} = +\infty, \tag{A.23}$$

where we have used Lemma A.4(iv). It follows that the trivial solution is unstable and that $\Gamma(k)$ lies above $\Psi(k,1)$ for positive values of k close to the origin. We know that $\Gamma(k)$ is concave and satisfies $\lim_{k\to\infty} \Gamma'(k) = 0$ while $\Psi(k,1)$ is strictly increasing with $\lim_{k\to\infty} \Psi_k(k,1) > 0$. Hence there is a unique positive and finite nontrivial root, $\hat{k}^{TO}$. At $k = \hat{k}^{TO}$, $\Psi(k,1)$ cuts $\Gamma(k)$ from below such that $0 < \Gamma'(\hat{k}^{TO})/\Psi_k(\hat{k}^{TO},1) < 1$, thus proving stability of the nontrivial equilibrium. Since $\Psi_{z_1}(k,z_1) > 0$ for all $k > 0$, increasing the perturbation parameter z_1 from 0 to 1 rotates the $\Psi(k,z_1)$ function in a counterclockwise fashion. As a consequence $\hat{k}^{TO} < \hat{k}^{WE}$. □

A.2.3 TY model

If a steady state equilibrium of the TY model satisfying $k_{t+1} = k_t = \hat{k}$ exists, then it is a solution to the following equation:

$$\Psi(\hat{k}) - \Gamma(\hat{k}, 1) = 0, \tag{A.24}$$

where $\Psi(k)$ has been defined in (A.13) and $\Gamma(k, z_2)$ is given by:

$$\Gamma(k, z_2) \equiv \frac{[1 - \alpha(1 - z_2\pi)]\,\Omega_0 k^{\alpha+\eta} + z_2\pi(1-\delta)\,k}{1+n}, \tag{A.25}$$

with z_2 a perturbation parameter ($0 \leq z_2 \leq 1$). Note that (A.25) is obtained from (20) in the text by setting $z_3^- = 0$ and dropping the superfluous argument. The following lemma covers the properties of $\Gamma(k, z_2)$.

Lemma A.5. *[Properties of the $\Gamma(k, z_2)$ function] Let $\Gamma(k, z_2)$ be defined as in* (A.25)*. This function has the following properties:*

(i) $\Gamma(0, z_2) = 0$;

(ii) $\Gamma_k(k, z_2) \equiv \partial\Gamma(k, z_2)/\partial k = (\alpha+\eta)\frac{[1-\alpha(1-z_2\pi)]\,\Omega_0}{1+n}k^{\alpha+\eta-1} + \frac{z_2\pi(1-\delta)}{1+n} > 0$ *and* $\Gamma_{z_2}(k, z_2) \equiv \partial\Gamma(k, z_2)/\partial z_2 = \frac{\alpha\pi\Omega_0 k^{\alpha+\eta} + \pi(1-\delta)\,k}{1+n} > 0$ *for* $k > 0$;

(iii) $\Gamma_{kk}(k, z_2) \equiv \partial^2\Gamma(k, z_2)/\partial k^2 = -(1-\alpha-\eta)(\alpha+\eta)\frac{[1-\alpha(1-z_2\pi)]\,\Omega_0}{1+n}k^{\alpha+\eta-2} < 0$ *and* $\Gamma_{kz_2} \equiv \partial^2\Gamma(k, z_2)/\partial k\partial z_2 = \frac{\alpha(\alpha+\eta)\pi\Omega_0 k^{\alpha+\eta-1} + \pi(1-\delta)}{1+n} > 0$ *for* $k > 0$;

(iv) $\lim_{k\to 0}\Gamma_k(k, z_2) = +\infty, \qquad \lim_{k\to\infty}\Gamma_k(k, z_2) = 0.$

Proof. Obvious by differentiation. □

We now proceed to the proof of Proposition 2 for the TY model.

Proposition 2. *[Existence and stability of the TY model] Consider the TY model and adopt Assumption 1. The following properties can be established:*

(i) The model has two steady-state solutions, the trivial one features $k_{t+1} = k_t = 0$, and the economically relevant one satisfies $k_{t+1} = k_t = \hat{k}^{TY}$, where $\hat{k}^{TY}$ is the solution to:

$$\frac{\hat{k}^{TY}}{1 - \Phi(\hat{k}^{TY})} = \frac{[1-\alpha(1-\pi)]\,\Omega_0(\hat{k}^{TY})^{\alpha+\eta} + \pi(1-\delta)\,\hat{k}^{TY}}{1+n}.$$

(ii) The trivial steady-state solution is unstable whilst the non-trivial solution is stable:

$$0 < \frac{dk_{t+1}}{dk_t} < 1, \qquad \textit{for } k_{t+1} = k_t = \hat{k}^{TY}.$$

For any positive initial value the capital intensity converges monotonically to $\hat{k}^{TY}$.

(iii) The steady-state capital intensity satisfies the following inequality:

$$0 < \hat{k}^{WE} < \hat{k}^{TY}.$$

Proof. The steady-state equation (A.24) has two roots. By Lemmas A.3(i) and A.5(i) one root is at $\hat{k} = 0$. Since $\lim_{k\to 0} \Gamma_k(k,0)/\Psi'(k) = +\infty$ by the proof of Proposition 1 and $\Gamma_{kz_2}(k,z_2) > 0$ for all $k > 0$ we have:

$$\left.\frac{dk_{t+1}}{dk_t}\right|_{\hat{k}=0} = \lim_{k\to 0} \frac{\Gamma_k\,(k,1)}{\Psi'(k)} = +\infty. \tag{A.26}$$

It follows that the trivial solution is unstable and that $\Gamma(k,1)$ lies above $\Psi(k)$ for positive values of k close to the origin. We know that $\Gamma(k,1)$ satisfies $\lim_{k\to\infty} \Gamma_k\,(k,1) = 0$ while $\Psi(k)$ is strictly increasing with $\lim_{k\to\infty} \Psi'(k) > 0$. Hence there is a unique positive and finite nontrivial root, $\hat{k}^{TY}$. At $k = \hat{k}^{TY}$, $\Psi\,(k)$ cuts $\Gamma\,(k,1)$ from below such that $0 < \Gamma_k(\hat{k}^{TY},1)/\Psi'(\hat{k}^{TY}) < 1$, thus proving stability of the nontrivial equilibrium. Since $\Gamma_{z_2}(k,z_2) > 0$ for all k, increasing the perturbation parameter z_2 from 0 to 1 rotates the $\Gamma(k,z_2)$ function in a counterclockwise fashion. As a consequence $\hat{k}^{TY} > \hat{k}^{WE}$. □

A.2.4 PA model

If a steady state equilibrium of the PA model satisfying $k_{t+1} = k_t = \hat{k}$ exists, then it is a solution to the following equation:

$$\Psi(\hat{k}, 1) - \Gamma(\hat{k}) = 0, \tag{A.27}$$

where $\Gamma(k)$ has been defined in (A.14) and $\Psi(k, z_3)$ is given by:

$$\Psi(k, z_3) \equiv \begin{cases} \dfrac{k}{1 - \Phi(k, z_3)} & \text{for } k > 0 \\ 0 & \text{for } k = 0 \end{cases}, \tag{A.28}$$

$$\Phi(k, z_3) \equiv \frac{1}{1 + \left(\frac{1-\pi}{1+\rho}\right)^{\sigma} \left(\frac{1-\delta+\alpha\Omega k^{\alpha+\eta-1}}{1-z_3\pi}\right)^{\sigma-1}}, \quad k > 0 \tag{A.29}$$

with z_3 a perturbation parameter ($0 \leq z_3 \leq 1$). Note that (A.28) and (A.29) are obtained from, respectively, (19) and (21) in the text by setting $z_1 = 0$ and dropping the superfluous argument.

We define:

$$\bar{\Phi}(\sigma, z_3) = \frac{1}{1 + \left(\frac{1-\pi}{1+\rho}\right)^{\sigma} \left(\frac{1-\delta}{1-z_3\pi}\right)^{\sigma-1}} \tag{A.30}$$

The following lemmas cover the properties of $\Phi(k, z_3)$ and $\Psi(k, z_3)$.

Lemma A.6. *[Properties of the $\Phi(k, z_3)$ function] Let $\Phi(k, z_3)$ be defined as in* (A.29)*. This function has the following properties:*

(i) $\lim_{k\to 0} \Phi(k, z_3) = \begin{cases} 1 & \textit{for } 0 < \sigma < 1 \\ \bar{\Phi}(1, z_3) & \textit{for } \sigma = 1 \\ 0 & \textit{for } 1 < \sigma \leq \bar{\sigma} \end{cases};$

(ii) $\lim_{k\to\infty} \Phi(k, z_3) = \bar{\Phi}(\sigma, z_3)$;

(iii) For $0 < \sigma \leq 1$*:*

$$\lim_{k\to 0} \frac{1 - \Phi(k, z_3)}{k} = +\infty;$$

(iv) For $1 \leq \sigma \leq \bar{\sigma}$*:*

$$\lim_{k\to 0} \frac{\Phi(k, z_3)}{k} = +\infty;$$

(v) $\Phi_k(k, z_3) \equiv \partial\Phi(k, z_3)/\partial k \lesseqqgtr 0 \Leftrightarrow \sigma \lesseqqgtr 1$ *and* $\Phi_{z_3}(k, z_3) \equiv \partial\Phi(k, z_3)/\partial z_3 \gtreqqless 0 \Leftrightarrow \sigma \lesseqqgtr 1$ *for* $k > 0$;

(vi) $\lim_{k\to 0} \Phi_k(k, z_3) = \begin{cases} -\infty & \textit{for } 0 < \sigma < 1 \\ 0 & \textit{for } \sigma = 1 \\ +\infty & \textit{for } 1 < \sigma \leq \bar{\sigma} \end{cases}$.

Proof. For future reference we provide the following two equivalent representations of $\Phi(k, z_3)$:

$$\Phi(k, z_3) = \frac{1}{1 + (1 - z_3\pi)^{1-\sigma}\, \phi_0\, (\phi_1 + k^{\alpha+\eta-1})^{\sigma-1}}, \tag{A.31}$$

$$= \frac{1}{1 + (1 - z_3\pi)^{1-\sigma} \phi_0 k^{(1-\sigma)(1-\alpha-\eta)} (\phi_1 k^{1-\alpha-\eta} + 1)^{\sigma-1}}, \tag{A.32}$$

where ϕ_0 and ϕ_1 are defined in equation (A.6) above. Part (i) can be proved by using expression (A.32) and noting (A.7) and (A.8). Part (ii) follows directly from (A.31) as $\lim_{k\to\infty} k^{\alpha+\eta-1} = 0$. To prove part (iii) and (iv) we note that the term $(1 - z_3\pi)^{1-\sigma}$ does not affect the limiting behaviour of the $\Phi(k, z_3)$ function such that Lemma A.1(iii) and A.1(iv) apply.

To prove part (v) we compute:

$$\Phi_k(k, z_3) = -(1 - \sigma)(1 - \alpha - \eta) \frac{\Phi(k, z_3)[1 - \Phi(k, z_3)]}{k} \frac{1}{\phi_1 k^{1-\alpha-\eta} + 1}, \tag{A.33}$$

$$\Phi_{z_3}(k, z_3) = \frac{\pi(1 - \sigma)\Phi(k, z_3)[1 - \Phi(k, z_3)]}{1 - z_3\pi}. \tag{A.34}$$

The sign follows from the fact that $0 < \alpha + \eta < 1$, $0 < \Phi(k, z_3) < 1$, $1/(\phi_1 k^{1-\alpha-\eta} + 1)$, and $(1 - \sigma) \geq 0$ for $0 < \sigma < 1$, $(1 - \sigma) = 0$ for $\sigma = 1$, and $(1 - \sigma) < 0$ for $\sigma > 1$.

We can prove part (v) by combining parts (iii), (iv), and (v). □

Lemma A.7. *[Properties of the $\Psi(k, z_3)$ function] Let $\Psi(k, z_3)$ be defined as in* (A.28)*. This function has the following properties:*

(i) $\lim_{k\to 0} \Psi(k, z_3) = 0$;

(ii) $\Psi_k(k, z_3) \equiv \partial\Psi(k, z_3)/\partial k > 0$ *and* $\Psi_{z_3}(k, z_3) \equiv \partial\Phi(k, z_3)/\partial z_3 \lesseqqgtr 0 \Leftrightarrow \sigma \gtreqqless 1$ *for* $k > 0$;

(iii) $\lim_{k\to 0} \Psi_k(k, z_3) = \begin{cases} +\infty & \textit{for } 0 < \sigma < 1 \\ \dfrac{1}{1 - \bar{\Phi}(1, z_3)} & \textit{for } \sigma = 1 \\ 1 & \textit{for } 1 < \sigma \leq \bar{\sigma} \end{cases}$;

(iv) $\lim_{k\to 0} \dfrac{\Gamma'(k)}{\Psi_k(k,1)} = +\infty$;

(v) $\lim_{k\to\infty} \Psi_k(k,1) = \dfrac{1}{1-\bar{\Phi}(\sigma,1)} > 0$.

Proof. Part (i) follows readily from Lemma A.6(i) for $1 < \sigma \leq \bar{\sigma}$ and from Lemma A.6(iii) for $0 < \sigma \leq 1$.

To prove part (ii) we compute:

$$\Psi_k(k, z_3) = \frac{1-(1-\sigma)(1-\alpha-\eta)\Phi(k,z_3)\frac{1}{\phi_1 k^{1-\alpha-\eta+1}}}{1-\Phi(k,z_3)}, \tag{A.35}$$

$$\Psi_{z_3}(k, z_3) = \frac{\pi\,(1-\sigma)\,\Phi\,(k,z_3)}{1-z_3\pi}\frac{k}{1-\Phi\,(k,z_3)}, \tag{A.36}$$

for which the signs follow from the fact that $0 < \alpha+\eta < 1$, $0 < \Phi(k, z_3) < 1$, $1/(\phi_1 k^{1-\alpha-\eta+1}) < 1$, and $\sigma > 0$.

Part (iii) follows from (A.35) by using Lemma A.6(i).

To prove part (iv) we use (A.35) and Lemma A.2(ii) to write:

$$\frac{\Gamma'(k)}{\Psi_k(k,1)} = \frac{(1-\alpha)(\alpha+\eta)\Omega_0}{1+n}\frac{[1-\Phi(k,1)]k^{\alpha+\eta-1}}{1-(1-\sigma)(1-\alpha+\eta)\Phi(k,1)\frac{1}{\phi_1 k^{1-\alpha-\eta+1}}}. \tag{A.37}$$

For $1 \leq \sigma < \bar{\sigma}$ the result follows immediately from Lemma A.6 and the fact that $\lim_{k\to 0} k^{\alpha+\eta-1} = +\infty$. For $0 < \sigma < 1$ we use that $\lim_{k\to 0}[1-\Phi(k,1)]k^{\alpha+\eta-1} = +\infty$.

Part (v) follows from (A.35) and Lemma A.6(ii). □

We now proceed to the proof of Proposition 2 for the PA model.

Proposition 2. *[Existence and stability of the PA model] Consider the PA model and adopt Assumption 1. The following properties can be established:*

(i) The model has two steady-state solutions, the trivial one features $k_{t+1} = k_t = 0$, and the economically relevant one satisfies $k_{t+1} = k_t = \hat{k}^{PA}$, where $\hat{k}^{PA}$ is the solution to:

$$\frac{\hat{k}^{PA}}{1-\Phi(\hat{k}^{PA},1)} = \frac{(1-\alpha)\,\Omega_0(\hat{k}^{PA})^{\alpha+\eta}}{1+n}.$$

(ii) The trivial steady-state solution is unstable whilst the non-trivial solution is stable:

$$0 < \frac{dk_{t+1}}{dk_t} < 1, \qquad \textit{for } k_{t+1} = k_t = \hat{k}^{PA}.$$

For any positive initial value the capital intensity converges monotonically to $\hat{k}^{PA}$.

(iii) The steady-state capital intensity satisfies the following inequality:

$$\hat{k}^{PA} \lesseqqgtr \hat{k}^{WE} \quad \Leftrightarrow \quad \sigma \lesseqqgtr 1$$

Proof. The steady-state equation (A.27) has two roots. By Lemmas A.2(i) and A.7(i) one root is at $\hat{k} = 0$. To investigate the stability of that root we write:

$$\left.\frac{dk_{t+1}}{dk_t}\right|_{\hat{k}=0} = \lim_{k \to 0} \frac{\Gamma'(k)}{\Psi_k(k,1)} = +\infty. \tag{A.38}$$

where we have used Lemma A.7(iv). It follows that the trivial solution is unstable and that $\Gamma(k)$ lies above $\Psi(k,1)$ for positive values of k close to the origin. We know that $\Gamma(k)$ is concave and satisfies $\lim_{k\to\infty} \Gamma'(k) = 0$ while $\Psi(k,1)$ is strictly increasing with $\lim_{k\to\infty} \Psi_k(k,1) > 0$. Hence there is a unique positive and finite nontrivial root, $\hat{k}^{PA}$. At $k = \hat{k}^{PA}$, $\Psi(k,1)$ cuts $\Gamma(k)$ from below such that $0 < \Gamma'(\hat{k}^{PA})/\Psi_k(\hat{k}^{PA},1) < 1$, thus proving stability of the nontrivial equilibrium. Since $\Psi_{z_3}(k,z_3) \gtreqqless 0$ for $\sigma \lesseqqgtr 1$, increasing the pertubation parameters z_3 from 0 to 1 rotates the $\Psi(k,z_3)$ function in a counterclockwise fashion ($0 \le \sigma < 1$), a clockwise fashion ($1 < \sigma \le \bar{\sigma}$), or not at all ($\sigma = 1$). As a consequence $\hat{k}^{PA} \lesseqqgtr \hat{k}^{WE}$ for $\sigma \lesseqqgtr 1$. □

A.3 Proof of Lemma 1

Lemma 1. *[Implications of the factor price frontier] Assume that the economy is initially in the steady state associated with the WE or TY scenario, and is dynamically efficient ($\hat{r} > n$). Let $dk_{t+\infty}/dz_i$ denote the long-run effect on the capital intensity of a unit perturbation in z_i occurring at shock-time $\tau = 0$ and evaluated at $z_i = 0$. It follows that the long-run effect on weighted factor prices can be written as:*

$$\frac{\hat{C}^o}{(1+\hat{r})^2}\frac{dr_{t+\infty}}{dz_i} + \frac{dw_{t+\infty}}{dz_i} = \Delta\frac{dk_{t+\infty}}{dz_i}, \tag{L1.1}$$

where Δ is a positive constant:

$$\Delta \equiv \left[\eta + \alpha\left(1-\alpha-\eta\right)\frac{\hat{r}-n}{1+\hat{r}}\right]\frac{\hat{r}+\delta}{\alpha} > 0. \tag{L1.2}$$

Proof. We first note that – evaluated at the initial steady state – we can write:

$$\begin{aligned}
\frac{dr_{t+\infty}}{dz_i} &= -\left(1-\alpha-\eta\right)\alpha\Omega_0\hat{k}^{\alpha+\eta-2}\frac{dk_{t+\infty}}{dz_i}, \\
\frac{dw_{t+\infty}}{dz_i} &= \left(\alpha+\eta\right)\left(1-\alpha\right)\Omega_0\hat{k}^{\alpha+\eta-1}\frac{dk_{t+\infty}}{dz_i}.
\end{aligned}$$

By using $\hat{r}+\delta = \alpha\Omega_0\hat{k}^{\alpha+\eta-1}$ these expressions can be rewritten as follows:

$$\begin{aligned}
\frac{dr_{t+\infty}}{dz_i} &= -\left(1-\alpha-\eta\right)\frac{\hat{r}+\delta}{\hat{k}}\frac{dk_{t+\infty}}{dz_i}, \\
\frac{dw_{t+\infty}}{dz_i} &= \left(\alpha+\eta\right)\left(1-\alpha\right)\frac{\hat{r}+\delta}{\alpha}\frac{dk_{t+\infty}}{dz_i}.
\end{aligned}$$

By substituting these expressions into (L1.1) and noting that $\hat{C}^o/(1+\hat{r}) = (1+n)\,\hat{k}$ for both WE and TY we obtain::

$$\begin{aligned}
\Delta &\equiv \left[-\alpha\left(1-\alpha-\eta\right)\frac{1+n}{1+\hat{r}} + \left(\alpha+\eta\right)\left(1-\alpha\right)\right]\frac{\hat{r}+\delta}{\alpha} \\
&= \left[\eta + \alpha\left(1-\alpha-\eta\right)\frac{\hat{r}-n}{1+\hat{r}}\right]\frac{\hat{r}+\delta}{\alpha}.
\end{aligned}$$

Since $\eta \geq 0$ and $\hat{r} > n$ it follows that Δ is strictly positive. □

A.4 Scenario changes: Allocation effects

Note the following steady-state relationship:

$$\phi_1 \hat{k}^{1-\alpha-\eta} + 1 = \frac{1+\hat{r}}{\hat{r}+\delta}, \tag{A.39}$$

where $\phi_1 \equiv (1-\delta)/(\alpha\Omega_0)$ as given in (A.6).

We define:

$$\gamma_0 \equiv (1-\alpha-\eta)\frac{\hat{r}+\delta}{1+\hat{r}}, \tag{A.40}$$

such that $0 < \gamma_0 < 1$ as $0 \leq \eta < 1-\alpha$ and $\delta \leq 1$.

The analytical expressions for the derivatives used below and their sign can be found in Section A.2.

A.4.1 From WE to TO

The relevant fundamental difference equation is given by:

$$\Psi(k_{t+1}, z_1) - \Gamma(k_t) = 0.$$

We start from the steady state of WE, such that initially $k_{t+1} = k_t = \hat{k}$ and $z_1 = 0$.

- Impact effect on the future capital intensity:

$$\frac{dk_{t+1}}{dz_1} = -\frac{\Psi_{z_1}(\hat{k},0)}{\Psi(\hat{k},0)} = -\frac{\pi}{1-\pi}\frac{\hat{k}\Psi(\hat{k})}{1-(1-\sigma)\gamma_0\Psi(\hat{k})} < 0. \tag{A.41}$$

- Long-run effect on the capital intensity:

$$\frac{dk_{t+\infty}}{dz_1} = -\frac{\Psi_{z_1}(\hat{k},0)}{\Psi_k(\hat{k},0) - \Gamma'(\hat{k})} = -\frac{\pi}{1-\pi}\frac{\hat{k}\Phi(\hat{k})}{1-\alpha-\eta-(1-\sigma)\gamma_0\Phi(\hat{k})} < 0, \tag{A.42}$$

  since in the initial steady state we have $\Gamma'(\hat{k}) < \Psi_k(\hat{k},0)$ by Proposition 2 and:

$$\Gamma'(\hat{k}) = (\alpha+\eta)\frac{\Gamma(\hat{k})}{\hat{k}} = (\alpha+\eta)\frac{\Psi(\hat{k},0)}{\hat{k}} = \frac{\alpha+\eta}{1-\Phi(\hat{k})}.$$

A.4.2 From WE to TY

The relevant fundamental difference equation is given by:

$$\Psi(k_{t+1}) - \Gamma(k_t, z_2) = 0.$$

We start from the steady state of WE, such that initially $k_{t+1} = k_t = \hat{k}$ and $z_2 = 0$.

- Impact effect on the future capital intensity:

$$\frac{dk_{t+1}}{dz_2} = \frac{\Gamma_{z_2}(\hat{k}, 0)}{\Psi'(\hat{k})} = \frac{\pi(1+\hat{r})\hat{k}}{1+n} \frac{1-\Phi(\hat{k})}{1-(1-\sigma)\gamma_0\Phi(\hat{k})} > 0. \tag{A.43}$$

- Long-run effect on the capital intensity:

$$\frac{dk_{t+\infty}}{dz_2} = \frac{\Gamma_{z_2}(\hat{k}, 0)}{\Psi'(\hat{k}) - \Gamma_k(\hat{k}, 0)} = \frac{\pi(1+\hat{r})\hat{k}}{1+n} \frac{1-\Phi(\hat{k})}{1-\alpha-\eta-(1-\sigma)\gamma_0\Phi(\hat{k})} > 0, \tag{A.44}$$

  since in the initial steady state we have $\Gamma_k(\hat{k}, 0) < \Psi'(\hat{k})$ by Proposition 2 and:

$$\Gamma_k(\hat{k}, 0) = (\alpha+\eta)\frac{\Gamma(\hat{k}, 0)}{\hat{k}} = (\alpha+\eta)\frac{\Psi(\hat{k})}{\hat{k}} = \frac{\alpha+\eta}{1-\Phi(\hat{k})}.$$

A.4.3 From WE to PA

The relevant fundamental difference equation is given by:

$$\Psi(k_{t+1}, z_3) - \Gamma(k_t) = 0.$$

We start from the steady state of WE, such that initially $k_{t+1} = k_t = \hat{k}$ and $z_3 = 0$.

- Impact effect on the future capital intensity:

$$\frac{dk_{t+1}}{dz_3} = -\frac{\Psi_{z_3}(\hat{k}, 0)}{\Psi_k(\hat{k}, 0)} = -\pi(1-\sigma)\frac{\hat{k}\Phi(\hat{k}, 0)}{1-(1-\sigma)\gamma_0\Phi(\hat{k}, 0)} \gtreqless 0 \quad \Leftrightarrow \quad \sigma \gtreqless 1. \tag{A.45}$$

- Long-run effect on the capital intensity:

$$\frac{dk_{t+\infty}}{dz_3} = -\frac{\Psi_{z_3}(\hat{k}, 0)}{\Psi_k(\hat{k}, 0) - \Gamma'(\hat{k})} = -\pi(1-\sigma)\frac{\hat{k}\Phi(\hat{k}, 0)}{1-\alpha-\eta-(1-\sigma)\gamma_0\Phi(\hat{k}, 0)} \gtreqless 0 \quad \Leftrightarrow \quad \sigma \gtreqless 1, \tag{A.46}$$

  since in the initial steady state we have $\Gamma'(\hat{k}) < \Psi_k(\hat{k}, 0)$ by Proposition 2 and:

$$\Gamma'(\hat{k}) = (\alpha+\eta)\frac{\Gamma(\hat{k})}{\hat{k}} = (\alpha+\eta)\frac{\Psi(\hat{k}, 0)}{\hat{k}} = \frac{\alpha+\eta}{1-\Phi(\hat{k}, 0)}.$$

A.4.4 From TY to PA

The relevant fundamental difference equation is given by:

$$\Psi(k_{t+1}, z_3) = \Gamma(k_t, z_2).$$

When we move from the TY to the PA scenario we administer two seperate shocks. As a consequence, the impact and long-run effects are more subtle than in the cases discussed above. At impact z_3 changes from 0 to 1, while z_2 jumps from 1 to 0 only after period $t+1$. Hence, only in the long run we have $z_2 = 1 - z_3$. We start from the steady state of TY, such that initially $k_{t+1} = k_t = \hat{k}$, $z_2 = 1$, and $z_3 = 0$.

- Impact effect on the future capital intensity.

  The fundamental difference equation in the impact period is:

  $$\Psi(k_{t+1}, 1) = \Gamma(k_t, 1).$$

  Using a first-order approximation of $\Psi(k_{t+1}, z_3)$ around the point $(\hat{k}, 0)$ we obtain:

  $$\Psi(k_{t+1}, z_3) \approx \Psi(\hat{k}, 0) + \Psi_k(\hat{k}, 0)[k_{t+1} - \hat{k}] + \Psi_{z_3}(\hat{k}, 0)z_3.$$

  It follows that:

  $$\begin{aligned} \Psi(\hat{k}, 0) - \Psi(k_{t+1}, 1) &\approx \Psi(\hat{k}, 0) - \Psi(\hat{k}, 0) - \Psi_k(\hat{k}, 0)[k_{t+1} - \hat{k}] - \Psi_{z_3}(\hat{k}, 0), \\ 0 &\approx -\Psi(\hat{k}, 0)[k_{t+1} - \hat{k}] - \Psi_{z_3}(\hat{k}, 0), \end{aligned}$$

  since $\Psi(k_{t+1}, 1) = \Gamma(\hat{k}, 1)$ by the fundamental difference equation in the impact period and $\Psi(\hat{k}, 0) = \Gamma(\hat{k}, 1)$ as we start in the steady state of TY. The impact effect on the future capital intensity $dk_{t+1}/dz_3 \approx k_{t+1} - \hat{k}$ can then be approximated as:

  $$\frac{dk_{t+1}}{dz_3} = -\frac{\Psi_{z_3}(\hat{k}, 0)}{\Psi_k(\hat{k}, 0)}.$$

- Long-run effect on the capital intensity.

  The fundamental difference equation immediately after the impact period is:

  $$\Psi(k_{t+2}, 1) = \Gamma(k_{t+1}).$$

Using a first-order approximation of $\Gamma(k_{t+1})$ around $\hat{k}$ we obtain:

$$\Gamma(k_{t+1}) \approx \Gamma(\hat{k}) + \Gamma'(\hat{k})[k_{t+1} - \hat{k}].$$

We also know that:

$$\Psi(\hat{k}, 0) = \Gamma(\hat{k}, 1) = \Gamma(\hat{k}) + \frac{\pi\,(1+\hat{r})\,\hat{k}}{1+n} = \Gamma(\hat{k}) + \Gamma_{z_2}(\hat{k}, 1).$$

It follows that the fundamental difference equation can be approximated by:

$$\begin{aligned} \Psi(\hat{k}, 0) + \Psi_k(\hat{k}, 0)[k_{t+2} - \hat{k}] + \Psi_{z_3}(\hat{k}, 0) &\approx \Gamma(\hat{k}) + \Gamma'(\hat{k})[k_{t+1} - \hat{k}], \\ \Psi_k(\hat{k}, 0)[k_{t+2} - \hat{k}] + \Psi_{z_3}(\hat{k}, 0) &\approx \Gamma'(\hat{k})[k_{t+1} - \hat{k}] - \Gamma_{z_2}(\hat{k}, 1). \end{aligned}$$

since $\Psi(\hat{k}, 0) = \Gamma(\hat{k}) + \Gamma_{z_2}(\hat{k}, 1)$. Solving for $k_{t+2} - \hat{k}$ yields:

$$k_{t+2} - \hat{k} = -\frac{\Psi_{z_3}(\hat{k}, 0) + \Gamma_{z_2}(\hat{k}, 1)}{\Psi_k(\hat{k}, 0)} + \frac{\Gamma'(\hat{k})}{\Psi_k(\hat{k}, 0)}[k_{t+1} - \hat{k}].$$

Forward iteration gives:

$$\begin{aligned} k_{t+\tau} - \hat{k} &= -\frac{\Psi_{z_3}(\hat{k}, 0) + \Gamma_{z_2}(\hat{k}, 1)}{\Psi_k(\hat{k}, 0)} \left[1 + \frac{\Gamma'(\hat{k})}{\Psi_k(\hat{k}, 0)} + \left(\frac{\Gamma'(\hat{k})}{\Psi_k(\hat{k}, 0)}\right)^2 + \cdots + \left(\frac{\Gamma'(\hat{k})}{\Psi_k(\hat{k}, 0)}\right)^{\tau-2} \right] \\ &\quad + \left(\frac{\Gamma'(\hat{k})}{\Psi_k(\hat{k}, 0)}\right)^{\tau-1} [k_{t+1} - \hat{k}] \end{aligned}$$

Note that $0 < \Gamma'(\hat{k})/\Psi_k(\hat{k}, 0) < \Gamma_k(\hat{k}, 1)/\Psi_k(\hat{k}, 0) < 1$ by Proposition 2. Hence the long run effect on the capital intensity can be approximated as:

$$k_{t+\infty} - \hat{k} = -\frac{\Psi_{z_3}(\hat{k}, 0) - \Gamma_{z_2}(\hat{k}, 1)}{\Psi_k(\hat{k}, 0)} \frac{1}{1 - \Gamma'(\hat{k})/\Psi_k(\hat{k}, 0)} = -\frac{\Psi_{z_3}(\hat{k}, 0) + \Gamma_{z_2}(\hat{k}, 1)}{\Psi_k(\hat{k}, 0) - \Gamma'(\hat{k})}.$$

The long-run effect on the capital intensity $dk_{t+\infty}/dz_3 \approx k_{t+\infty} - \hat{k}$ can then be approximated as:

$$\frac{dk_{t+\infty}}{dz_3} = -\frac{\Psi_{z_3}(\hat{k}, 0) + \Gamma_{z_2}(\hat{k}, 1)}{\Psi_k(\hat{k}, 0) - \Gamma'(\hat{k})}.$$

A.4.5 From TO to PA

- In period t the FDE is:

$$k_{t+1} = [1 - \Phi(k_{t+1}, z_3)]\,\Gamma(k_t) - \frac{z_1 \pi}{1-\pi} \Phi(k_{t+1})\, k_{t+1}$$

– future old-age transfers cannot be annuitized by definition

– at time t we have $z_1 = 1 - z_3$ and $\Phi\left(k_{t+1}\right) k_{t+1} = \Phi(\hat{k})\hat{k}$.

- Define:

$$\Psi(k_{t+1}, z_1, z_3) \equiv \frac{1 + \frac{z_1 \pi}{1-\pi}\Phi\left(k_{t+1}\right)}{1 - \Phi\left(k_{t+1}, z_3\right)} k_{t+1},$$

and note that:

$$\begin{aligned}
\Psi_k(k_{t+1}, z_1, z_3) &> 0, \\
\Psi_{z_1}(k_{t+1}, z_1, z_3) &\equiv \frac{\frac{\pi}{1-\pi}\Phi\left(k_{t+1}\right)}{1 - \Phi\left(k_{t+1}, z_3\right)} k_{t+1} > 0, \\
\Psi_{z_3}(k_{t+1}, z_1, z_3) &\equiv \Phi_{z_3}\left(k_{t+1}, z_3\right) \frac{1 + \frac{z_1 \pi}{1-\pi}\Phi\left(k_{t+1}\right)}{\left[1 - \Phi\left(k_{t+1}, z_3\right)\right]^2} k_{t+1} \gtreqless 0 \;\Leftrightarrow\; \sigma \lesseqgtr 1.
\end{aligned}$$

- In the initial steady state we have:

$$\Psi(\hat{k}^{TO}, 1, 0) = \Gamma(\hat{k}^{TO}).$$

- At time t we have:

$$\Psi(k_{t+1}, 0, 1) = \Gamma(\hat{k}^{TO}).$$

- Approximate $\Psi(k_{t+1}, z_1, z_3)$ around $(\hat{k}^{TO}, \bar{z}_1, \bar{z}_3)$:

$$\begin{aligned}
\Psi(k_{t+1}, z_1, z_3) \approx\; & \Psi(\hat{k}^{TO}, \bar{z}_1, \bar{z}_3) + \Psi_k(\hat{k}^{TO}, \bar{z}_1, \bar{z}_3)\left[k_{t+1} - \hat{k}^{TO}\right] \\
& + \Psi_{z_1}(\hat{k}^{TO}, \bar{z}_1, \bar{z}_3)\left[z_1 - \bar{z}_1\right] + \Psi_{z_3}(\hat{k}^{TO}, \bar{z}_1, \bar{z}_3)\left[z_3 - \bar{z}_3\right].
\end{aligned}$$

- Hence around $(\hat{k}^{TO}, 1, 0)$ we find:

$$\begin{aligned}
\Psi(k_{t+1}, 0, 1) \approx\; & \Psi(\hat{k}^{TO}, 1, 0) + \Psi_k(\hat{k}^{TO}, 1, 0)\left[k_{t+1} - \hat{k}^{TO}\right] \\
& + \Psi_{z_1}(\hat{k}^{TO}, 1, 0)\left[z_1 - 1\right] + \Psi_{z_3}(\hat{k}^{TO}, 1, 0) z_3.
\end{aligned}$$

- It follows that:

$$\begin{aligned}
\Gamma(\hat{k}^{TO}) &= \Psi(\hat{k}^{TO}, 1, 0) + \Psi_k(\hat{k}^{TO}, 1, 0)\left[k_{t+1} - \hat{k}^{TO}\right] - \Psi_{z_1}(\hat{k}^{TO}, 1, 0) z_3 + \Psi_{z_3}(\hat{k}^{TO}, 1, 0) z_3 \\
0 &= \Psi_k(\hat{k}^{TO}, 1, 0)\left[k_{t+1} - \hat{k}^{TO}\right] - \Psi_{z_1}(\hat{k}^{TO}, 1, 0) z_3 + \Psi_{z_3}(\hat{k}^{TO}, 1, 0) z_3,
\end{aligned}$$

or:

$$\frac{k_{t+1}-\hat{k}^{TO}}{z_3}=\frac{\Psi_{z_1}(\hat{k}^{TO},1,0)-\Psi_{z_3}(\hat{k}^{TO},1,0)}{\Psi_k(\hat{k}^{TO},1,0)}.$$

This is the discrete counterpart to:

$$\left.\frac{dk_{t+1}}{dz_3}\right|_{k_t=\hat{k}^{TO}}=\frac{\Psi_{z_1}(\hat{k}^{TO},1,0)-\Psi_{z_3}(\hat{k}^{TO},1,0)}{\Psi_k(\hat{k}^{TO},1,0)}.$$

- Note that:

$$\begin{aligned}\Psi_{z_1}(\hat{k}^{TO},1,0) &\equiv \frac{\frac{\pi}{1-\pi}\Phi(\hat{k}^{TO})}{1-\Phi(\hat{k}^{TO})}\hat{k}^{TO},\\ \Psi_{z_3}(\hat{k}^{TO},1,0) &\equiv \Phi_{z_3}(\hat{k}^{TO},0)\frac{1+\frac{\pi}{1-\pi}\Phi(\hat{k}^{TO})}{\left[1-\Phi(\hat{k}^{TO},1)\right]^2}\hat{k}^{TO},\\ \Phi_{z_3}(\hat{k}^{TO},0) &\equiv \pi\,(1-\sigma)\,\Phi(\hat{k}^{TO})[1-\Phi(\hat{k}^{TO})].\end{aligned}$$

Combining these results we thus obtain:

$$\left.\frac{dk_{t+1}}{dz_3}\right|_{k_t=\hat{k}^{TO}}=\frac{\pi}{1-\pi}\frac{\hat{k}^{TO}\Phi(\hat{k}^{TO})}{\Psi_k(\hat{k}^{TO},1,0)}\frac{1-(1-\sigma)\left[1-\pi(1-\Phi(\hat{k}^{TO}))\right]}{[1-\Phi(\hat{k}^{TO})]\Psi_k(\hat{k}^{TO},1,0)}>0.$$

- For period $t+2$ we note that:

$$\Gamma\left(k_{t+1}\right)=\Psi(k_{t+2},0,1).$$

- Approximating both sides:

$$\begin{aligned}\Gamma(\hat{k}^{TO})+\Gamma'(\hat{k}^{TO})\left[k_{t+1}-\hat{k}^{TO}\right] &= \Psi(\hat{k}^{TO},1,0)+\Psi_k(\hat{k}^{TO},1,0)\left[k_{t+2}-\hat{k}^{TO}\right]\\ &\quad-\left[\Psi_{z_1}(\hat{k}^{TO},1,0)-\Psi_{z_3}(\hat{k}^{TO},1,0)\right]z_3,\end{aligned}$$

or:

$$\begin{aligned}\frac{k_{t+2}-\hat{k}^{TO}}{z_3} &= \frac{\Gamma'(\hat{k}^{TO})}{\Psi_k(\hat{k}^{TO},1,0)}\frac{k_{t+1}-\hat{k}^{TO}}{z_3}\\ &\quad+\frac{1}{\Psi_k(\hat{k}^{TO},1,0)}\left[\Psi_{z_1}(\hat{k}^{TO},1,0)-\Psi_{z_3}(\hat{k}^{TO},1,0)\right].\end{aligned}$$

- This is the discrete counterpart to:

$$\left.\frac{dk_{t+2}}{dz_3}\right|_{k_t=\hat{k}^{TO}}.$$

- After some iterations we find:

$$\left.\frac{dk_{t+2}}{dz_3}\right|_{k_t=\hat{k}^{TO}}=\frac{\Psi_{z_1}(\hat{k}^{TO},1,0)-\Psi_{z_3}(\hat{k}^{TO},1,0)}{\Psi_k(\hat{k}^{TO},1,0)-\Gamma'(\hat{k}^{TO})}.$$

A.5 Scenario changes: Welfare effects

Note that in the steady state of the WE scenario and the TY scenario the following relationship holds:

$$\frac{\hat{C}^o}{1+\hat{r}} = \hat{S} = (1+n)\hat{k}. \tag{A.47}$$

Similarly, for the TO scenario we have:

$$\frac{\hat{C}^o}{1+\hat{r}} = \hat{S} + \frac{\hat{Z}^o}{1+\hat{r}} = (1+n)\hat{k} + \frac{\pi(1+n)}{1-\pi}\hat{k} = \frac{1+n}{1-\pi}\hat{k}. \tag{A.48}$$

The impact effect on the future interest rate (evaluated in the initial steady state) can be written as:

$$\frac{dr_{t+1}}{dz_i} = -(1-\alpha-\eta)\frac{\hat{r}+\delta}{\hat{k}}\frac{dk_{t+1}}{dz_i}, \tag{A.49}$$

which is exactly opposite in sign to the impact effect on the future capital stock.

We define:

$$\Theta \equiv \left[\frac{\eta}{\alpha} + (1-\alpha-\eta)\frac{\hat{r}-n}{1+\hat{r}}\right]\frac{\hat{r}+\delta}{1+n}\frac{\Phi(\hat{k})}{1-\alpha-\eta-(1-\sigma)\,\gamma_0\Phi(\hat{k})}, \tag{A.50}$$

$$= \left[\frac{\eta}{\alpha\,(1-\alpha-\eta)} + \frac{\hat{r}-n}{1+\hat{r}}\right]\frac{\hat{r}+\delta}{1+n}\frac{\frac{\hat{r}+\delta}{1+\hat{r}}\Phi(\hat{k})}{1-(1-\sigma)\,\frac{\hat{r}+\delta}{1+\hat{r}}\Phi(\hat{k})} \geq 0. \tag{A.51}$$

A.5.1 From WE to TO

We want to derive analytical expressions for the welfare effect at impact and in the long run of a change in redistribution regime from WE to TO.

A.5.1.1 Impact effect

In order to find the impact effect on welfare we write:

$$\mathbb{E}\Lambda_t^y(z_1) \equiv U(C_t^y) + \frac{1-\pi}{1+\rho}U(C_{t+1}^o) - \lambda_t\left[C_t^y + \frac{C_{t+1}^o}{1+r_{t+1}} - w_t - z_1\frac{\pi\,(1+n)}{1-\pi}k_{t+1}\right],$$

where λ_t is the Lagrange multiplier and z_1 is a perturbation parameter which equals zero for the WE scenario and one for TO. The choice variables C_t^y and C_{t+1}^o can be considered functions of z_1 in a neighbourhood of the optimal solution. The first-order conditions for the private optimum are:

$$\begin{aligned} U'(C_t^y) &= \lambda_t, \\ \frac{1-\pi}{1+\rho}U'(C_{t+1}^o) &= \frac{\lambda_t}{1+r_{t+1}}, \\ C_t^y + \frac{C_{t+1}^o}{1+r_{t+1}} &= w_t + z_1\frac{\pi(1+n)}{1-\pi}k_{t+1}. \end{aligned}$$

At impact (i.e. in period t), k_t and w_t are predetermined. Differentiation with respect to z_1 yields:

$$\begin{aligned} \frac{d\mathbb{E}\Lambda_t^y(z_1)}{dz_1} &= U'(C_t^y)\frac{dC_t^y}{dz_1} + \frac{1-\pi}{1+\rho}U'(C_{t+1}^o)\frac{dC_{t+1}^o}{dz_1} \\ &\quad + \left[C_t^y + \frac{C_{t+1}^o}{1+r_{t+1}} - w_t - z_1\frac{\pi(1+n)}{1-\pi}k_{t+1}\right]\frac{d\lambda_t}{dz_1} - \lambda_t\left[\frac{dC_t^y}{dz_1} + \frac{1}{1+r_{t+1}}\frac{dC_{t+1}^o}{dz_1}\right. \\ &\quad \left. - \frac{C_{t+1}^o}{(1+r_{t+1})^2}\frac{dr_{t+1}}{dz_1} - \frac{\pi(1+n)}{1-\pi}k_{t+1} - z_1\frac{\pi(1+n)}{1-\pi}\frac{dk_{t+1}}{dz_1}\right]. \end{aligned} \tag{A.52}$$

We start from the steady state of WE, such that initially $k_{t+1} = k_t = \hat{k}$ (the superscript *WE* is omitted for convenience) and $z_1 = 0$. Incorporating the first-order conditions in (A.52) and evaluating at $z_1 = 0$ gives:

$$\frac{d\mathbb{E}\Lambda_t^y(z_1)}{dz_1} = U'(\hat{C}^y)\left[\frac{\pi(1+n)}{1-\pi}\hat{k} + \frac{\hat{C}^o}{(1+\hat{r})^2}\frac{dr_{t+1}}{dz_1}\right], \tag{A.53}$$

where $\hat{C}^y$ and $\hat{C}^o$ are the steady-state values of youth and old-age consumption, respectively, in the WE scenario. We can rewrite (A.53) using (A.47) to obtain:

$$\frac{d\mathbb{E}\Lambda_t^y(z_1)}{dz_1} = U'(\hat{C}^y)(1+n)\hat{k}\left[\frac{\pi}{1-\pi} + \frac{1}{1+\hat{r}}\frac{dr_{t+1}}{dz_1}\right]. \tag{A.54}$$

Equation (A.54) coincides with expression (27) in the text.

Substituting the results from section A.4 we find:

$$\begin{aligned} \frac{d\mathbb{E}\Lambda_t^y(z_1)}{dz_1} &= U'(\hat{C}^y)(1+n)\hat{k}\left[\frac{\pi}{1-\pi} - (1-\alpha-\eta)\frac{\hat{r}+\delta}{1+\hat{r}}\frac{1}{\hat{k}}\frac{dk_{t+1}}{dz_1}\right], \\ &= U'(\hat{C}^y)\frac{\pi}{1-\pi}(1+n)\hat{k}\left[1 + \frac{\gamma_0\Phi(\hat{k})}{1-(1-\sigma)\gamma_0\Phi(\hat{k})}\right], \\ &= U'(\hat{C}^y)\frac{\pi}{1-\pi}(1+n)\hat{k}\frac{1+\sigma\gamma_0\Phi(\hat{k})}{1-(1-\sigma)\gamma_0\Phi(\hat{k})} > 0. \end{aligned} \tag{A.55}$$

A.5.1.2 Long-run effect

In order to find the long-run welfare effect we write:

$$\mathbb{E}\Lambda^y_{t+\infty}(z_1) \equiv U(C^y_{t+\infty})+\frac{1-\pi}{1+\rho}U(C^o_{t+\infty})-\lambda_{t+\infty}\left[C^y_{t+\infty}+\frac{C^o_{t+\infty}}{1+r_{t+\infty}}-w_{t+\infty}-z_1\frac{\pi\left(1+n\right)}{1-\pi}k_{t+\infty}\right],$$

where $k_{t+\infty}$ and $w_{t+\infty}$ are no longer predetermined. Differentiation with respect to z_1 yields:

$$\begin{aligned}\frac{d\mathbb{E}\Lambda^y_{t+\infty}(z_1)}{dz_1} &= U'(C^y_{t+\infty})\frac{dC^y_{t+\infty}}{dz_1}+\frac{1-\pi}{1+\rho}U'(C^o_{t+\infty})\frac{dC^o_{t+\infty}}{dz_1} \\ &\quad+\left[C^y_{t+\infty}+\frac{C^o_{t+\infty}}{1+r_{t+\infty}}-w_{t+\infty}-z_1\frac{\pi\left(1+n\right)}{1-\pi}k_{t+\infty}\right]\frac{d\lambda_{t+\infty}}{dz_1} \\ &\quad-\lambda_{t+\infty}\left[\frac{dC^y_{t+\infty}}{dz_1}+\frac{1}{1+r_{t+\infty}}\frac{dC^o_{t+\infty}}{dz_1}-\frac{C^o_{t+\infty}}{\left(1+r_{t+\infty}\right)^2}\frac{dr_{t+\infty}}{dz_1}-\frac{dw_{t+\infty}}{dz_1}\right. \\ &\quad\left.-\frac{\pi\left(1+n\right)}{1-\pi}k_{t+\infty}-z_1\frac{\pi\left(1+n\right)}{1-\pi}\frac{dk_{t+\infty}}{dz_1}\right]. \qquad \text{(A.56)}\end{aligned}$$

We start from the steady state of WE, such that initially $k_{t+\infty}=\hat{k}$ and $z_1=0$. Incorporating the first-order conditions in (A.56) and evaluating at $z_1=0$ gives:

$$\frac{d\mathbb{E}\Lambda^y_{t+\infty}(z_1)}{dz_1}=U'(\hat{C}^y)\left[\frac{\pi\left(1+n\right)}{1-\pi}\hat{k}+\frac{\hat{C}^o}{\left(1+\hat{r}\right)^2}\frac{dr_{t+\infty}}{dz_1}+\frac{dw_{t+\infty}}{dz_1}\right]. \qquad \text{(A.57)}$$

We can rewrite (A.57) using (L1.1) to obtain:

$$\frac{d\mathbb{E}\Lambda^y_{t+\infty}(z_1)}{dz_1}=U'(\hat{C}^y)\left[\frac{\pi\left(1+n\right)}{1-\pi}\hat{k}+\Delta\frac{dk_{t+\infty}}{dz_1}\right]. \qquad \text{(A.58)}$$

Substituting the results from section A.4 we find:

$$\begin{aligned}\frac{d\mathbb{E}\Lambda^y_{t+\infty}(z_1)}{dz_1} &= U'(\hat{C}^y)\left[\frac{\pi(1+n)}{1-\pi}\hat{k}-\frac{\pi}{1-\pi}\left[\eta+\alpha(1-\alpha-\eta)\frac{\hat{r}-n}{1+\hat{r}}\right]\frac{\hat{r}+\delta}{\alpha}\right. \\ &\quad\left.\cdot\frac{\hat{k}\Phi(\hat{k})}{1-\alpha-\eta-(1-\sigma)\gamma_0\Phi(\hat{k})}\right], \\ &= U'(\hat{C}^y)\frac{\pi}{1-\pi}(1+n)\hat{k}\left[1-\left[\frac{\eta}{\alpha}+(1-\alpha-\eta)\frac{\hat{r}-n}{1+\hat{r}}\right]\frac{\hat{r}+\delta}{1+n}\right. \\ &\quad\left.\cdot\frac{\Phi(\hat{k})}{1-\alpha-\eta-(1-\sigma)\gamma_0\Phi(\hat{k})}\right], \\ &= U'(\hat{C}^y)\frac{\pi}{1-\pi}(1+n)\hat{k}[1-\Theta]\lesseqgtr 0 \quad\Leftrightarrow\quad \Theta\gtreqless 1. \qquad \text{(A.59)}\end{aligned}$$

A.5.2 From WE to TY

We want to derive analytical expressions for the welfare effect at impact and in the long run of a change in redistribution regime from WE to TY.

A.5.2.1 Impact effect

In order to find the impact effect on welfare we write:

$$\mathbb{E}\Lambda_t^y(z_2) \equiv U(C_t^y) + \frac{1-\pi}{1+\rho}U(C_{t+1}^o) - \lambda_t \left[C_t^y + \frac{C_{t+1}^o}{1+r_{t+1}} - w_t - z_2\pi(1+r_t)k_t\right],$$

where λ_t is the Lagrange multiplier and z_2 is a perturbation parameter which equals zero for the WE scenario and one for TY. The choice variables C_t^y and C_{t+1}^o can be considered functions of z_1 in a neighbourhood of the optimal solution. The first-order conditions for the private optimum are:

$$\begin{aligned} U'(C_t^y) &= \lambda_t, \\ \frac{1-\pi}{1+\rho}U'(C_{t+1}^o) &= \frac{\lambda_t}{1+r_{t+1}}, \\ C_t^y + \frac{C_{t+1}^o}{1+r_{t+1}} &= w_t + z_2\pi\left(1+r_t\right)k_t. \end{aligned}$$

At impact (i.e. in period t), k_t, w_t, and r_t are predetermined. Differentiation with respect to z_2 yields:

$$\begin{aligned} \frac{d\mathbb{E}\Lambda_t^y(z_2)}{dz_2} &= U'(C_t^y)\frac{dC_t^y}{dz_2} + \frac{1-\pi}{1+\rho}U'(C_{t+1}^o)\frac{dC_{t+1}^o}{dz_2} \\ &\quad + \left[C_t^y + \frac{C_{t+1}^o}{1+r_{t+1}} - w_t - z_2\pi(1+r_t)k_t\right]\frac{d\lambda_t}{dz_2} - \lambda_t\left[\frac{dC_t^y}{dz_2} + \frac{1}{1+r_{t+1}}\frac{dC_{t+1}^o}{dz_2}\right. \\ &\quad \left. - \frac{C_{t+1}^o}{\left(1+r_{t+1}\right)^2}\frac{dr_{t+1}}{dz_2} - \pi\left(1+r_t\right)k_t\right]. \end{aligned} \tag{A.60}$$

We start from the steady state of WE, such that initially $k_{t+1} = k_t = \hat{k}$ (the superscript WE is omitted for convenience) and $z_2 = 0$. Incorporating the first-order conditions in (A.60) and evaluating at $z_2 = 0$ gives:

$$\frac{d\mathbb{E}\Lambda_t^y(z_2)}{dz_2} = U'(\hat{C}^y)\left[\pi\left(1+\hat{r}\right)\hat{k} + \frac{\hat{C}^o}{(1+\hat{r})^2}\frac{dr_{t+1}}{dz_2}\right], \tag{A.61}$$

where $\hat{C}^y$ and $\hat{C}^o$ are the steady-state values of youth and old-age consumption, respectively, in the WE scenario. We can rewrite (A.61) using (A.47) to obtain:

$$\frac{d\mathbb{E}\Lambda_t^y(z_2)}{dz_2} = U'(\hat{C}^y)(1+n)\hat{k}\left[\frac{\pi(1+\hat{r})}{1+n} + \frac{1}{1+\hat{r}}\frac{dr_{t+1}}{dz_2}\right]. \tag{A.62}$$

Equation (A.62) coincides with expression (32) in the text.

Substituting the results from section A.4 we find:

$$\begin{aligned}\frac{d\mathbb{E}\Lambda_t^y(z_2)}{dz_2} &= U'(\hat{C}^y)(1+n)\hat{k}\left[\frac{\pi(1+\hat{r})}{1+n} - (1-\alpha-\eta)\frac{\hat{r}+\delta}{1+\hat{r}}\frac{1}{\hat{k}}\frac{dk_{t+1}}{dz_2}\right], \\ &= U'(\hat{C}^y)\pi(1+\hat{r})\hat{k}\left[1 - \frac{\gamma_0[1-\Phi(\hat{k})]}{1-(1-\sigma)\gamma_0\Phi(\hat{k})}\right], \\ &= U'(\hat{C}^y)\pi(1+\hat{r})\hat{k}\frac{1-\gamma_0[1-\sigma\Phi(\hat{k})]}{1-(1-\sigma)\gamma_0\Phi(\hat{k})} > 0. \end{aligned} \tag{A.63}$$

A.5.2.2 Long-run effect

In order to find the long-run welfare effect we write:

$$\mathbb{E}\Lambda_{t+\infty}^y(z_2) \equiv U(C_{t+\infty}^y) + \frac{1-\pi}{1+\rho}U(C_{t+\infty}^o) - \lambda_{t+\infty}\left[C_{t+\infty}^y + \frac{C_{t+\infty}^o}{1+r_{t+\infty}} - w_{t+\infty} - z_2\pi\left(1+r_{t+\infty}\right)k_{t+\infty}\right],$$

where $k_{t+\infty}$, $w_{t+\infty}$, and $r_{t+\infty}$ are no longer predetermined. Differentiation with respect to z_2 yields:

$$\begin{aligned}\frac{d\mathbb{E}\Lambda_{t+\infty}^y(z_2)}{dz_2} &= U'(C_{t+\infty}^y)\frac{dC_{t+\infty}^y}{dz_2} + \frac{1-\pi}{1+\rho}U'(C_{t+\infty}^o)\frac{dC_{t+\infty}^o}{dz_2} \\ &\quad + \left[C_{t+\infty}^y + \frac{C_{t+\infty}^o}{1+r_{t+\infty}} - w_{t+\infty} - z_2\pi\left(1+r_{t+\infty}\right)k_{t+\infty}\right]\frac{d\lambda_{t+\infty}}{dz_2} \\ &\quad -\lambda_{t+\infty}\left[\frac{dC_{t+\infty}^y}{dz_2} + \frac{1}{1+r_{t+\infty}}\frac{dC_{t+\infty}^o}{dz_2} - \frac{C_{t+\infty}^o}{(1+r_{t+\infty})^2}\frac{dr_{t+\infty}}{dz_2} - \frac{dw_{t+\infty}}{dz_2}\right. \\ &\quad \left. -\pi(1+r_{t+\infty})k_{t+\infty} - z_2\pi\frac{d(1+r_{t+\infty})k_{t+\infty}}{dz_2}\right]. \end{aligned} \tag{A.64}$$

We start from the steady state of WE, such that initially $k_{t+\infty} = \hat{k}$ and $z_2 = 0$. Incorporating the first-order conditions in (A.64) and evaluating at $z_2 = 0$ gives:

$$\frac{d\mathbb{E}\Lambda_{t+\infty}^y(z_2)}{dz_2} = U'(\hat{C}^y)\left[\pi(1+\hat{r})\hat{k} + \frac{\hat{C}^o}{(1+\hat{r})^2}\frac{dr_{t+\infty}}{dz_2} + \frac{dw_{t+\infty}}{dz_2}\right]. \tag{A.65}$$

We can rewrite (A.65) using (L1.1) to obtain:

$$\frac{d\mathbb{E}\Lambda_{t+\infty}^y(z_2)}{dz_2} = U'(\hat{C}^y)\left[\pi(1+\hat{r})\hat{k} + \Delta\frac{dk_{t+\infty}}{dz_2}\right]. \tag{A.66}$$

Substituting the results from section A.4 we find:

$$\begin{aligned}
\frac{d\mathbb{E}\Lambda^{y}_{t+\infty}(z_2)}{dz_2} &= U'(\hat{C}^y)\left[\pi(1+\hat{r})\hat{k} + \left[\eta + \alpha(1-\alpha-\eta)\frac{\hat{r}-n}{1+\hat{r}}\right]\frac{\hat{r}+\delta}{\alpha}\right.\\
&\quad \left.\cdot\frac{\pi(1+\hat{r})\hat{k}}{1+n}\frac{1-\Phi(\hat{k})}{1-\alpha-\eta-(1-\sigma)\gamma_0\Phi(\hat{k})}\right],\\
&= U'(\hat{C}^y)\pi(1+\hat{r})\hat{k}\left[1 + \left[\frac{\eta}{\alpha} + (1-\alpha-\eta)\frac{\hat{r}-n}{1+\hat{r}}\right]\frac{\hat{r}+\delta}{1+n}\right.\\
&\quad \left.\cdot\frac{1-\Phi(\hat{k})}{1-\alpha-\eta-(1-\sigma)\gamma_0\Phi(\hat{k})}\right],\\
&= U'(\hat{C}^y)\pi(1+\hat{r})\hat{k}\left[1 + \frac{1-\Phi(\hat{k})}{\Phi(\hat{k})}\Theta\right] > 0. \qquad \text{(A.67)}
\end{aligned}$$

A.5.3 From WE to PA

We want to derive analytical expressions for the welfare effect at impact and in the long run of a change from the WE to the PA scenario.

A.5.3.1 Impact effect

In order to find the impact effect on welfare we write:

$$\mathbb{E}\Lambda_t^y(z_3) \equiv U(C_t^y) + \frac{1-\pi}{1+\rho}U(C_{t+1}^o) - \lambda_t\left[C_t^y + \frac{(1-z_3\pi)\,C_{t+1}^o}{1+r_{t+1}} - w_t\right],$$

where λ_t is the Lagrange multiplier and z_3 is a perturbation parameter which equals zero for the WE scenario and one for PA. The first-order conditions for the private optimum are:

$$\begin{aligned} U'\left(C_t^y\right) &= \lambda_t, \\ \frac{1-\pi}{1+\rho}U'(C_{t+1}^o) &= \frac{1-z_3\pi}{1+r_{t+1}}\lambda_t, \\ C_t^y + \frac{(1-z_3\pi)C_{t+1}^o}{1+r_{t+1}} &= w_t. \end{aligned}$$

At impact (i.e. in period t), w_t is predetermined. Differentiation with respect to z_3 yields:

$$\begin{aligned} \frac{d\mathbb{E}\Lambda_t^y(z_3)}{dz_3} &= U'(C_t^y)\frac{dC_t^y}{dz_3} + \frac{1-\pi}{1+\rho}U'(C_{t+1}^o)\frac{dC_{t+1}^o}{dz_3} \\ &\quad + \left[C_t^y + \frac{(1-z_3\pi)\,C_{t+1}^o}{1+r_{t+1}} - w_t\right]\frac{d\lambda_t}{dz_3} - \lambda_t\left[\frac{dC_t^y}{dz_3} + \frac{(1-z_3\pi)}{1+r_{t+1}}\frac{dC_{t+1}^o}{dz_3}\right. \\ &\quad \left. - \frac{\pi C_{t+1}^o}{1+r_{t+1}} - \frac{(1-z_3\pi)\,C_{t+1}^o}{(1+r_{t+1})^2}\frac{dr_{t+1}}{dz_3}\right]. \end{aligned} \tag{A.68}$$

We start from the steady state of WE, such that initially $k_{t+1} = k_t = \hat{k}$ (the superscript *WE* is omitted for convenience) and $z_3 = 0$. Incorporating the first-order conditions in (A.68) and evaluating at $z_3 = 0$ gives:

$$\frac{d\mathbb{E}\Lambda_t^y(z_3)}{dz_3} = \frac{U'(\hat{C}^y)\hat{C}^o}{1+\hat{r}}\left[\pi + \frac{1}{1+\hat{r}}\frac{dr_{t+1}}{dz_3}\right]. \tag{A.69}$$

where $\hat{C}^y$ and $\hat{C}^o$ are the steady-state values of youth and old-age consumption, respectively, in the WE scenario. We can rewrite (A.69) using (A.47) to obtain:

$$\frac{d\mathbb{E}\Lambda_t^y(z_3)}{dz_3} = U'(\hat{C}^y)(1+n)\hat{k}\left[\pi + \frac{1}{1+\hat{r}}\frac{dr_{t+1}}{dz_3}\right]. \tag{A.70}$$

Equation (A.70) coincides with expression (36) in the text.

Substituting the results from section A.4 we find:

$$\begin{aligned}
\frac{d\mathbb{E}\Lambda_t^y(z_3)}{dz_3} &= U'(\hat{C}^y)(1+n)\hat{k}\left[\pi - (1-\alpha-\eta)\frac{\hat{r}+\delta}{1+\hat{r}}\frac{1}{\hat{k}}\frac{dk_{t+1}}{dz_3}\right], \\
&= U'(\hat{C}^y)\pi(1+n)\hat{k}\left[1+\frac{(1-\sigma)\gamma_0\Phi(\hat{k},0)}{1-(1-\sigma)\gamma_0\Phi(\hat{k},0)}\right], \\
&= U'(\hat{C}^y)\frac{\pi(1+n)\hat{k}}{1-(1-\sigma)\gamma_0\Phi(\hat{k},0)} > 0. \qquad \text{(A.71)}
\end{aligned}$$

A.5.3.2 Long-run effect

In order to find the long-run welfare effect we write:

$$\mathbb{E}\Lambda_{t+\infty}^y(z_3) \equiv U(C_{t+\infty}^y) + \frac{1-\pi}{1+\rho}U(C_{t+\infty}^o) - \lambda_{t+\infty}\left[C_{t+\infty}^y + \frac{(1-z_3\pi)\,C_{t+\infty}^o}{1+r_{t+\infty}} - w_{t+\infty}\right],$$

where $w_{t+\infty}$ is no longer predetermined. Differentiation with respect to z_3 yields:

$$\begin{aligned}
\frac{d\mathbb{E}\Lambda_{t+\infty}^y(z_3)}{dz_3} &= U'(C_{t+\infty}^y)\frac{dC_{t+\infty}^y}{dz_3} + \frac{1-\pi}{1+\rho}U'(C_{t+\infty}^o)\frac{dC_{t+\infty}^o}{dz_3} \\
&\quad + \left[C_{t+\infty}^y + \frac{(1-z_3\pi)\,C_{t+\infty}^o}{1+r_{t+\infty}} - w_{t+\infty}\right]\frac{d\lambda_{t+\infty}}{dz_3} \\
&\quad - \lambda_{t+\infty}\left[\frac{dC_{t+\infty}^y}{dz_3} + \frac{1-z_3\pi}{1+r_{t+\infty}}\frac{dC_{t+\infty}^o}{dz_3} - \frac{\pi C_{t+\infty}^o}{1+r_{t+\infty}}\right. \\
&\quad \left. - \frac{(1-z_3\pi)\,C_{t+\infty}^o}{(1+r_{t+\infty})^2}\frac{dr_{t+\infty}}{dz_3} - \frac{dw_{t+\infty}}{dz_3}\right]. \qquad \text{(A.72)}
\end{aligned}$$

We start from the steady state of WE, such that initially $k_{t+\infty} = \hat{k}$ and $z_3 = 0$. Incorporating the first-order conditions in (A.72) and evaluating at $z_3 = 0$ gives:

$$\frac{d\mathbb{E}\Lambda_{t+\infty}^y(z_3)}{dz_3} = U'(\hat{C}^y)\left[\frac{\pi\hat{C}^o}{1+\hat{r}} + \frac{\hat{C}^o}{(1+\hat{r})^2}\frac{dr_{t+\infty}}{dz_3} + \frac{dw_{t+\infty}}{dz_3}\right]. \qquad \text{(A.73)}$$

We can rewrite (A.73) using (A.47) and (L1.1) to obtain:

$$\frac{d\mathbb{E}\Lambda_{t+\infty}^y(z_3)}{dz_3} = U'(\hat{C}^y)\left[\pi(1+n)\hat{k} + \Delta\frac{dk_{t+\infty}}{dz_3}\right]. \qquad \text{(A.74)}$$

Substituting the results from section A.4 we find:

$$\begin{aligned}
\frac{d\mathbb{E}\Lambda^y_{t+\infty}(z_3)}{dz_3} &= U'(\hat{C}^y)\left[\pi(1+n)\hat{k} - \left[\eta + \alpha(1-\alpha-\eta)\frac{\hat{r}-n}{1+\hat{r}}\right]\frac{\hat{r}+\delta}{\alpha}\right. \\
&\left. \cdot\pi(1-\sigma)\frac{\hat{k}\Phi(\hat{k},0)}{1-\alpha-\eta-(1-\sigma)\gamma_0\Phi(\hat{k},0)}\right], \\
&= U'(\hat{C}^y)\pi(1+n)\hat{k}\left[1-(1-\sigma)\left[\frac{\eta}{\alpha}+(1-\alpha-\eta)\frac{\hat{r}-n}{1+\hat{r}}\right]\right. \\
&\left. \cdot\frac{\hat{r}+\delta}{1+n}\frac{\Phi(\hat{k},0)}{1-\alpha-\eta-(1-\sigma)\gamma_0\Phi(\hat{k},0)}\right], \\
&= U'(\hat{C}^y)\pi(1+n)\hat{k}[1-(1-\sigma)\Theta], \qquad \text{(A.75)}
\end{aligned}$$

since $\Phi(k, z_3)$ evaluated in $(\hat{k}, 0)$ is equivalent to the function $\Phi(\hat{k})$ as given in the definition of Θ in (A.50).

A.5.4 From TY to PA

We want to derive analytical expressions for the welfare effect at impact and in the long run of a change from the TY to the PA scenario.

A.5.4.1 Impact effect

In order to find the impact effect on welfare we write:

$$\mathbb{E}\Lambda_t^y(z_3) \equiv U(C_t^y) + \frac{1-\pi}{1+\rho}U(C_{t+1}^o) - \lambda_t\left[C_t^y + \frac{(1-z_3\pi)\,C_{t+1}^o}{1+r_{t+1}} - w_t - \pi\,(1+r_t)\,k_t\right],$$

where λ_t is the Lagrange multiplier and z_3 is a perturbation parameter which equals zero for the TY scenario and one for PA. The choice variables C_t^y and C_{t+1}^o can be considered functions of z_3 in a neighbourhood of the optimal solution. Note that in the impact period the transfers to the young (the last term between square brackets) have not yet been abolished.

The first-order conditions for the private optimum are:

$$\begin{aligned} U'(C_t^y) &= \lambda_t, \\ \frac{1-\pi}{1+\rho}U'(C_{t+1}^o) &= \frac{1-z_3\pi}{1+r_{t+1}}\lambda_t, \\ C_t^y + \frac{(1-z_3\pi)C_{t+1}^o}{1+r_{t+1}} &= w_t + \pi\,(1+r_t)\,k_t. \end{aligned}$$

At impact (i.e. in period t), k_t and w_t are predetermined. Differentiation with respect to z_3 yields:

$$\begin{aligned} \frac{d\mathbb{E}\Lambda_t^y(z_3)}{dz_3} &= U'(C_t^y)\frac{dC_t^y}{dz_3} + \frac{1-\pi}{1+\rho}U'(C_{t+1}^o)\frac{dC_{t+1}^o}{dz_3} \\ &\quad + \left[C_t^y + \frac{(1-z_3\pi)\,C_{t+1}^o}{1+r_{t+1}} - w_t - \pi\,(1+r_t)\,k_t\right]\frac{d\lambda_t}{dz_3} - \lambda_t\left[\frac{dC_t^y}{dz_3}\right. \\ &\quad \left. + \frac{(1-z_3\pi)}{1+r_{t+1}}\frac{dC_{t+1}^o}{dz_3} - \frac{\pi C_{t+1}^o}{1+r_{t+1}} - \frac{(1-z_3\pi)\,C_{t+1}^o}{(1+r_{t+1})^2}\frac{dr_{t+1}}{dz_3}\right]. \end{aligned} \tag{A.76}$$

We start from the steady state of TY, such that initially $k_{t+1} = k_t = \hat{k}$ (the superscript TY is omitted for convenience) and $z_3 = 0$. Incorporating the first-order conditions in (A.76) and evaluating at $z_3 = 0$ gives:

$$\frac{d\mathbb{E}\Lambda_t^y(z_3)}{dz_3} = \frac{U'(\hat{C}^y)\hat{C}^o}{1+\hat{r}}\left[\pi + \frac{1}{1+\hat{r}}\frac{dr_{t+1}}{dz_3}\right]. \tag{A.77}$$

where $\hat{C}^y$ and $\hat{C}^o$ are the steady-state values of youth and old-age consumption, respectively, in the TY scenario. We can rewrite (A.77) using (A.47) to obtain:

$$\frac{d\mathbb{E}\Lambda_t^y(z_3)}{dz_3} = U'(\hat{C}^y)(1+n)\hat{k}\left[\pi + \frac{1}{1+\hat{r}}\frac{dr_{t+1}}{dz_3}\right]. \tag{A.78}$$

Equation (A.78) coincides with expression (38) in the text.

Substituting the results from section A.4 we find:

$$\begin{aligned}
\frac{d\mathbb{E}\Lambda_t^y(z_3)}{dz_3} &= U'(\hat{C}^y)(1+n)\hat{k}\left[\pi - (1-\alpha-\eta)\frac{\hat{r}+\delta}{1+\hat{r}}\frac{1}{\hat{k}}\frac{dk_{t+1}}{dz_3}\right], \\
&= U'(\hat{C}^y)\pi(1+n)\hat{k}\left[1 + \frac{(1-\sigma)\gamma_0\Phi(\hat{k},0)}{1-(1-\sigma)\gamma_0\Phi(\hat{k},0)}\right], \\
&= U'(\hat{C}^y)\frac{\pi(1+n)\hat{k}}{1-(1-\sigma)\gamma_0\Phi(\hat{k},0)} > 0.
\end{aligned} \tag{A.79}$$

A.5.4.2 Long-run welfare effect

In order to find the long-run welfare effect we write:

$$\begin{aligned}
\mathbb{E}\Lambda_{t+\infty}^y(z_3) \equiv\; & U(C_{t+\infty}^y) + \frac{1-\pi}{1+\rho}U(C_{t+\infty}^o) - \lambda_{t+\infty}\Bigg[C_{t+\infty}^y + \frac{(1-z_3\pi)\,C_{t+\infty}^o}{1+r_{t+\infty}} \\
& -w_{t+\infty} - (1-z_3)\,\pi\,(1+r_{t+\infty})\,k_{t+\infty}\Bigg],
\end{aligned}$$

where $k_{t+\infty}$, $w_{t+\infty}$, and $r_{t+\infty}$ are no longer predetermined. The first-order conditions for the private optimum are:

$$\begin{aligned}
U'(C_{t+\infty}^y) &= \lambda_{t+\infty}, \\
\frac{1-\pi}{1+\rho}U'(C_{t+\infty}^o) &= \frac{1-z_3\pi}{1+r_{t+\infty}}\lambda_{t+\infty}, \\
C_{t+\infty}^y + \frac{(1-z_3\pi)C_{t+\infty}^o}{1+r_{t+\infty}} &= w_{t+\infty} + (1-z_3)\pi\,(1+r_{t+\infty})\,k_{t+\infty}.
\end{aligned}$$

Note that the last condition differs from the one used in the impact analysis since the transfers to the young are abolished in the PA scenario (with $z_3 = 1$) from period $t+1$ onwards. This implies that in deriving the long-run welfare effect we should treat the transfer term as a constant. Differentiation with respect to z_3 yields:

$$\begin{aligned}\frac{d\mathbb{E}\Lambda^y_{t+\infty}(z_3)}{dz_3} &= U'(C^y_{t+\infty})\frac{dC^y_{t+\infty}}{dz_3} + \frac{1-\pi}{1+\rho}U'(C^o_{t+\infty})\frac{dC^o_{t+\infty}}{dz_3} \\ &+ \left[C^y_{t+\infty} + \frac{(1-z_3\pi)\,C^o_{t+\infty}}{1+r_{t+\infty}} - w_{t+\infty} - (1-z_3)\,\pi\,(1+\hat{r})\,\hat{k}\right]\frac{d\lambda_{t+\infty}}{dz_3} \\ &- \lambda_{t+\infty}\left[\frac{dC^y_{t+\infty}}{dz_3} + \frac{1-z_3\pi}{1+r_{t+\infty}}\frac{dC^o_{t+\infty}}{dz_3} - \frac{\pi C^o_{t+\infty}}{1+r_{t+\infty}} - \frac{(1-z_3\pi)\,C^o_{t+\infty}}{(1+r_{t+\infty})^2}\frac{dr_{t+\infty}}{dz_3}\right. \\ &\left. - \frac{dw_{t+\infty}}{dz_3} + \pi\,(1+r_{t+\infty})\,k_{t+\infty}\right]. \qquad \text{(A.80)}\end{aligned}$$

We start from the steady state of TY, such that initially $k_{t+\infty} = \hat{k}$ and $z_3 = 0$. Incorporating the first-order conditions in (A.80) and evaluating at $z_3 = 0$ gives:

$$\frac{d\mathbb{E}\Lambda^y_{t+\infty}(z_3)}{dz_3} = U'(\hat{C}^y)\left[\frac{\pi\hat{C}^o}{1+\hat{r}} + \frac{\hat{C}^o}{(1+\hat{r})^2}\frac{dr_{t+\infty}}{dz_3} + \frac{dw_{t+\infty}}{dz_3} - \pi(1+\hat{r})\hat{k}\right]. \qquad \text{(A.81)}$$

We can rewrite (A.81) using (A.47) and (L1.1) to obtain:

$$\frac{d\mathbb{E}\Lambda^y_{t+\infty}(z_3)}{dz_3} = U'(\hat{C}^y)\left[-\pi(\hat{r}-n)\hat{k} + \Delta\frac{dk_{t+\infty}}{dz_3}\right]. \qquad \text{(A.82)}$$

Substituting the results from section A.4 we find:

$$\begin{aligned}\frac{d\mathbb{E}\Lambda^y_{t+\infty}(z_3)}{dz_3} &= U'(\hat{C}^y)\left[-\pi(\hat{r}-n)\hat{k} + \Delta\frac{dk_{t+\infty}}{dz_3}\right] \\ &= -U'(\hat{C}^y)\left[\pi(\hat{r}-n)\hat{k} + \left[\eta + \alpha(1-\alpha-\eta)\frac{\hat{r}-n}{1+\hat{r}}\right]\frac{\hat{r}+\delta}{\alpha}\frac{\Psi_{z_3}(\hat{k},0) + \Gamma_{z_2}(\hat{k},1)}{\Psi_k(\hat{k},0) - \Gamma'(\hat{k})}\right].\end{aligned}$$

We know that:

$$\begin{aligned}\Psi_k(\hat{k},0) - \Gamma'(\hat{k}) &= \frac{1-\alpha-\eta-(1-\sigma)\,\gamma_0\Phi(\hat{k},0)}{1-\Phi(\hat{k},0)}, \\ \Psi_{z_3}(\hat{k},0) + \Gamma_{z_2}(\hat{k},1) &= \pi\hat{k}\frac{(1-\sigma)\,\Phi(\hat{k},0) + \frac{1+\hat{r}}{1+n}\left(1-\Phi(\hat{k},0)\right)}{1-\Phi(\hat{k},0)}.\end{aligned}$$

It follows that:

$$\frac{d\mathbb{E}\Lambda^y_{t+\infty}(z_3)}{dz_3} = -U'(\hat{C}^y)\pi\hat{k}\left[\hat{r} - n + \Theta\left(\frac{(1+n)\,(1-\sigma)\,\Phi(\hat{k},0) + (1+\hat{r})(1-\Phi(\hat{k},0))}{\Phi(\hat{k},0)}\right)\right].$$

For $0 < \sigma \leq 1$ the welfare effect is unambiguously negative.

A.6 Additional visualization

In this section we present some additional visualization of the various cases considered in the paper.

Figure A.1 depicts the fundamental difference equations for the different scenarios and for different values of the intertemporal substitution elasticity. The pure scenarios are given by WE, TY, TO, and PA. The curve labeled TY_t is used in the transition from TY to PA. At the time of the shock, the young still receive transfers because the shock-time old did not have an opportunity to annuitize their savings. For $\sigma = 1$ we find that TY and TY_t coincide because the change in the interest rate has no effect on the savings propensity. For $\sigma < 1$ ($\sigma > 1$) we find that TY_t lies below (above) TY.

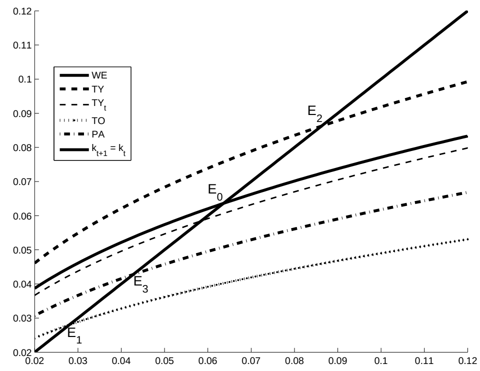
Figure A.2 shows the transitional dynamics for $k_{t+\tau}$, $\mathbb{E}\Lambda^y_{t+\tau}$, $C^y_{t+\tau}$, and $C^o_{t+\tau}$ for the change from WE to TO, TY, or PA. Again different values for σ are considered. This figure formed part of the original CESifo working paper. Figure A.2(f) depicts the strong version of the tragedy of annuitization. Generations $\tau = 0$ and $\tau = 1$ are better off as a result of annuitization but all subsequent generations are worse off.

Figure A.3 shows the transitional dynamics for $k_{t+\tau}$, $\mathbb{E}\Lambda^y_{t+\tau}$, $C^y_{t+\tau}$, and $C^o_{t+\tau}$ for the change from TY or TO to PA. Again different values for σ are considered. This figure also formed part of the original CESifo working paper. Figure A.3(b) depicts the weak version of the tragedy of annuitization. Generation $\tau = 0$ is better off as a result of annuitization but all subsequent generations are worse off.

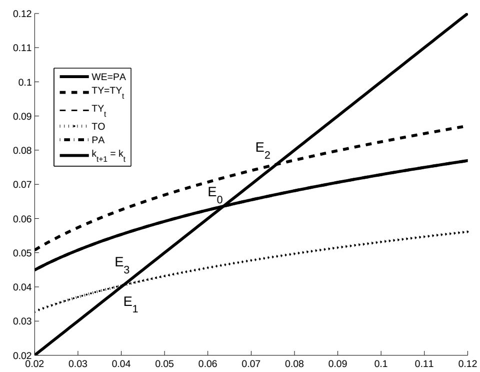
Figure A.3(f) depicts the welfare effects for the transition from TO to TY. This case is mentioned in Section 5.4 in the paper. Interestingly, the tragedy of annuitization now occurs at shock time. Indeed, all generations except generation $\tau = 0$ are better off as a result of annuitization.

Figure A.1: Transitional dynamics

(a) Weak intertemporal substitution effect ($\sigma = \frac{1}{2}$)



(b) Benchmark intertemporal substitution effect ($\sigma = 1$)



(c) Strong intertemporal substitution effect ($\sigma = \frac{3}{2}$)

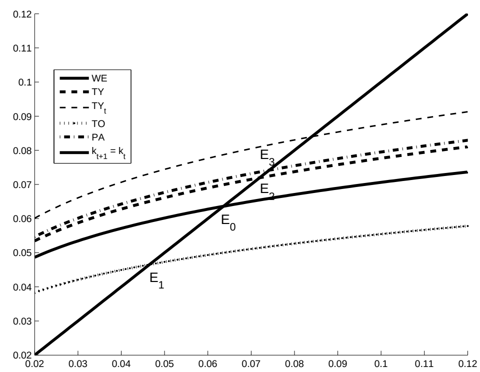
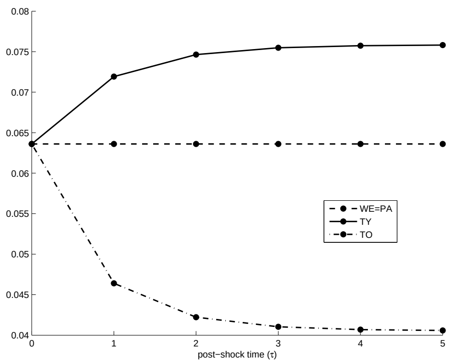


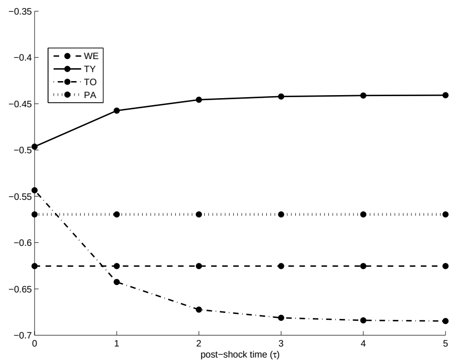
Figure A.2: Transitional dynamics in the exogenous growth model

Panel A: Benchmark: $\sigma = 1$

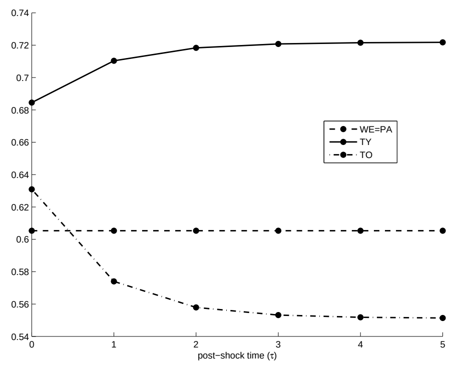
(a) capital intensity ($k_{t+\tau}$)



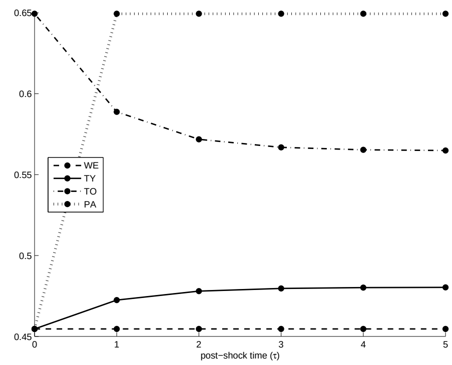
(b) expected lifetime utility ($\mathbb{E}\Lambda^{y}_{t+\tau}$)



(c) youth consumption ($C^{y}_{t+\tau}$)



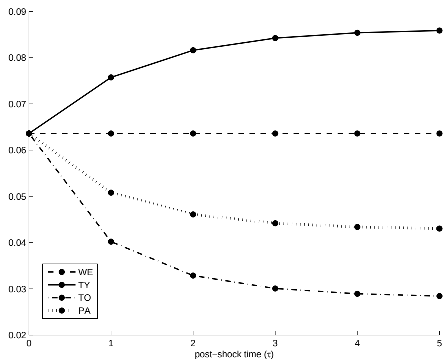
(d) old-age consumption ($C^{o}_{t+\tau}$)



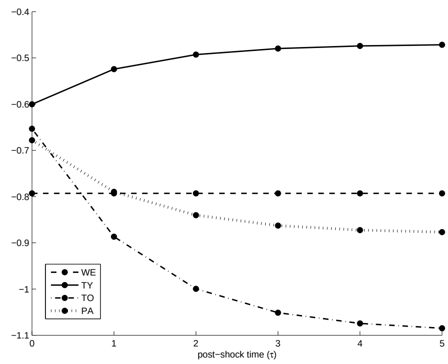
(Figure A.2, continued)

Panel B: Weak intertemporal substitution effect: $\sigma = \frac{1}{2}$

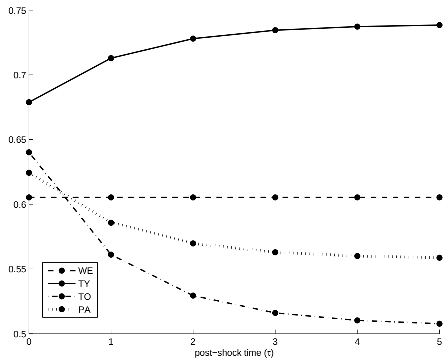
(e) capital intensity ($k_{t+\tau}$)



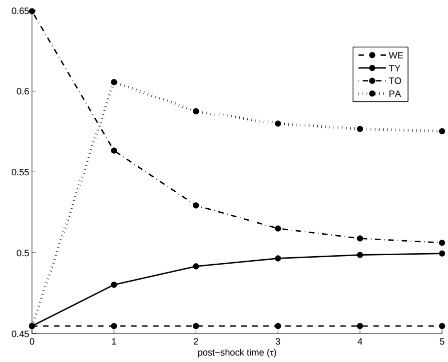
(f) expected lifetime utility ($\mathbb{E}\Lambda^y_{t+\tau}$)



(g) youth consumption ($C^y_{t+\tau}$)



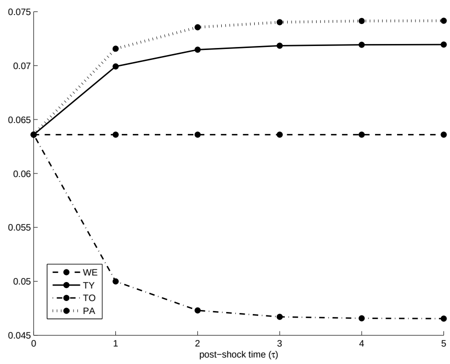
(h) old-age consumption ($C^o_{t+\tau}$)



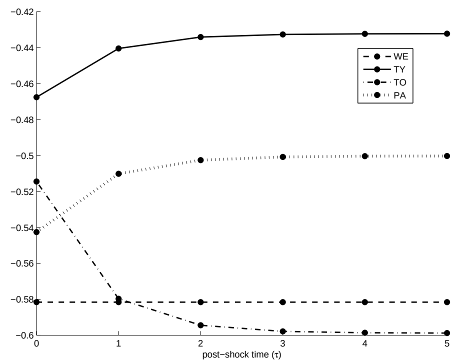
(Figure A.2, continued)

Panel C: Strong intertemporal substitution effect: $\sigma = \frac{3}{2}$

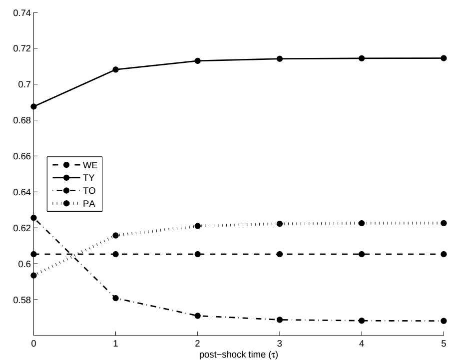
(i) capital intensity ($k_{t+\tau}$)



(j) expected lifetime utility ($\mathbb{E}\Lambda^{y}_{t+\tau}$)



(k) youth consumption ($C^{y}_{t+\tau}$)



(l) old-age consumption ($C^{o}_{t+\tau}$)

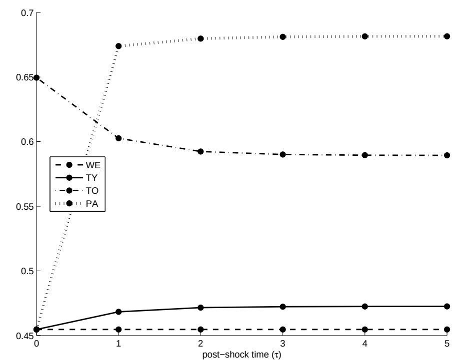
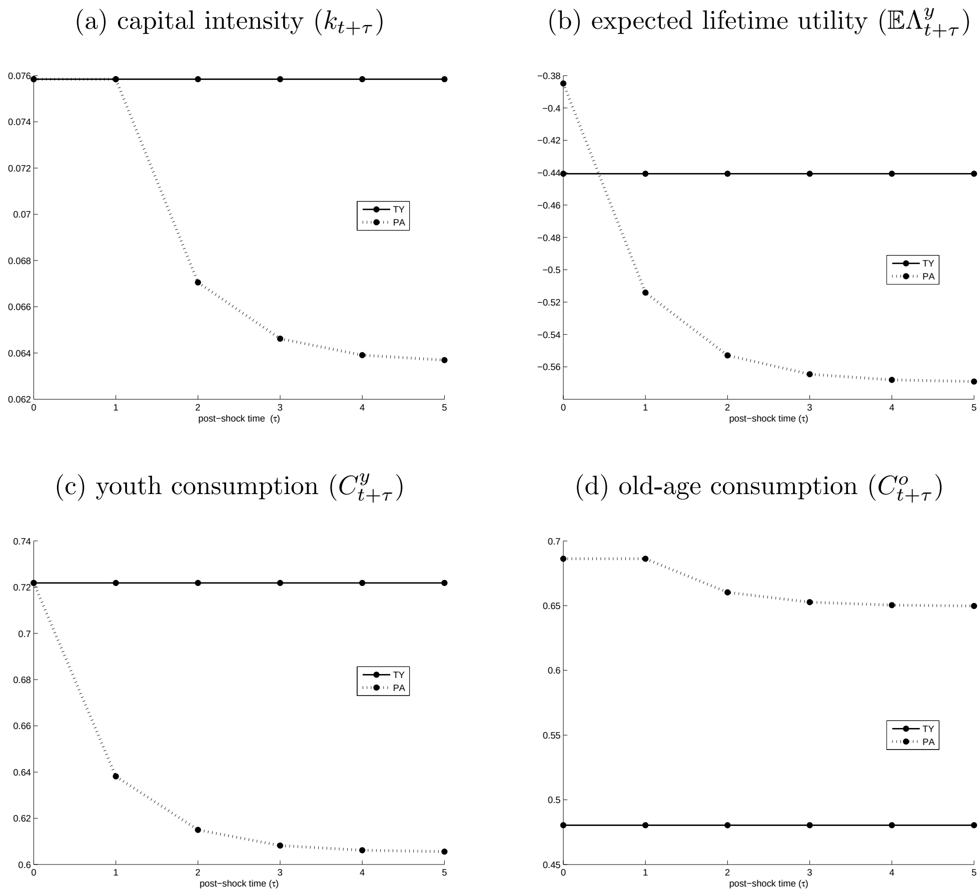


Figure A.3: Transition from transfers to annuities in the exogenous growth model

Panel A: from TY to PA ($\sigma = 1$)



(a) capital intensity ($k_{t+\tau}$)

(b) expected lifetime utility ($\mathbb{E}\Lambda^y_{t+\tau}$)

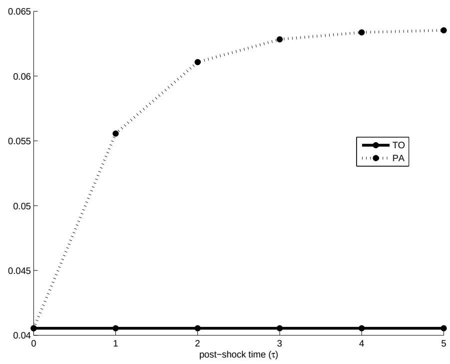
(c) youth consumption ($C^y_{t+\tau}$)

(d) old-age consumption ($C^o_{t+\tau}$)

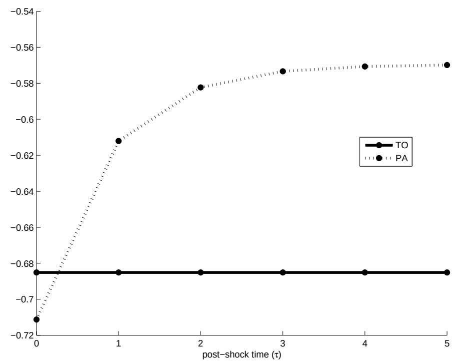
(Figure A.3, continued)

Panel B: from TO to PA ($\sigma = 1$)

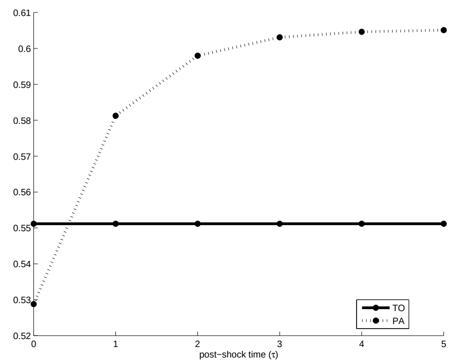
(e) capital intensity ($k_{t+\tau}$)



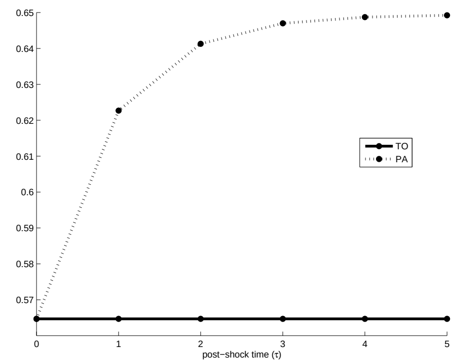
(f) expected lifetime utility ($\mathbb{E}\Lambda^y_{t+\tau}$)



(g) youth consumption ($C^y_{t+\tau}$)



(h) old-age consumption ($C^o_{t+\tau}$)



A.7 The role of scale economies

As is mentioned in Section 5.3 of the paper our conclusions are robust to non-zero values of η. Provided the externality is bounded away from the endogenous growth case ($0 \leq \eta \leq 1 - \alpha$), the model features exogenous growth and the analytical methods discussed in the paper are valid.

In Tables A.1 and A.2 we show the results for, respectively, $\eta = 0.3$ and $\eta = 0.6$. Note that the latter case is very close to the endogenous growth case (as $1 - \alpha = 0.7$). As is assereted in the paper, the presence of a capital externaility exacerbates the (negative or positive) effects of the transitions between scenarios. Qualitatively, however, the conclusions based on the case with $\eta = 0$ are unchallenged.

Table A.1: Steady-state equilibrium values*

	Panel A: $\eta = 0.3$, $\sigma = 1$				Panel B: $\eta = 0.3$, $\sigma = \frac{1}{2}$				Panel C: $\eta = 0.3$, $\sigma = \frac{3}{2}$			
	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	(j)	(k)	(l)
	WE	TO	TY	PA	WE	TO	TY	PA	WE	TO	TY	PA
$\hat{C}^y$	0.6053	0.3932	0.8237	0.6053	0.6053	0.2735	0.9291	0.4141	0.6053	0.4494	0.7838	0.6987
$\hat{C}^o$	0.4546	0.4028	0.5482	0.6495	0.4546	0.2726	0.6286	0.4263	0.4546	0.4662	0.5184	0.7647
$\hat{g}$	0.0916				0.0916				0.0916			
$\hat{Z}^o$		0.1209				0.0818				0.1399		
$\hat{Z}^y$			0.1105				0.1267				0.1044	
$\hat{y}$	1.0000	0.6232	1.2031	1.0000	1.0000	0.4230	1.3768	0.6591	1.0000	0.7203	1.1384	1.1752
$\hat{k}$	0.0636	0.0289	0.0866	0.0636	0.0636	0.0152	0.1084	0.0317	0.0636	0.0368	0.0789	0.0832
$\hat{w}$	0.7000	0.4363	0.8421	0.7000	0.7000	0.2961	0.9638	0.4614	0.7000	0.5042	0.7969	0.8226
$\hat{r}$	3.8010	5.5491	3.2541	3.8010	3.8010	7.4546	2.8954	5.3121	3.8010	4.9544	3.4106	3.3198
$\hat{r}_a$	4.00	4.81	3.69	4.00	4.00	5.48	3.46	4.71	4.00	4.56	3.78	3.73
$\hat{r}_a^A$				4.93				5.65				4.65
$\mathbb{E}\Lambda_t^y$	-0.6253^a	-0.5580	-0.4907	-0.5695	-0.7930^a	-0.6844	-0.5854	-0.6941	-0.5816^a	-0.5243	-0.4644	-0.5381
$\mathbb{E}\Lambda_t^y$			-0.2879^a	-0.2321			-0.1458^a	-0.0746			-0.3356^a	-0.2901
$\mathbb{E}\Lambda_t^y$		-1.0757^a		-1.0873		-2.9693^a		-2.9878		-0.8181^a		-0.8311
$\widehat{\mathbb{E}\Lambda}^y$	-0.6253	-1.0757	-0.2879	-0.5695	-0.7930	-2.9693	-0.1458	-1.5730	-0.5816	-0.8181	-0.3356	-0.3821

*Hats denote steady-state values. To facilitate interpretation, $\hat{r}_a$ and $\hat{r}_a^A$ are reported as annual percentage rates of return. In the rows for $\mathbb{E}\Lambda_t^y$ the superscript a denotes the initial steady-state equilibrium that is perturbed.

Table A.2: Steady-state equilibrium values*

	Panel A: $\eta = 0.6$, $\sigma = 1$				Panel B: $\eta = 0.6$, $\sigma = \frac{1}{2}$				Panel C: $\eta = 0.6$, $\sigma = \frac{3}{2}$			
	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	(j)	(k)	(l)
	WE	TO	TY	PA	WE	TO	TY	PA	WE	TO	TY	PA
$\hat{C}^y$	0.6053	0.0370	2.0760	0.6053	0.6053	0.0037	4.5966	0.0515	0.6053	0.0871	1.4983	1.5659
$\hat{C}^o$	0.4546	0.0379	1.3816	0.6495	0.4546	0.0037	3.1098	0.0530	0.4546	0.0904	0.9909	1.7139
$\hat{g}$	0.0916				0.0916				0.0916			
$\hat{Z}^o$		0.0114				0.0011				0.0271		
$\hat{Z}^y$			0.2784				0.6266				0.1997	
$\hat{y}$	1.0000	0.0586	3.0320	1.0000	1.0000	0.0057	6.8119	0.0820	1.0000	0.1396	2.1761	2.6337
$\hat{k}$	0.0636	0.0027	0.2181	0.0636	0.0636	0.0002	0.5362	0.0039	0.0636	0.0071	0.1509	0.1865
$\hat{w}$	0.7000	0.0410	2.1224	0.7000	0.7000	0.0040	4.7683	0.0574	0.7000	0.0977	1.5233	1.8436
$\hat{r}$	3.8010	5.5491	3.2541	3.8010	3.8010	7.4546	2.8954	5.3121	3.8010	4.9544	3.4106	3.3198
$\hat{r}_a$	4.00	4.81	3.69	4.00	4.00	5.48	3.46	4.71	4.00	4.56	3.78	3.73
$\hat{r}_a^A$				4.93				5.65				4.65
$\mathbb{E}\Lambda_t^y$	-0.6253^a	-0.5726	-0.4850	-0.5695	-0.7930^a	-0.7086	-0.5748	-0.7061	-0.5816^a	-0.5362	-0.4604	-0.5324
$\mathbb{E}\Lambda_t^y$			0.7810^a	0.8370			0.8622^a	0.8748			0.4312^a	0.4950
$\mathbb{E}\Lambda_t^y$		-3.8098^a		-3.8051		-300.5791^a		-256.9482		-1.9547^a		-1.9525
$\widehat{\mathbb{E}\Lambda}^y$	-0.6253	-3.8098	0.7810	-0.5695	-0.7930	-300.5791	0.8622	-20.5102	-0.5816	-1.9547	0.4312	0.5853

*Hats denote steady-state values. To facilitate interpretation, $\hat{r}_a$ and $\hat{r}_a^A$ are reported as annual percentage rates of return. In the rows for $\mathbb{E}\Lambda_t^y$ the superscript a denotes the initial steady-state equilibrium that is perturbed.

A.8 The endogenous growth model

In this section we briefly consider the knife-edge case featuring $\eta = 1 - \alpha$. The model then exhibits growth which is driven endogenously by the rate of capital accumulation. We can solve (T1.7) for the equilibrium growth rate:

$$(1+n)(1+\gamma) = [1 - \Phi(\bar{r}, z_3)] \left[(1-\alpha)\Omega_0 + \frac{Z_t^y}{k_t} \right] - \frac{\Phi(\bar{r}, z_3)}{1+\bar{r}} \frac{Z_{t+1}^o}{k_t}, \tag{A.83}$$

where $\gamma \equiv k_{t+1}/k_t - 1$ is the (time-invariant) equilibrium growth rate and we have used the fact that the interest rate is constant in this scenario such that $r_t = \bar{r} \equiv \alpha\Omega_0 - \delta$ for all t. Using the expressions in (A.83) we can derive the equilibrium growth rates under the three revenue recycling schemes and after the introduction of a private annuity market.

(WE) If the government uses the proceeds from the accidental bequests for wasteful government expenditures the growth rate becomes:

$$1 + \gamma^{WE} = \frac{1 - \Phi(\bar{r}, 0)}{1+n} (1-\alpha)\Omega_0. \tag{48a}$$

(TY) If instead the proceeds are redistributed to the young we find:

$$1 + \gamma^{TY} = \frac{1 - \Phi(\bar{r}, 0)}{1+n} [(1-\alpha)\Omega_0 + \pi(1+\bar{r})]. \tag{48b}$$

(TO) If the accidental bequests go the elderly then the growth rate is given by

$$1 + \gamma^{TO} = \frac{1 + \gamma^{WE}}{1 + \Phi(\bar{r}, 0)\frac{\pi}{1-\pi}}. \tag{48c}$$

(PA) Finally, if a private annuity market is introduced we have:

$$1 + \gamma^{PA} = \frac{1 - \Phi(\bar{r}, 1)}{1+n} (1-\alpha)\Omega_0. \tag{48d}$$

Straightforward inspection of the growth rates reveals that $\gamma^{TY} > \gamma^{WE} > \gamma^{TO}$ for all admissible values of σ. Hence, in terms of growth, it is better to give the accidental bequests to the young than to use them for wasteful expenditures, yet it is better to let the accidental bequests go to waste than to give them to the elderly.

Comparison with the private annuities scenario is more subtle. The introduction of private annuities increases the rate against which individuals save. The savings response of consumers, and thereby the growth rate in the perfect annuities scenario relative to the various recycling schemes, depends on the value of the intertemporal elasticity of substitution σ. For the benchmark case with $\sigma = 1$ savings are independent of the interest rate and $\gamma^{TY} > \gamma^{PA} = \gamma^{WE} > \gamma^{TO}$. If $0 < \sigma < 1$ the higher interest rate will lead to less savings than in the benchmark scenario so that we get $\gamma^{TY} > \gamma^{WE} > \gamma^{PA} > \gamma^{TO}$. Finally, if $\sigma > 1$ the higher interest rate will lead to more savings which results in $\gamma^{PA} > \gamma^{WE} > \gamma^{TO}$ and, depending on the exact magnitude of σ, $\gamma^{PA} \gtreqless \gamma^{TY}$.

In order to compare consumer welfare across the various scenarios we must recognize the fact that steady-state expected lifetime utility grows at a scenario-dependent rate in an endogenous growth model. To see this, note that if $\eta = 1 - \alpha$ we can write the consumption demand equations (5) and (6) under scenario i as:

$$C^{y,i}_{t+\tau} \equiv \Phi\left(r^i\right)\theta^i w^i_{t+\tau}, \qquad C^{o,i}_{t+\tau+1} \equiv \left(1+r^i\right)\left[1-\Phi\left(r^i\right)\right]\theta^i w^i_{t+\tau}, \tag{A.49}$$

where $r^i = \bar{r}$ for $i \in \{\text{WE,TY,TO}\}$ and $r^i = \bar{r}^A$ for $i = \text{PA}$. The value of the parameter θ^i depends on the specific scenario $i \in \{\text{WE,TY,TO,PA}\}$.[1] Wages grow over time according to the equilibrium growth rate associated with scenario i:

$$w^i_{t+\tau} = \left(1+\gamma^i\right)^\tau w_t. \tag{A.50}$$

Consider an economy that is initially in the WE scenario and features a wage rate at time t equal to w_t. Expected lifetime utility of future newborns under scenario i can then be written as:

$$\widehat{\mathbb{E}\Lambda}^{y,i}_{t+\tau} \equiv \begin{cases} \dfrac{\Phi\left(r^i\right)^{-1/\sigma}\left[\theta^i\left(1+\gamma^i\right)^\tau w_t\right]^{1-1/\sigma} - \dfrac{2+\rho-\pi}{1+\rho}}{1-1/\sigma} & \text{for } \sigma > 0,\ \sigma \neq 1 \\[2ex] \Xi_0 + \dfrac{2+\rho-\pi}{1+\rho}\left[\theta^i\left(1+\gamma^i\right)^\tau w_t\right] + \dfrac{1-\pi}{1+\rho}\ln\left(1+r^i\right) & \text{for } \sigma = 1 \end{cases} \tag{A.51}$$

We call this welfare metric normalized utility. Clearly, $\widehat{\mathbb{E}\Lambda}^{y,i}_{t+\tau}$ depends both on post-shock time τ and on the scenario-dependent (endogenous) value of γ^i. From equation (A.51) we observe that with the introduction of a transfer regime or an annuity market there is both a

[1] For the three public policy regimes we get $\theta^{WE} = 1$, $\theta^{TY} = \left[1 + \frac{\pi(1+\bar{r})}{(1-\alpha)\Omega_0}\right]$, and $\theta^{TO} = \left[1 + \pi\frac{1+n}{1-\pi}\frac{1+\gamma^{TO}}{(1-\alpha)\Omega_0}\right]$. For private annuities $r^i = \bar{r}^A$ and $\theta^{PA} = 1$.

level effect (represented by a change in the θ^i parameter) and a *growth* effect (induced by a change in γ^i). However, over time the growth effect will always dominate the level effect.

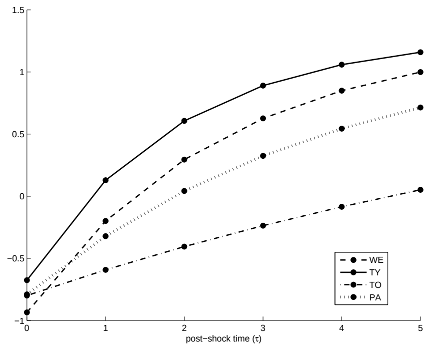
In order to quantify the growth and welfare effects we adopt the following approach. For n, π, α, δ, and r we use the same values as for the exogenous growth model (see the text below Proposition 2). We calibrate an annual growth rate of one percent in the WE scenario ($\gamma^{WE} = 0.49$) and obtain $\Omega_0 = 15.72$ and $\rho = 1.78$ (or 2.58% annually). The equilibrium growth rate under the various policy schemes is reported in Table 4 for different values of σ and the corresponding welfare paths are depicted in Figure A.4.

In line with the exogenous growth model we find that if the economy exhibits endogenous growth and the intertemporal substitution elasticity is in the realistic range ($0 < \sigma \leq 1$) then it is better to transfer the proceeds of accidental bequests to the young than to open up a private annuity market – see Table 4 in the paper and Figure A.4. In addition we find that for low values of σ it may even be better to waste the accidental bequests than to have a system of private annuities. Hence, both the weak and the strong version of the tragedy of annuitization show up in terms of economic growth rates.

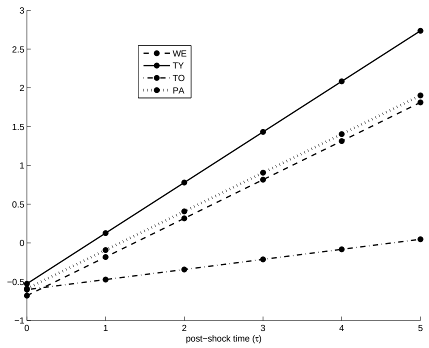
Finally, we find that only if σ is unrealistically high (e.g., $\sigma = \frac{3}{2}$) private annuities slightly outperform transfers to the young in terms of growth – see Table 4(c). However, in terms of welfare, PA only outpaces the TY scenario after three periods (i.e. 120 years) and even then only marginally so – see Figure A.4(c).

Figure A.4: Welfare paths in the endogenous growth model

(a) Weak intertemporal substitution effect: $\sigma = \frac{1}{2}$



(b) Benchmark: $\sigma = 1$



(c) Strong intertemporal substitution effect: $\sigma = \frac{3}{2}$