Macroeconomic Effects of Demographic Shocks: Mathematical Appendix

Ben J. Heijdra*

May 2004

NOT INTENDED FOR PUBLICATION IN THE JOURNAL
(Subsequent to publication downloadable from homepage)

In this appendix we present the derivations for the main results used in the paper. References to equations without the prefix "A" are to equations found in the paper. In Tables A.2 and A.3 there is an overview of, respectively, variables and parameters.

A.1 Households

A.1.1 Individual households

The optimization problem faced by the representative consumer can be solved in two stages. The method extends the procedure of Marini and van der Ploeg (1988) by allowing the intertemporal substitution elasticity to differ from unity and by incorporating endogenous labour supply. In step 1 the path of full consumption, $\bar{x}(v,\tau)$, is solved. In step 2 full consumption is allocated between its components $\bar{c}(v,\tau)$ and $1-\bar{l}(v,\tau)$.

Stage 1. Full consumption is defined as the sum of spending on consumption goods and on leisure:

$$\bar{x}(v,\tau) \equiv \bar{c}(v,\tau) + w(\tau)\left[1-\bar{l}(v,\tau)\right], \tag{A.1}$$

where $w(\tau)$ is the real wage rate and final output is the numeraire good (so that $P_Y(\tau)=1$). We define the ideal cost-of-living index as $P_U(\tau)$ as:

$$P_U(\tau)\bar{u}(v,\tau) = \bar{x}(v,\tau), \tag{A.2}$$

where $\bar{u}(v,\tau) \equiv \Omega[\bar{c}(v,\tau), 1-\bar{l}(v,\tau)]$ is the CES sub-utility function:

$$\bar{u}(v,\tau) \equiv \left[\varepsilon_C \bar{c}(v,\tau)^{(\sigma_C-1)/\sigma_C} + (1-\varepsilon_C)\left[1-\bar{l}(v,\tau)\right]^{(\sigma_C-1)/\sigma_C}\right]^{\sigma_C/(\sigma_C-1)}, \tag{A.3}$$

*Department of Economics, University of Groningen, P.O. Box 800, 9700 AV Groningen, The Netherlands. Phone: +31-50-363-7303, Fax: +31-50-363-7337, E-mail: b.j.heijdra@eco.rug.nl. I thank Peter Broer for his assistance with section A.5. The latest version of this appendix can be downloaded from http://www.eco.rug.nl/medewerk/heijdra/demoapp.pdf with $\sigma_C \geq 0$ and $0 < \varepsilon_C < 1$. In the first stage the household chooses the optimal time sequence of $\bar{u}(v,\tau)$ and $\bar{a}(v,\tau)$ (for $\tau \in [t,\infty)$) in order to maximize lifetime utility,

$$\Lambda(v,t) \equiv \int_t^\infty \left[\frac{\bar{u}(v,\tau)^{1-1/\sigma_X} - 1}{1-1/\sigma_X}\right] \exp\left[(\alpha+\beta)(t-\tau)\right] d\tau, \tag{A.4}$$

subject to the budget identity:

$$\dot{\bar{a}}(v,\tau) = \left[r(\tau)+\beta\right]\bar{a}(v,\tau) + w(\tau) - \bar{z}(\tau) - P_U(\tau)\bar{u}(v,\tau), \tag{A.5}$$

and taking as given the initial level of financial assets, $\bar{a}(v,t)$. Financial assets are consist of government bonds, $\bar{b}(v,\tau)$, plus the capital stock $\bar{k}(v,\tau)$, i.e. $\bar{a}(v,\tau) = \bar{b}(v,\tau) + \bar{k}(v,\tau)$.[1] As usual, a variable with a dot represents the time derivative of that variable, i.e. $\dot{\bar{a}}(v,\tau) \equiv d\bar{a}(v,\tau)/d\tau$. The (interesting) first-order conditions are (for $\tau \in [t,\infty)$):

$$\bar{u}(v,\tau)^{-1/\sigma_X} = \lambda_A(v,\tau)P_U(\tau), \tag{A.6}$$

$$\dot{\lambda}_A(v,\tau) = \left[\alpha - r(\tau)\right]\lambda_A(v,\tau), \tag{A.7}$$

where $\lambda_A(v,\tau)$ is the co-state variable of the flow budget identity (A.5). The life-time budget *restriction* is obtained by integrating the flow budget identity and imposing the NPG condition for the household, i.e. $\lim_{\tau\to\infty} \bar{a}(v,\tau) \times \exp[-\int_t^\tau [r(s)+\beta]\, ds] = 0$:

$$\begin{aligned}\bar{a}(v,t) + \bar{h}(t) &= \int_t^\infty P_U(\tau)\bar{u}(v,\tau)\exp\left[-\int_t^\tau [r(s)+\beta]\, ds\right] d\tau \\ &= \int_t^\infty \lambda_A(v,\tau)^{-\sigma_X} P_U(\tau)^{1-\sigma_X} \exp\left[-\int_t^\tau [r(s)+\beta]\, ds\right] d\tau,\end{aligned} \tag{A.8}$$

where $\bar{h}(t)$ is defined as:

$$\bar{h}(t) \equiv \int_t^\infty \left[w(\tau) - \bar{z}(\tau)\right] \exp\left[-\int_t^\tau [r(s)+\beta]\, ds\right] d\tau. \tag{A.9}$$

The path of $\lambda_A(v,\tau)$ is described by (A.7) which can be solved to yield the following:

$$\lambda_A(v,\tau) = \exp\left[-\int_t^\tau [r(s)-\alpha]\, ds\right] \lambda_A(v,t), \qquad \tau \geq t. \tag{A.10}$$

Using (A.10) in (A.8) yields the following:

$$\begin{aligned}\bar{a}(v,t) + \bar{h}(t) = \int_t^\infty &\left[\exp\left[\int_t^\tau [\alpha - r(s)]\, ds\right] \lambda_A(v,t)\right]^{-\sigma_X} \\ &\times P_U(\tau)^{1-\sigma_X} \exp\left[-\int_t^\tau [r(s)+\beta]\, ds\right] d\tau.\end{aligned} \tag{A.11}$$

We know that:

$$P_U(t)\,\bar{u}(v,t) = \bar{x}(v,t) = \lambda_A(v,t)^{-\sigma_X} P_U(t)^{1-\sigma_X}, \tag{A.12}$$

so that we can rewrite (A.11) as follows:

$$\bar{x}(v,t) = \left[\Delta(t)\right]^{-1} \left[\bar{a}(v,t) + \bar{h}(t)\right], \tag{A.13}$$

[1] Below we show that, in the absence of adjustment costs for investment, Tobin's q is unity so that the value of claims on the capital stock is just equal to the capital stock itself.

where $\Delta(t)$ is defined as:

$$\Delta(t) \equiv \int_t^\infty \left(\frac{P_U(\tau)}{P_U(t)}\right)^{1-\sigma_X} \exp\left[-\int_t^\tau [(1-\sigma_X) r(s) + \alpha\sigma_X + \beta] \, ds\right] d\tau. \tag{A.14}$$

The differential equation for $\Delta(t)$ is found by differentiating (A.14) with respect to t:

$$\begin{aligned}
\dot{\Delta}(t) &= -1 - (1-\sigma_X) \frac{\dot{P}_U(t)}{P_U(t)} \Delta(t) + [(1-\sigma_X) r(t) + \alpha\sigma_X + \beta] \Delta(t) \\
&= -1 + \left[(1-\sigma_X)\left(r(t) - \frac{\dot{P}_U(t)}{P_U(t)}\right) + \alpha\sigma_X + \beta\right] \Delta(t).
\end{aligned} \tag{A.15}$$

Equation (A.13)-(A.15) generalise the various expressions found in Blanchard (1985, pp. 233-4) to the case of endogenous labour supply.

Stage 2. The household now chooses goods consumption ($\bar{c}(v,t)$) and leisure ($1-\bar{l}(v,t)$) in order to maximize

$$\bar{u}(v,t) = \Omega[\bar{c}(v,t), 1-\bar{l}(v,t)], \tag{A.16}$$

subject to the constraint (A.1). Using the sub-utility function (A.3) we find that the (interesting) first-order condition is:

$$\frac{\bar{c}(v,t)}{1-\bar{l}(v,t)} = \left(\frac{\varepsilon_C}{1-\varepsilon_C}\right)^{\sigma_C} w(t)^{\sigma_C}. \tag{A.17}$$

By substituting (A.17) into the constraint (A.1) we obtain the expressions for $\bar{l}(v,t)$ and $\bar{c}(v,t)$ in terms of full consumption $\bar{x}(v,t)$.

$$w(t)\left[1-\bar{l}(v,t)\right] = [1-\omega_{CX}(t)] \, \bar{x}(v,t), \tag{A.18}$$

$$\bar{c}(v,t) = \omega_{CX}(t) \, \bar{x}(v,t), \tag{A.19}$$

where $\omega_{CX}(t)$ is defined as:

$$\omega_{CX}(t) \equiv \frac{\varepsilon_C^{\sigma_C}}{\varepsilon_C^{\sigma_C} + (1-\varepsilon_C)^{\sigma_C} w(t)^{1-\sigma_C}}. \tag{A.20}$$

The expression for the true price index, $P_U(t)$, is obtained by substituting (A.18)-(A.19) into the direct sub-utility function (A.3) and noting (A.2):

$$P_U(t) \equiv \begin{cases} \left[(\varepsilon_C)^{\sigma_C} + (1-\varepsilon_C)^{\sigma_C} w(t)^{1-\sigma_C}\right]^{1/(1-\sigma_C)} & \text{for } \sigma_C \neq 1 \\ \left(\frac{1}{\varepsilon_C}\right)^{\varepsilon_C} \left(\frac{w(t)}{1-\varepsilon_C}\right)^{1-\varepsilon_C} & \text{for } \sigma_C = 1 \end{cases}. \tag{A.21}$$

Finally, by using (A.2) and (A.6)-(A.7) we obtain the expression for the Euler equation for the household:

$$\frac{\dot{\bar{x}}(v,t)}{\bar{x}(v,t)} = \sigma_X [r(t) - \alpha] + (1-\sigma_X) \frac{\dot{P}_U(t)}{P_U(t)}. \tag{A.22}$$

A.1.2 Demography

In order to allow for non-zero population growth, we employ the analytical framework developed by Buiter (1988). This framework makes a distinction between the probability of death β (≥ 0)

and the birth rate η (≥ 0) and thus allows for net population growth or decline. We denote the population size at time t by $N(t)$. In the absence of international migration, the growth rate of the population, n_N, is equal to the difference between the birth and death rates:[2]

$$\frac{\dot{N}(t)}{N(t)} = \eta - \beta \equiv n_N. \tag{A.23}$$

By solving (A.23) subject to the initial condition $N(0) = 1$, we find the path for the aggregate population:

$$N(t) = e^{n_N t}. \tag{A.24}$$

We assume that the size of a newborn generation is proportional to the current population:

$$N(v,v) = \eta N(v), \tag{A.25}$$

where $N(v,v)$ is the size of the cohort born at time v. Since the death rate is constant and cohorts are assumed to be large, the size of each generation falls exponentially according to:

$$N(v,t) = e^{\beta(v-t)} N(v,v), \qquad t \geq v. \tag{A.26}$$

By substituting (A.24) and (A.25) into (A.26) we finally obtain:[3]

$$N(v,t) = \eta e^{\eta v} e^{-\beta t}. \tag{A.27}$$

A.1.3 Aggregate household sector

Aggregate variables are calculated as the integral of the generation-specific values weighted by the corresponding generation sizes (given in (A.27) above).

A.1.3.1 Full consumption

Aggregate full consumption, $X(t)$, is defined as:

$$X(t) \equiv \int_{-\infty}^{t} N(v,t)\bar{x}(v,t)dv, \tag{A.28}$$

where $N(v,t)$ and $\bar{x}(v,t)$ are given in, respectively, (A.27) and (A.13). It follows that aggregate full consumption can be written as:

$$X(t) = \Delta(t)^{-1}\left[A(t) + H(t)\right], \tag{A.29}$$

where $A(t)$ is aggregate financial wealth and $H(t) \equiv N(t)\,\bar{h}(t)$ is *aggregate* human wealth.

The aggregate Euler equation for full consumption is obtained by differentiating (A.28) with respect to time and noting (A.27):

$$\dot{X}(t) = \int_{-\infty}^{t} N(v,t)\dot{\bar{x}}(v,t)\,dv + \eta N(t)\bar{x}(t,t) - \beta X(t). \tag{A.30}$$

[2]Below we shall allow β and η to be time-dependent. Age-dependency cannot be incorporated as it would destroy the simple aggregation properties of the model.

[3]An attractive feature of the Buiter formulation is that it nests two influential OLG models as special cases. Indeed, by setting $\eta = \beta$ the Blanchard (1985) model is obtained and by setting $\beta = 0$ the Weil (1989) model is obtained.

By substituting (A.22) into (A.30) and dividing by $X(t)$ we obtain:

$$\frac{\dot{X}(t)}{X(t)} = \left[\sigma_U \left[r(t) - \alpha\right] + (1-\sigma_U)\frac{\dot{P}_U(t)}{P_U(t)}\right] + \left[\frac{\eta N(t)\bar{x}(t,t) - \beta X(t)}{X(t)}\right]. \tag{A.31}$$

The first term in square brackets on the right-hand side is the growth rate of individual full consumption ($\dot{\bar{x}}(v,t)/\bar{x}(v,t)$) whilst the second term is the *generational turnover* term which itself depends on the demographic parameters. Growth in aggregate full consumption is boosted because of the arrival of new agents (who start to consume out of human wealth) but it is slowed down by the death of a cross-section of the population.

Newborn generations are born without financial assets ($\bar{a}(t,t) = 0$) and human wealth is age-independent ($H(t) = N(t)\bar{h}(t)$) so the numerator of the generational turnover term simplifies to:

$$\begin{aligned} \eta N(t)\bar{x}(t,t) - \beta X(t) &= \eta N(t)\left[\Delta(t)\right]^{-1}\bar{h}(t) - \beta\Delta(t)^{-1}\left[A(t) + H(t)\right] \\ &= \frac{(\eta-\beta)H(t) - \beta A(t)}{\Delta(t)} \\ &= \frac{n_N\left[A(t) + H(t)\right] - \eta A(t)}{\Delta(t)}. \end{aligned} \tag{A.32}$$

By substituting (A.32) into (A.31) and noting (A.29) we find:

$$\frac{\dot{X}(t)}{X(t)} = \left[\sigma_U\left[r(t) - \alpha\right] + n_N + (1-\sigma_U)\frac{\dot{P}_U(t)}{P_U(t)}\right] - \frac{\eta A(t)}{\Delta(t)X(t)}, \tag{A.33}$$

where we have used the fact that $n_N \equiv \eta - \beta$. According to (A.33), aggregate full consumption growth differs from individual full consumption growth (given in (A.22) above) for two reasons. First, aggregate growth exceeds individual growth if there is positive net population growth (i.e. if $n_N > 0$). Second, aggregate growth falls short of individual growth if the group of newborns consume less than existing households because they are less wealthy (i.e. if $A(t) > 0$).

A.1.3.2 Financial wealth

Aggregate financial wealth is defined as $A(t) \equiv \int_{-\infty}^{t} N(v,t)\bar{a}(v,t)dv$. By differentiating this expression with respect to t and noting that $\bar{a}(t,t) = 0$ (newborns are born without financial wealth) we find an expression for national saving:

$$\dot{A}(t) = -\beta A(t) + \int_{-\infty}^{t} N(v,t)\dot{\bar{a}}(v,t)dv. \tag{A.34}$$

By using (A.1)-(A.2) and (A.5) in (A.34) and simplifying we find:

$$\dot{A}(t) = r(t)A(t) + w(t)N(t) - N(t)\bar{z}(t) - X(t), \tag{A.35}$$

$$X(t) = C(t) + w(t)\left[N(t) - L(t)\right], \tag{A.36}$$

where $C(t)$ and $L(t)$ are, respectively, aggregate consumption and aggregate labour supply. It follows from (A.35) that aggregate saving, unlike individual saving, does not feature the death rate, β. The annuity payments are simply transfers, via the life insurance companies, from households which pass away to households who continue to enjoy life.

A.1.3.3 Human wealth

Aggregate human wealth is defined as $H(t) \equiv \int_{-\infty}^{t} N(v,t)\bar{h}(t)dv = N(t)\,\bar{h}(t)$. It follows that:

$$\dot{H}(t) = N(t)\,\dot{\bar{h}}(t) + \bar{h}(t)\,\dot{N}(t)\,. \tag{A.37}$$

By differentiating (A.9) with respect to time we find:

$$\dot{\bar{h}}(t) = [r(t)+\beta]\,\bar{h}(t) - [w(t) - \bar{z}(t)]\,. \tag{A.38}$$

By substituting (A.38) into (A.37) and noting that $n_N \equiv \dot{N}(t)/N(t) = \eta - \beta$ we find:

$$\begin{aligned}\dot{H}(t) &= N(t)\Big[-[w(t)-\bar{z}(t)] + [r(t)+\beta]\,\bar{h}(t)\Big] + N(t)\,\bar{h}(t)\,\frac{\dot{N}(t)}{N(t)} \\ &= [r(t)+\eta]\,H(t) - N(t)\,[w(t)-\bar{z}(t)]\,. \end{aligned} \tag{A.39}$$

While *individual* household human wealth grows at the annuity rate of interest $(r(t)+\beta)$, *aggregate* human wealth accumulates at the rate $r(t)+\eta$.

A.1.3.4 Portfolio investment

Households purchase final goods in order to augment the capital stock. Since all households face the same trade-offs, we treat the investment problem in the aggregate (and thus minimize the notation). The objective function of the representative household-investor is:

$$V(t) = \int_{t}^{\infty} \left[w^K(\tau)K(\tau) - I(\tau)\right] e^{-R(t,\tau)} d\tau, \tag{A.40}$$

where $K(\tau)$ is the capital stock, $w^K(\tau)$ is the real rental rate on existing capital, $I(\tau)$ is gross investment, and $R(t,\tau) \equiv \int_t^{\tau} r(s)ds$ is a discounting factor. The capital accumulation identity is:

$$\dot{K}(\tau) = I(\tau) - \delta K(\tau), \tag{A.41}$$

where $\delta > 0$ is the constant depreciation rate of capital. The household-investor chooses paths for gross investment and the capital stock in order to maximize (A.40) subject to (A.41) and taking as given the path of the real rental rate $(w^K(\tau))$, and the initial capital stock $(K(t))$. The (interesting) first-order conditions of this control problem are:

$$\lambda_I(\tau) = 1, \tag{A.42}$$

$$\dot{\lambda}_I(\tau) = [r(\tau)+\delta]\,\lambda_I(\tau) - w^K(\tau)\,, \tag{A.43}$$

where $\lambda_I(t)$ is the co-state variable associated with the constraint (A.41). By combining (A.42)-(A.43) we obtain the usual expression for the rental rate on capital:

$$w^K(\tau) = r(\tau) + \delta. \tag{A.44}$$

Note that the cash flow can be written as:

$$\begin{aligned} w^K(\tau)\,K(\tau) - I(\tau) &= w^K(\tau)\,K(\tau) - \dot{K}(\tau) - \delta K(\tau) \\ &= r(\tau)\,K(\tau) - \dot{K}(\tau)\,. \end{aligned} \tag{A.45}$$

By substituting (A.45) into the objective function of the household-investor and simplifying we obtain:

$$V(t) = \int_t^\infty \left[r(\tau) K(\tau) - \dot{K}(\tau) \right] e^{-R(t,\tau)} d\tau = K(t). \tag{A.46}$$

The value of the capital stock to households is just the replacement value of this stock, i.e. Tobin's q equals unity and $V(t) = K(t)$.

A.2 Firms

The final goods sector uses capital and labour to produce a homogeneous good. This good is either consumed by households or the government, or invested by households to augment the aggregate capital stock. The final goods sector is characterized by perfect competition. There are many identical firms and for convenience we normalize their number to unity. Technology is described by a constant-elasticity-of-substitution (CES) production function:

$$Y(t) = F(K(t), L(t)) \equiv \Psi_Y \left[\varepsilon_K K(t)^{(\sigma_K - 1)/\sigma_K} + (1 - \varepsilon_K) L(t)^{(\sigma_K - 1)/\sigma_K} \right]^{\sigma_K/(\sigma_K - 1)}, \tag{A.47}$$

where $Y(t)$ is output, Ψ_Y is an index of general technology, $K(t)$ is capital, and $L(t)$ is labour. The elasticity of substitution in non-negative ($\sigma_K \geq 0$) and the efficiency parameter satisfies $0 < \varepsilon_K < 1$.

Real profit of the representative firm is defined in the usual way:

$$\Pi(t) \equiv [1 - t_Y(t)] F(K(t), L(t)) - w^K(t) K(t) - w(t) L(t), \tag{A.48}$$

where $t_Y(t)$ is a tax levied on the output of the final goods sector.[4] The representative producer in the final goods sector chooses $K(t)$ and $L(t)$ in order to maximize $\Pi(t)$, taking input prices as given. The first-order conditions for this static problem are:

$$[1 - t_Y(t)] \frac{\partial F(K(t), L(t))}{\partial K(t)} = r(t) + \delta, \tag{A.49}$$

$$[1 - t_Y(t)] \frac{\partial F(K(t), L(t))}{\partial L(t)} = w(t). \tag{A.50}$$

A.3 Loose ends

The government consumes $G(\tau)$ units of the final good and its periodic budget identity is given by:

$$\dot{B}(\tau) = r(\tau)B(\tau) + G(\tau) - N(\tau)\bar{z}(\tau) - t_Y(\tau)Y(\tau), \tag{A.51}$$

where $B(\tau)$ is real government debt at time τ. The government can finance its expenditure on goods plus debt services by issuing more debt ($\dot{B}(\tau)$), or by changing one or both of its tax instruments, viz. the lump-sum tax ($\bar{z}(\tau)$) or the tax rate on output of the final goods sector ($t_Y(\tau)$). After dividing both sides by $N(\tau)$ and simplifying we find the budget identity in per capita terms:

$$\dot{b}(\tau) = [r(\tau) - n_N] b(\tau) + g(\tau) - \bar{z}(\tau) - t_Y(\tau)y(\tau), \tag{A.52}$$

[4] The output tax is equivalent to a uniform tax on labour and capital income. See Baxter and King (1993, p. 318) and below.

where $b(\tau) \equiv B(\tau)/N(\tau)$, $g(\tau) \equiv G(\tau)/N(\tau)$, $y(\tau) \equiv Y(\tau)/N(\tau)$, and we have used the fact that $n_N \equiv \dot{N}(\tau)/N(\tau)$. We restrict attention to the dynamically efficient case so that $r(\tau) > n_N$. Since the government must remain solvent, the NPG condition is $\lim_{\tau\to\infty} b(\tau) \times \exp[-\int_t^\tau [r(s) - n_N]\, ds] = 0$, so that (A.52) can be integrated forward to derive the government budget restriction:

$$b(t) = \int_t^\infty [\bar{z}(\tau) + t_Y(\tau)y(\tau) - g(\tau)] \exp\left[-\int_t^\tau [r(s) - n_N]\, ds\right] d\tau. \tag{A.53}$$

Solvency of the government implies that the level of government debt must equal the present value of present and future primary surpluses (all measured in per capita terms).

The homogeneous good, $Y(\tau)$, is used by the private sector (for consumption and investment purposes) and by the government. The goods market clearing condition is thus given by:

$$Y(\tau) = C(\tau) + I(\tau) + G(\tau). \tag{A.54}$$

A.4 Summary of the growth model

In the presence of non-zero population growth, the model developed thus far will give rise to ongoing economic growth also in the steady state. In order to study the growth process, we must thus rewrite the model in a stationary format by expressing all growing variables relative to the *population*, $N(t)$. The key equations of the full model are stated in Table A.1, whilst the variable and parameter definitions are stated in, respectively, Tables A.2 and A.3. The endogenous variables are w, r, k, l, y, c, x, Δ, ω_{CX}, P_U and one of $\bar{z}$, t_Y, and b (11 in total).[5] Since there are also 11 equations, the model is fully determinate (in principle). We demonstrate below that the model is saddle-point stable, with one predetermined state variable (k) and two non-predetermined "jumping" state variables (x and Δ). The exogenous variables are g, β, η, and two of $\bar{z}$, t_Y, and b.

We briefly discuss the key equations in Table A.1. Consider the dynamic equations first. Equation (T1.1) is the accumulation equation for the per capita stock of capital, $k(t) \equiv K(t)/N(t)$. It is obtained by combining (A.41) and (A.54) and dividing by $N(t)$. Equation (T1.2) shows the optimum time path of per capita full consumption, $x(t) \equiv X(t)/N(t)$. It is obtained by expressing (A.33) in per capita format and noting that $a(t) = k(t) + b(t)$. Finally, equation (T1.3), describing the dynamic behaviour of $\Delta(t)$, is a slightly rewritten version of (A.15).

Equations (T1.4)-(T1.10) are essentially static equations, in the sense that they can be related uniquely to the endogenous state variables and the exogenous variables of the model. Equation (T1.4) is the intensive-form production function, obtained from (A.47). Equations (T1.5)-(T1.6) are obtained by rewriting the expressions in (A.49)-(A.50) in the intensive form and noting (A.44). Equations (T1.7)-(T1.8) are obtained by using (A.18)-(A.19). Equations (T1.9) and (T1.10) are identical to, respectively, (A.20) and (A.21).

Finally, equation (T1.11) is the solvency condition of the government restricting the types of fiscal policies which the government can pursue. In the paper we distinguish various policies which all satisfy (T1.11).

[5] See below for a discussion of the different closure rules for the government budget restriction.

Table A.1: Short-run version of the model$^{(\sharp)}$

(a) Dynamic equations:

$$\dot{k}(t) = y(t) - c(t) - g(t) - (\delta + n_N)\, k(t) \tag{T1.1}$$

$$\dot{x}(t) = \left[\sigma_X \left(r\,(t) - \alpha\right) + (1-\sigma_X)\frac{\dot{P}_U\,(t)}{P_U\,(t)}\right] x(t) - \eta \left[\frac{k(t) + b\,(t)}{\Delta(t)}\right] \tag{T1.2}$$

$$\dot{\Delta}(t) = -1 + \left[(1-\sigma_X)\left(r\,(t) + \beta - \frac{\dot{P}_U(t)}{P_U(t)}\right) + \sigma_X\,(\alpha+\beta)\right]\Delta(t) \tag{T1.3}$$

(b) Static equations:

$$y(t) = \Psi_Y \left[\varepsilon_K k\,(t)^{(\sigma_K-1)/\sigma_K} + (1-\varepsilon_K) l\,(t)^{(\sigma_K-1)/\sigma_K}\right]^{\sigma_K/(\sigma_K-1)} \tag{T1.4}$$

$$\frac{w\,(t)}{1-t_Y} = (1-\varepsilon_K)\,\Psi_Y^{1-1/\sigma_K}\left(\frac{y\,(t)}{l\,(t)}\right)^{1/\sigma_K} \tag{T1.5}$$

$$\frac{r\,(t)+\delta}{1-t_Y} = \varepsilon_K \Psi_Y^{1-1/\sigma_K}\left(\frac{y\,(t)}{k\,(t)}\right)^{1/\sigma_K} \tag{T1.6}$$

$$c(t) = \omega_{CX}\,(t)\,x(t) \tag{T1.7}$$

$$w\,(t)\,[1-l(t)] = [1-\omega_{CX}\,(t)]\,x(t) \tag{T1.8}$$

(c) Miscellaneous equations:

$$\omega_{CX}\,(t) \equiv \frac{\varepsilon_C^{\sigma_C}}{\varepsilon_C^{\sigma_C} + (1-\varepsilon_C)^{\sigma_C}\, w\,(t)^{1-\sigma_C}} \tag{T1.9}$$

$$P_U(t) \equiv \begin{cases} \left[(\varepsilon_C)^{\sigma_C} + (1-\varepsilon_C)^{\sigma_C} w(t)^{1-\sigma_C}\right]^{1/(1-\sigma_C)} & \text{for } \sigma_C \neq 1 \\ \left(\frac{1}{\varepsilon_C}\right)^{\varepsilon_C}\left(\frac{w(t)}{1-\varepsilon_C}\right)^{1-\varepsilon_C} & \text{for } \sigma_C = 1 \end{cases} \tag{T1.10}$$

$$b\,(t) = \int_t^\infty \left[\bar{z}(\tau) + t_Y(\tau)y(\tau) - g(\tau)\right]\exp\left[-\int_t^\tau \left[r(s) - n_N\right] ds\right] d\tau \tag{T1.11}$$

($\sharp$) See Table A.2 for the variable definitions and Table A.3 for the parameters.

Table A.2: Key variable definitions

a	$(A, \bar{a})$	per capita (aggregate, individual) financial assets
b	$(B, \bar{b})$	per capita (aggregate, individual) government bonds
c	$(C, \bar{c})$	per capita (aggregate, individual) goods consumption
g	(G)	per capita (aggregate) government goods consumption
h	$(H, \bar{h})$	per capita (aggregate, individual) human wealth
I		aggregate gross investment
k	$(K, \bar{k})$	per capita (aggregate, individual) capital stock
l	$(L, \bar{l})$	per capita (aggregate, individual) labour supply
Λ		life-time utility
N	(n_N)	level (growth rate of) the population
P_U		true cost-of-living index
P_Y		output price (numeraire, $P_Y = 1$)
Π		firm profit
r		real interest rate
t_Y		output tax
$\bar{u}$		individual sub-utility
V		value of the capital stock ($V = K$)
w		real wage rate
w^K		real rental rate on capital
x	$(X, \bar{x})$	per capita (aggregate, individual) full consumption
y	(Y)	per capita (aggregate) output
$\bar{z}$	$(N\bar{z}, \bar{z})$	per capita (aggregate, individual) lump-sum tax

Table A.3: Key structural parameters

α	pure rate of time preference
β	instantaneous death probability
δ	capital depreciation rate
Δ	propensity to full-consume out of total wealth
ε_C	parameter for consumption in the sub-felicity function
ε_K	efficiency parameter for capital in the production function
η	instantaneous birth rate
n_N	population growth rate ($n_N \equiv \eta - \beta$)
σ_C	substitution elasticity between consumption and leisure
σ_K	substitution elasticity between capital and labour
σ_X	intertemporal substitution elasticity
Ψ	index of general technology
ω_{CX}	share of goods consumption in full consumption

A.5 Phase diagram

In this section we construct the phase diagram for a special case of the model. This so-called unit-elastic model has the following features:

- all elasticities in preferences and technology are set equal to unity: $\sigma_C = \sigma_K = \sigma_X = 1$
- initial per capita debt is zero ($b = 0$), the output tax is constant, and lump-sum taxes balance the government budget
- the output share of public consumption ($\omega_G \equiv g/y$) is held constant
- the birth rate exceeds the death rate ($\eta > \beta$) and there is thus a constant rate of population growth ($n_N \equiv \eta - \beta > 0$)

For convenience we present the equations of the unit-elastic model in Table A.4.

A.5.1 Employment as a function of the state variables

By using labour demand (T4.4), labour supply (T4.6), and the aggregate production function (T4.3), we obtain the following expression relating equilibrium employment to full consumption and capital (we drop the time index for convenience). Note that "LME" designates labour market equilibrium.

$$\text{LME:} \qquad (f(l) \equiv) \quad \frac{1-l}{l^{\varepsilon_K}} = \frac{\omega_0 x}{k^{\varepsilon_K}}, \tag{A.55}$$

where ω_0 is defined as:

$$\omega_0 \equiv \frac{1-\varepsilon_C}{(1-\varepsilon_K)(1-t_Y)\Psi_Y}. \tag{A.56}$$

Table A.4: The unit-elastic model$^{(\sharp)}$

(a) Dynamic equations:

$$\dot{k}(t) = (1 - \omega_G)\, y(t) - \varepsilon_C x(t) - (\delta + n_N)\, k(t) \tag{T4.1}$$
$$\dot{x}(t) = [r\,(t) - \alpha]\, x(t) - \eta\,(\alpha + \beta)\, k(t) \tag{T4.2}$$

(b) Static equations:

$$y(t) = \Psi_Y k\,(t)^{\varepsilon_K}\, l\,(t)^{1-\varepsilon_K} \tag{T4.3}$$
$$\frac{w\,(t)}{1 - t_Y} = (1 - \varepsilon_K) \left(\frac{y\,(t)}{l\,(t)}\right) \tag{T4.4}$$
$$\frac{r\,(t) + \delta}{1 - t_Y} = \varepsilon_K \left(\frac{y\,(t)}{k\,(t)}\right) \tag{T4.5}$$
$$w\,(t)\,[1 - l(t)] = (1 - \varepsilon_C)\, x(t) \tag{T4.6}$$
$$\bar{z}(t) = (\omega_G - t_Y)\, y(t) \tag{T4.7}$$

($\sharp$) See Table A.2 for the variable definitions and Table A.3 for the parameters.

We are interested in the properties of $f(l)$ in the *economically meaningful* interval $0 \leq l \leq 1$. Clearly, it follows from (A.55) that $f(1) = 0$ and $f(0) \to \infty$. Furthermore, $f(l)$ has the following first- and second-order derivatives:

$$f'(l) = -\left[\frac{\varepsilon_K\,(1 - l) + l}{l^{1+\varepsilon_K}}\right] < 0, \tag{A.57}$$
$$f''(l) = \varepsilon_K \left[\frac{1 + l + \varepsilon_K (1 - l)}{l^{2+\varepsilon_K}}\right] > 0. \tag{A.58}$$

It follows that for $l \in [0, 1]$, $f\,(l)$ is a convex and downward sloping function as drawn in Figure A.1.

The important conclusion to be drawn from (A.55) is that, for any non-negative value of x/k^{ε_K}, there is exactly **one** equilibrium employment level which solves equation (A.55). Furthermore, $f\,(\cdot)$ can be inverted to obtain a relationship between l and $\omega_0 x/k^{\varepsilon_k}$: $l = f^{-1}\,(\omega_0 x/k^{\varepsilon_k})$. This inverse function is used below—see, e.g., equations (A.72) and (A.107) below.

A.5.2 Capital stock equilibrium

It follows from (T4.1) that $\dot{k} = 0$ holds if and only if $(\delta + n_N)\, k = (1 - \omega_G)\, y - \varepsilon_C x$. By combining labour demand and labour supply we find that x/y can be written as:

$$\frac{x}{y} = \frac{(1 - \varepsilon_K)\,(1 - t_Y)}{1 - \varepsilon_C} \left(\frac{1 - l}{l}\right). \tag{A.59}$$

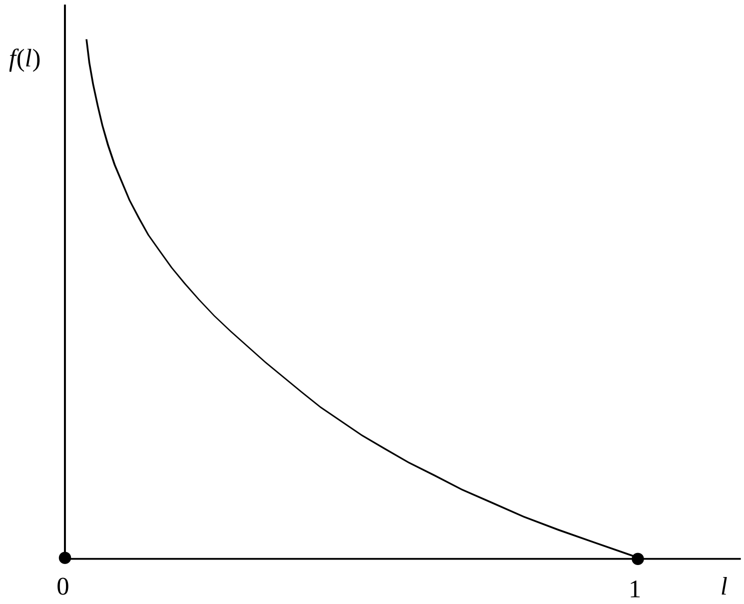

Figure A.1: Labour Market Equilibrium

It follows that the capital stock equilibrium (CSE) locus (for which $\dot{k} = 0$) can be written as:[6]

$$\text{CSE:} \qquad (\delta + n_N)\, k = \left[1 - \omega_G - \frac{\varepsilon_C\,(1-\varepsilon_K)\,(1-t_Y)}{1-\varepsilon_C}\left(\frac{1-l}{l}\right)\right] y. \tag{A.60}$$

We are clearly only interested in positive values of output and capital so that the term in square brackets on the right-hand side of (A.60) must be non-negative. This furnishes a lower bound for employment, i.e. l must exceed l_{MIN} which is defined as follows:

$$l_{\text{MIN}} \equiv \frac{\varepsilon_C\,(1-\varepsilon_K)\,(1-t_Y)}{\varepsilon_C\,(1-\varepsilon_K)\,(1-t_Y) + (1-\omega_G)(1-\varepsilon_C)}, \quad 0 < l_{\text{MIN}} < 1. \tag{A.61}$$

By using the definition of l_{MIN} in equation (A.60) and substituting (T4.3) we obtain the following expression for the CSE locus:

$$\text{CSE:} \qquad \left(\frac{k}{l}\right)^{1-\varepsilon_K} = \left(\frac{\varepsilon_C\,(1-\varepsilon_K)\,(1-t_Y)\Psi_Y}{(\delta+n_N)\,(1-\varepsilon_C)}\right)\left(\frac{l-l_{\text{MIN}}}{l_{\text{MIN}} l}\right). \tag{A.62}$$

Note that (A.62) can be further rewritten as:

$$\text{CSE:} \qquad k = g\,(l) \equiv \left(\frac{\varepsilon_C\,(1-\varepsilon_K)\,(1-t_Y)\Psi_Y}{(\delta+n_N)\,(1-\varepsilon_C)}\right)^{1/(1-\varepsilon_K)} \left(\frac{l-l_{\text{MIN}}}{l_{\text{MIN}} l^{\varepsilon_K}}\right)^{1/(1-\varepsilon_K)}, \tag{A.63}$$

where $g\,(l)$ represents a function relating k to l (and the constants) which is defined over the interval $l \in [l_{\text{MIN}}, 1]$. We find from (A.63) that this function is upward sloping:

$$\frac{dk}{dl} = g'\,(l) = \left(\frac{g\,(l)}{1-\varepsilon_K}\right)\left[\frac{(1-\varepsilon_K)\,l + \varepsilon_K l_{\text{MIN}}}{l\,(l-l_{\text{MIN}})}\right] > 0, \tag{A.64}$$

[6] If $\varepsilon_C = 1$ labour supply is exogenous, $l = 1$, and the CSE curve is:

$$(\delta + n_N)\, k = (1-\omega_G)\Psi_Y k^{\varepsilon_K} - \varepsilon_C x.$$

where the sign follows from the fact that $l \geq l_{\text{MIN}} > 0$ so that the term in square brackets on the right-hand side is strictly positive. It follows from (A.63) that $k = 0$ for $l = l_{\text{MIN}}$, so as l rises from $l = l_{\text{MIN}}$ to $l = 1$, k rises from $k = 0$ to $k = k_K$, the expression for which is obtained by substituting $l = 1$ in equation (A.63):

$$k_K \equiv \left(\frac{(1-\omega_G)\Psi_Y}{\delta + n_N} \right)^{1/(1-\varepsilon_K)} > 0. \tag{A.65}$$

To summarize, we have found:

$$g\left(l_{\text{MIN}}\right) = 0, \qquad g\left(1\right) = k_K. \tag{A.66}$$

We now have two zeros for the $\dot{k} = 0$ line, i.e. both $(k, l) = (0, l_{\text{MIN}})$ and $(k, l) = (k_K, 1)$ solve equation (A.60). By using (A.55) we find the corresponding values for x, i.e. $(x, k, l) = (0, 0, l_{\text{MIN}})$ and $(x, k, l) = (0, k_K, 1)$ are both zeros for the CSE-line–see Figure A.2. The slope of the CSE-line is computed as follows. We note that the CSE-line can be written as:

$$\varepsilon_C x = (1-\omega_G)\Psi_Y \left[g^{-1}(k)\right]^{1-\varepsilon_K} k^{\varepsilon_K} - (\delta + n_N)\, k, \tag{A.67}$$

where $l = g^{-1}(k)$ is the implicit function defined in (A.63). By taking the derivative of (A.67) we obtain in a few steps:

$$\varepsilon_C \left(\frac{dx}{dk} \right)_{\dot{k}=0} = (1-\omega_G) \left[\varepsilon_K + (1-\varepsilon_K)\, \eta_g(k)\right] \Psi_Y \left[\frac{g^{-1}(k)}{k} \right]^{1-\varepsilon_K} - (\delta + n_N)\,, \tag{A.68}$$

where $\eta_g(k) \geq 0$ is the elasticity of the $g^{-1}(k)$ function:

$$\begin{aligned} \eta_g(k) &\equiv \frac{k}{g^{-1}(k)} \frac{dg^{-1}\left(k\right)}{dk} \\ &= \frac{(1-\varepsilon_K)\left[g^{-1}\left(k\right) - l_{\text{MIN}}\right]}{(1-\varepsilon_K)\, g^{-1}(k) + \varepsilon_K l_{\text{MIN}}}, \end{aligned} \tag{A.69}$$

where we have used (A.64) to get to the second line. We follows from (A.63) that $g^{-1}(k) \to l_{\text{MIN}} > 0$ as $k \to 0$, and thus from (A.69) that $\eta_g(0) = 0$. Using these results in (A.68) we find that:

$$\lim_{k \to 0} \varepsilon_C \left(\frac{dx}{dk} \right)_{\dot{k}=0} = (1-\omega_G)\Psi_Y \varepsilon_K \lim_{k \to 0} \left[\frac{l_{\text{MIN}}}{k} \right]^{1-\varepsilon_K} = +\infty. \tag{A.70}$$

We thus find the usual Inada-style result that the CSE locus is vertical for k near zero. The "golden-rule" point–for which full consumption is at its maximum value–is obtained by setting $dx/dk = 0$ in (A.67):[7]

$$(1-\omega_G)\left[\varepsilon_K + (1-\varepsilon_K)\, \eta_g\left(k^{GR}\right)\right] \left(\frac{y}{k} \right)^{GR} = \delta + n_N, \tag{A.71}$$

In order to study the dynamics of the capital stock we write (T4.1) as follows:

$$\dot{k} = (1-\omega_G)\, \Psi_Y k^{\varepsilon_K} \left[f^{-1}\left(\frac{\omega_0 x}{k^{\varepsilon_K}} \right) \right]^{1-\varepsilon_K} - \varepsilon_C x - (\delta + n_N)\, k, \tag{A.72}$$

[7] For exogenous labour supply, $\eta_g\left(k\right) \equiv 0$ so that (A.71) reduces to the usual expression:

$$(1-\omega_G)\varepsilon_K \left(\frac{y}{k} \right) = \delta + n_N.$$

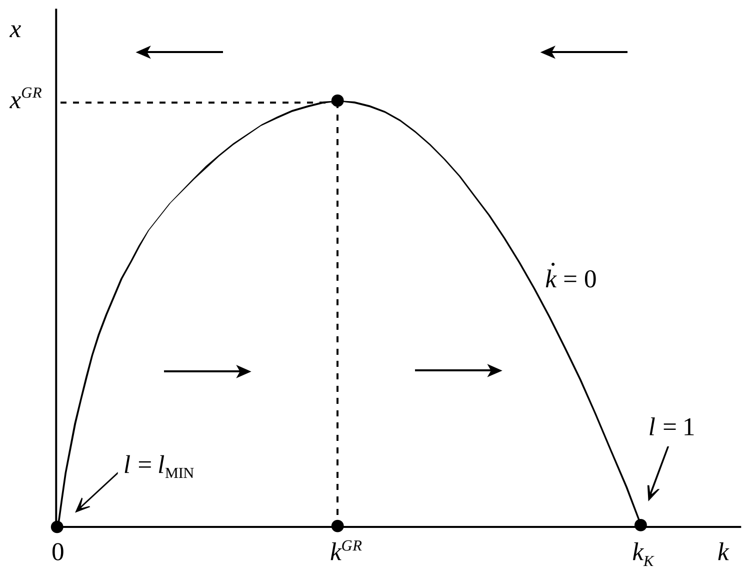

Figure A.2: Capital Equilibrium Locus

where $f^{-1}(\cdot)$ is the inverse of $f(\cdot)$ defined in (A.55) above. By partially differentiating () with respect to full consumption we find:

$$\frac{\partial \dot{k}}{\partial x} = (1-\omega_G)\,\frac{y}{l}\frac{\partial}{\partial x}\left[f^{-1}\left(\frac{\omega_0 x}{k^{\varepsilon_K}}\right)\right] - \varepsilon_C < 0, \tag{A.73}$$

where the sign follows from the fact that $\partial f^{-1}(\cdot)/\partial x < 0$. Hence, for points above (below) the $\dot{k} = 0$ line, employment is too low (high) and full consumption is too high (low) so that the capital stock falls (rises). This has been illustrated with horizontal arows in Figure A.2.

A.5.3 Consumption equilibrium

The consumption equilibrium (CE) locus describes (x,k)-combinations for which $\dot{x} = 0$. By using (T4.2) (in steady-state format), (T4.5) and (A.59) we can write the $\dot{x} = 0$ line as follows:

$$\eta(\alpha+\beta) = \frac{(1-\varepsilon_K)(1-t_Y)}{1-\varepsilon_C}\left(\frac{1-l}{l}\right)\left(\frac{y}{k}\right)\left[\varepsilon_K(1-t_Y)\left(\frac{y}{k}\right) - (\alpha+\delta)\right] \tag{A.74}$$

$$\frac{y}{k} = \Psi_Y\left(\frac{l}{k}\right)^{1-\varepsilon_K}. \tag{A.75}$$

Equations (A.74)-(A.75) together define the $\dot{x} = 0$ line in the (k,l)-plane. We distinguish two cases, viz. the representative-agent case ($\eta = 0$) and the overlapping-generations case ($\eta > 0$).

A.5.3.1 Representative agents ($\eta = 0$)

When $\eta = 0$, the expression in (A.74) has two solutions: full employment of labour ($l = 1$), or equality between the rate of interest and the rate of time preference ($r = \alpha$). The former solution constitutes the origin and the latter solution defines a unique output-capital ratio and thus (via

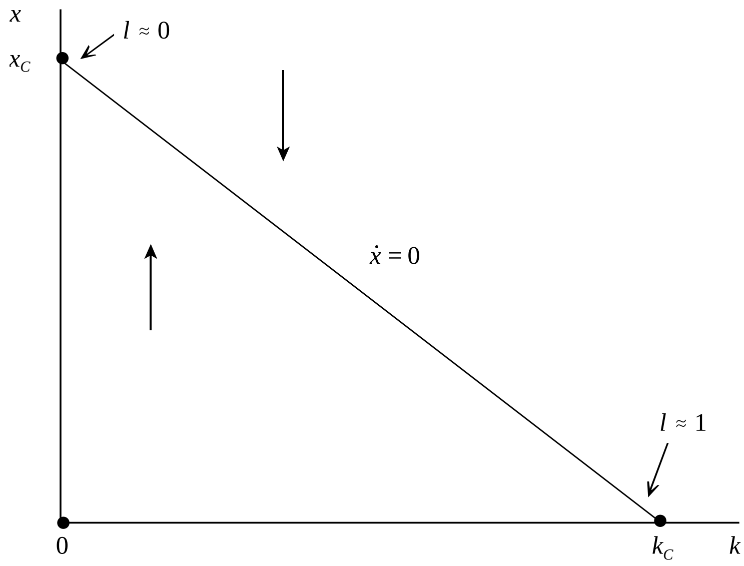

Figure A.3: Consumption equilibrium locus (RA model)

the production function (A.75)) an expression relating employment and capital:

$$\frac{y}{k} = \left(\frac{y}{k}\right)^* = \frac{\alpha + \delta}{\varepsilon_K(1 - t_Y)}, \tag{A.76}$$

$$l = \left[\frac{1}{\Psi_Y}\left(\frac{y}{k}\right)^*\right]^{1/(1-\varepsilon_K)} k. \tag{A.77}$$

Equation (A.77) expresses l as an upward sloping linear function of k along the CE-line. By using (A.77) in (A.55) we obtain the solution in the (x, k)-plane:

$$x = \left(\frac{(1 - \varepsilon_K)(\alpha + \delta)}{(1 - \varepsilon_C)\varepsilon_K}\right)\left[\left(\frac{\Psi_Y}{(y/k)^*}\right)^{1/(1-\varepsilon_K)} - k\right]. \tag{A.78}$$

Equation (A.78) defines a linear function with coordinates $(x, k) = (0, k_C)$ and $(x, k) = (x_C, 0)$—see Figure A.3:

$$k_C \equiv \left(\frac{\Psi_Y}{(y/k)^*}\right)^{1/(1-\varepsilon_K)}, \tag{A.79}$$

$$x_C \equiv \left(\frac{(1 - \varepsilon_K)(\alpha + \delta)}{(1 - \varepsilon_C)\varepsilon_K}\right) k_C. \tag{A.80}$$

A.5.3.2 Overlapping generations ($\eta > 0$)

The derivation of the $\dot{x} = 0$ line is much more tedious for the overlapping-generations model in which $\eta > 0$. In fact, we can only describe the $\dot{x} = 0$ line parametrically, i.e. by varying l in the interval $[0, 1]$. First, we can solve equation (A.74) for the economically sensible (positive) solution for the output-capital ratio:

$$\frac{y}{k} = \left(\frac{y}{k}\right)^* \left[\frac{1}{2} + \left(\frac{1}{4} + \zeta\left(\frac{l}{1 - l}\right)\right)^{1/2}\right], \tag{A.81}$$

where ζ is a positive constant:

$$\zeta \equiv \frac{\eta(\alpha+\beta)\varepsilon_K(1-\varepsilon_C)}{(1-\varepsilon_K)(\alpha+\delta)^2}. \tag{A.82}$$

By substituting (A.75) into (A.81) we obtain an expression for the $\dot{x}=0$ line in the (k,l)-plane:

$$k^{1-\varepsilon_K} = \left(\frac{\Psi_Y}{(y/k)^*}\right) h(l), \tag{A.83}$$

$$h(l) \equiv \frac{l^{1-\varepsilon_K}}{\frac{1}{2}+\left(\frac{1}{4}+\zeta\left(\frac{l}{1-l}\right)\right)^{1/2}}. \tag{A.84}$$

We derive in a straightforward fashion from (A.83)-(A.84) that:

$$\lim_{l\to 0} \frac{k}{l} = \left(\frac{\Psi_Y}{(y/k)^*}\right)^{1/(1-\varepsilon_K)}, \tag{A.85}$$

$$\lim_{l\to 1} \frac{k}{l} = 0. \tag{A.86}$$

Note that (A.55) can be written as follows:

$$x = \left(\frac{(1-\varepsilon_K)(1-t_Y)\Psi_Y}{1-\varepsilon_C}\right)(1-l)\left(\frac{k}{l}\right)^{\varepsilon_K}. \tag{A.87}$$

By using (A.85)-(A.86) in (A.87) we find the following limiting results for x:

$$\begin{aligned}\lim_{l\to 0} x &= \left(\frac{(1-\varepsilon_K)(1-t_Y)\Psi_Y}{1-\varepsilon_C}\right)\lim_{l\to 0}(1-l)\left(\frac{k}{l}\right)^{\varepsilon_K} \\ &= \left(\frac{(1-\varepsilon_K)(1-t_Y)\Psi_Y}{1-\varepsilon_C}\right)\left(\frac{\Psi_Y}{(y/k)^*}\right)^{\varepsilon_K/(1-\varepsilon_K)} \equiv x_C,\end{aligned} \tag{A.88}$$

$$\lim_{l\to 1} x = 0, \tag{A.89}$$

where x_C is defined in (A.80) above. The $\dot{x}=0$ line has two points on the x-axis, namely $x=0$ (for $l\to 1$) and $x=x_C$ (for $l\to 0$)—see Figure A.4.

Next we must determine the slope of the $\dot{x}=0$ line. We first look at the region of low employment ($l\approx 0$). We can derive from (A.83) and (A.87) that:

$$\lim_{l\to 0} \frac{dk}{dl} = \left(\frac{\Psi_Y}{(y/k)^*}\right)^{1/(1-\varepsilon_K)}, \tag{A.90}$$

$$\lim_{l\to 0} \frac{dx}{dl} = -\left(\frac{(1-\varepsilon_K)(\alpha+\delta)}{(1-\varepsilon_C)\varepsilon_K}\right)\left(\frac{\Psi_Y}{(y/k)^*}\right)^{1/(1-\varepsilon_K)}\left[1+\zeta\left(\frac{\varepsilon_K}{1-\varepsilon_K}\right)\right], \tag{A.91}$$

so that:

$$\lim_{l\to 0} \frac{dx}{dk} = -\left(\frac{(1-\varepsilon_K)(\alpha+\delta)}{(1-\varepsilon_C)\varepsilon_K}\right)\left[1+\zeta\left(\frac{\varepsilon_K}{1-\varepsilon_K}\right)\right] < 0. \tag{A.92}$$

It follows from the comparison of (A.78) and (A.92) that as l approaches zero, the CE line for the OLG model approaches the CE line for the RA model from below. This has been illustrated in Figure A.4.

We next look at the region of near full employment ($l\approx 1$). It is clear from the definition of $h(l)$ in (A.84) that $h(0)=\lim_{l\to 1} h(l)=0$ so both $(k,l)=(0,0)$ and $(k,l)=(0,1)$ are solutions which lie on the $\dot{x}=0$ line. The derivatives of $h(l)$ around $l=0$ and $l=1$ are:

$$\lim_{l\to 0} h'(l) = (1-\varepsilon_K)\lim_{l\to 0} l^{-\varepsilon_K} = +\infty, \tag{A.93}$$

$$\lim_{l\to 1} h'(l) = -\lim_{l\to 1}\left(\frac{4\zeta l^{1-\varepsilon_K}}{(1-l)^2\left[1+4\zeta\left(\frac{l}{1-l}\right)\right]^{1/2}\left[1+\left[1+4\zeta\left(\frac{l}{1-l}\right)\right]^{1/2}\right]^2}\right)$$
$$= -\lim_{l\to 1}\left(\frac{4\zeta l^{1-\varepsilon_K}}{\left[1-l+4\zeta l\right]^{1/2}\left[(1-l)^{3/4}+\left((1-l)^{3/2}+4\zeta l\,(1-l)^{1/2}\right)^{1/2}\right]^2}\right)$$
$$= -\infty. \tag{A.94}$$

By totally differentiating (A.83), the elasticity of the $\dot{x}=0$ line in the (k,l)-plane can be written as:

$$(1-\varepsilon_K)\left(\frac{dk}{k}\right) = \eta_h(l)\left(\frac{dl}{l}\right), \tag{A.95}$$

where $\eta_h(l) \equiv lh'(l)/h(l)$ is the elasticity of the $h(\cdot)$ function:

$$\eta_h(l) = (1-\varepsilon_K) - \frac{2\zeta l}{(1-l)^2\left[1+4\zeta\left(\frac{l}{1-l}\right)\right]^{1/2}\left[1+\left[1+4\zeta\left(\frac{l}{1-l}\right)\right]^{1/2}\right]}. \tag{A.96}$$

We readily obtain the following limiting results for $\eta_h(l)$:

$$\lim_{l\to 0} \eta_h(l) = 1-\varepsilon_K, \tag{A.97}$$
$$\lim_{l\to 1} \eta_h(l) = -\infty. \tag{A.98}$$

In a similar fashion, equation (A.55) can be written in elasticity form as:

$$\eta_f(l)\left(\frac{dl}{l}\right) = \left(\frac{dx}{x}\right) - \varepsilon_K\left(\frac{dk}{k}\right), \tag{A.99}$$

where $\eta_f(l)$ is the elasticity of the $f(\cdot)$ function:

$$\eta_f(l) \equiv \frac{lf'(l)}{f(l)} = -\left[\frac{l+(1-l)\,\varepsilon_K}{1-l}\right]. \tag{A.100}$$

The limiting results for $\eta_f(l)$ are:

$$\lim_{l\to 0} \eta_f(l) = -\varepsilon_K, \tag{A.101}$$
$$\lim_{l\to 1} \eta_f(l) = -\infty, \tag{A.102}$$
$$\lim_{l\to 1}\left(\frac{\eta_f(l)}{\eta_h(l)}\right) = 2. \tag{A.103}$$

By combining (A.95) and (A.99) we obtain the elasticity of the $\dot{x}=0$ line in the (x,k)-plane for the corner case, $l\to 1$:

$$\lim_{l\to 1}\left(\frac{k}{x}\frac{dx}{dk}\right)_{\dot{x}=0} = \varepsilon_K + (1-\varepsilon_K)\lim_{l\to 1}\left(\frac{\eta_f(l)}{\eta_h(l)}\right) = 2-\varepsilon_K > 1. \tag{A.104}$$

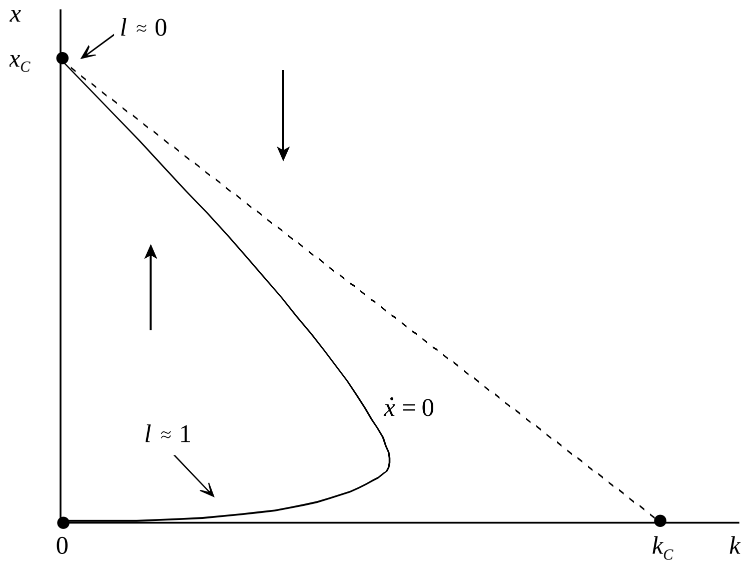

Figure A.4: onsumption equilibrium locus for the OLG model

Furthermore, by using (A.55) and (A.83) we derive the following limiting result:

$$\lim_{l \to 1} \left(\frac{x}{k} \right)_{\dot{x}=0} = \left(\frac{(1-\varepsilon_K)(1-t_Y)(y/k)^*}{1-\varepsilon_C} \right) \lim_{l \to 1} \left(\frac{(1-l)l^{-\varepsilon_K}}{h(l)} \right) = 0. \tag{A.105}$$

Combining (A.104) and (A.105) shows that:

$$\lim_{l \to 1} \left(\frac{dx}{dk} \right)_{\dot{x}=0} = 0. \tag{A.106}$$

Hence, the $\dot{x} = 0$ line is horizontal for $l \approx 1$, i.e. near the origin in Figure A.4.

In order to study the dynamics of full consumption we write (T4.2) as follows:

$$\begin{aligned} \dot{x} &= \left[\varepsilon_K (1-t_Y) \frac{y}{k} - (\alpha+\delta) \right] x - \eta(\alpha+\beta) k \\ &= \left[\varepsilon_K (1-t_Y) \Psi_Y \left(\frac{f^{-1}(\omega_0 x / k^{\varepsilon_K})}{k} \right)^{1-\varepsilon_K} - (\alpha+\delta) \right] x - \eta(\alpha+\beta) k, \end{aligned} \tag{A.107}$$

where we have used (A.75) and $l = f^{-1}(\omega_0 x / k^{\varepsilon_K})$ to get to the second line. By partially differentiating (A.108) with respect to the capital stock we find:

$$\frac{\partial \dot{x}}{\partial k} = x \varepsilon_K (1-t_Y)(1-\varepsilon_K) \Psi_Y \left(\frac{f^{-1}(\omega_0 x / k^{\varepsilon_K})}{k} \right)^{-\varepsilon_K} \frac{\partial}{\partial k} \left(\frac{f^{-1}(\omega_0 x / k^{\varepsilon_K})}{k} \right) - \eta(\alpha+\beta) < 0, \tag{A.108}$$

where the sign follows from the fact that $\frac{\partial}{\partial k} \left(\frac{f^{-1}(\omega_0 x / k^{\varepsilon_K})}{k} \right) < 0$.[8] Hence, full consumption rises (falls) for points to the left (right) of the $\dot{x} = 0$ line. This has been illustrated with vertical arrows in Figures A.3 and A.4.

[8] This can be seen most readily from (A.115) below.

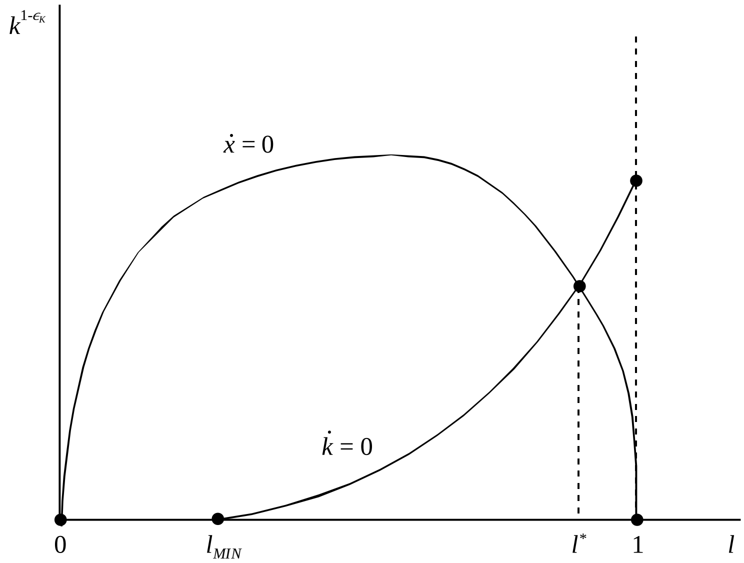

Figure A.5: Existence and Uniqueness of the Equilibrium

A.5.4 Uniqueness of the equilibrium

The equilibrium is unique as can be shown most easily in the (k, l)-plane. Equations (A.62) and (A.83) define the steady-state equilibrium:

$$\text{CSE } (\dot{k} = 0): \qquad k^{1-\varepsilon_K} = \left(\frac{\varepsilon_C (1 - \varepsilon_K)(1 - t_Y)\Psi_Y}{(\delta + n_N)(1 - \varepsilon_C)} \right) \left(\frac{l - l_{\text{MIN}}}{l_{\text{MIN}} l^{\varepsilon_K}} \right) \tag{A.109}$$

$$\text{CE } (\dot{x} = 0): \qquad k^{1-\varepsilon_K} = \left(\frac{\Psi_Y}{(y/k)^*} \right) h(l). \tag{A.110}$$

By equating the two expressions a single equation in l is obtained which possesses a unique root $l^* \in (l_{\text{MIN}}, 1)$. This has been illustrated in Figure A.5. Once l^* is known so is k^* and (by equation (A.55)) so is x^*. Hence, (k^*, x^*, l^*) is unique. □

A.6 Model solution

In the previous section we have demonstrated that the steady-state equilibrium for the unit-elastic model is unique. The main text of the paper contains the phase diagrams associated with the various prototypical cases distinguished by the model. In this section we show how the comparative dynamic effects of the various shocks can be computed. We also demonstrate saddle-point stability for the various cases.

A.6.1 Linearization

We log-linearize a slightly more general version of the unit-elastic model[9] around an initial steady state growth path–see Table A.5 for the resulting expressions. The following definitions are adopted for the flow variables and the capital stock:

$$\begin{array}{lllll} \tilde{k}\,(t) \equiv \frac{dk(t)}{k_0} & \dot{\tilde{k}}(t) \equiv \frac{d\dot{k}(t)}{k_0} & \tilde{l}\,(t) \equiv \frac{dl(t)}{l_0} & \tilde{y}\,(t) \equiv \frac{dy(t)}{y_0} & \tilde{z}\,(t) \equiv \frac{dz(t)}{z_0} \\ \tilde{x}\,(t) \equiv \frac{dx(t)}{x_0} & \dot{\tilde{x}}(t) \equiv \frac{d\dot{x}(t)}{x_0} & \tilde{w}\,(t) \equiv \frac{dW(t)}{W_0} & \tilde{r}\,(t) \equiv \frac{dr(t)}{r_0} & \end{array}, \tag{A.111}$$

where the variables with a subscript "0" refer to steady-state values. Intuitively, a variable with a tilde ('˜') denotes the proportional rate of change in that variable (relative to the initial steady state), and a variable with a tilde and a dot is the time rate of change in terms of the initial level. For the output tax rate and debt the following alternative conventions are used:

$$\tilde{t}_Y(t) \equiv \frac{dt_Y(t)}{1-t_{Y0}} \quad \tilde{b}(t) \equiv \frac{r_0 db(t)}{y_0} \quad \dot{\tilde{b}}(t) \equiv \frac{r_0 d\dot{b}(t)}{y_0} \ . \tag{A.112}$$

These conventions have the advantage that the resulting expressions are meaningful even if the initial level of the variable is zero, as is indeed the case for b_0 in Table A.5 (but not for t_{Y0}).

The following initial steady-state shares are used in Table A.5:

$$\begin{array}{ll} \omega_A \equiv \left(\frac{rk}{y}\right)_0 & \text{output share of income from financial assets (recall } b_0 = 0) \\ \omega_C \equiv \left(\frac{c}{y}\right)_0 & \text{output share of private consumption} \\ \omega_G \equiv \left(\frac{g}{y}\right)_0 & \text{output share of government consumption} \\ \omega_I \equiv \left(\frac{(\delta+n_N)k}{y}\right)_0 & \text{output share of private investment} \\ \omega_{LL} \equiv \left(\frac{1-l}{l}\right)_0 & \text{leisure/labour supply ratio} \\ \omega_Z \equiv \left(\frac{z}{y}\right)_0 & \text{share of lump-sum taxes} \end{array} \tag{A.113}$$

It is easy to deduce the following relationships between the steady-state shares and the various parameters:

$$\begin{aligned} \left(\frac{r-n_N}{r}\right)_0 \omega_A &= \omega_C + \omega_Z - (1-t_Y)(1-\varepsilon_K) \\ &= \varepsilon_K(1-t_Y) - \omega_I \\ \omega_G &= \omega_Z + t_Y \\ 1 &= \omega_C + \omega_I + \omega_G \\ \left(\frac{y}{k}\right)_0 &= \frac{r_0}{\omega_A} = \frac{\delta+n_N}{\omega_I} = \frac{r_0+\delta}{\varepsilon_K(1-t_Y)} \end{aligned} \tag{A.114}$$

The loglinearized model can be reduced to a two-dimensional system of first-order differential equations in the capital stock, $\tilde{k}(t)$, and full consumption, $\tilde{x}(t)$. Of these state variables, the

[9] Instead of (T4.1), (T4.2) and (T4.7) we use:

$$\begin{aligned} \dot{k}\,(t) &= y\,(t) - \varepsilon_C x\,(t) - g\,(t) - (\delta + n_N)\,k\,(t)\,, \\ \dot{x}\,(t) &= [r\,(t) - \alpha]\,x\,(t) - \eta\,(\alpha+\beta)\,[k\,(t) + b\,(t)] \\ b(t) &= \int_t^\infty [\bar{z}(\tau) + t_Y(\tau)y(\tau) - g(\tau)] \exp\left[-\int_t^\tau [r(s) - n_N]\,ds\right] d\tau. \end{aligned}$$

In the construction of the phase diagram it was useful to assume a constant ω_G and to abstract from debt altogether.

capital stock is predetermined, whilst full consumption is a non-predetermined 'jump' variable. Conditional upon the state variables and the policy shocks, the static part of the log-linearized model, consisting of equations (T5.3)-(T5.6) in Table A.5, can be used to derive the following useful 'quasi-reduced form' expressions:

$$\tilde{y}(t) = \phi\varepsilon_K\tilde{k}(t) - (\phi-1)\left[\tilde{x}(t)+\tilde{t}_Y(t)\right], \tag{A.115}$$

$$(1-\varepsilon_K)\,\tilde{l}(t) = (\phi-1)\left[\varepsilon_K\tilde{k}(t) - \tilde{x}\,(t) - \tilde{t}_Y\,(t)\right], \tag{A.116}$$

$$(1-\varepsilon_K)\left[\tilde{w}(t)+\tilde{t}_Y(t)\right] = -\varepsilon_K\left[\tilde{y}(t)-\tilde{k}(t)\right] \tag{A.117}$$

$$= \varepsilon_K\left[(1-\phi\varepsilon_K)\,\tilde{k}\,(t) + (\phi-1)\left[\tilde{x}(t)+\tilde{t}_Y(t)\right]\right] \tag{A.118}$$

where ϕ is a crucial parameter representing the intertemporal labour supply elasticity:

$$\phi \equiv \frac{1+\omega_{LL}}{1+\varepsilon_K\omega_{LL}} \geq 1, \tag{A.119}$$

where ω_{LL} ($\equiv (1-l)/l \geq 0$) is the ratio between leisure and labour, which also represents the intertemporal substitution elasticity of labour supply. Note that $\phi = 1$ if labour supply is exogenous (since $l = 1$ implies that $\omega_{LL} = 0$) but is strictly greater than unity otherwise. Note furthermore that:

$$0 < 1-\phi\varepsilon_K = \frac{1-\varepsilon_K}{1+\varepsilon_K\omega_{LL}} < 1-\varepsilon_K. \tag{A.120}$$

Table A.5: The log-linearized unit-elastic model

$$\dot{\tilde{k}}(t) = \left(\frac{y}{k}\right)_0\left[\tilde{y}\,(t) - \omega_C\tilde{x}\,(t) - \omega_G\tilde{g}\,(t) - \omega_I\tilde{k}\,(t)\right] - d\eta + d\beta \tag{T5.1}$$

$$\dot{\tilde{x}}(t) = r_0\tilde{r}(t) + (r_0-\alpha)\left[\tilde{x}(t) - \tilde{k}(t) - (1/\omega_A)\tilde{b}(t) - \frac{d\eta}{\eta_0} - \frac{d\beta}{\alpha+\beta_0}\right] \tag{T5.2}$$

$$\tilde{y}(t) = (1-\varepsilon_K)\,\tilde{l}(t) + \varepsilon_K\tilde{k}(t) \tag{T5.3}$$

$$\tilde{l}(t) = \tilde{y}(t) - \tilde{w}(t) - \tilde{t}_Y(t) \tag{T5.4}$$

$$\tilde{k}(t) = \tilde{y}(t) - \left(\frac{r_0}{r_0+\delta}\right)\tilde{r}(t) - \tilde{t}_Y(t) \tag{T5.5}$$

$$\tilde{l}(t) = \omega_{LL}\left[\tilde{w}(t) - \tilde{x}(t)\right] \tag{T5.6}$$

$$\tilde{b}(0) = (1-t_Y)\left[\mathcal{L}\left\{\tilde{t}_Y, (r-n_N)_0\right\} + \left(\frac{t_Y}{1-t_Y}\right)\mathcal{L}\left\{\tilde{y}, (r-n_N)_0\right\}\right] + \omega_Z\mathcal{L}\left\{\tilde{z}, (r-n_N)_0\right\} - \omega_G\mathcal{L}\left\{\tilde{g}, (r-n_N)_0\right\} \tag{T5.7}$$

A.6.2 General solution approach

The Laplace transform[10] technique of Judd (1982, 1998) is used to solve the model. In its most general form the dynamical system can be written as:

$$\begin{bmatrix} \dot{\tilde{k}}(t) \\ \dot{\tilde{x}}(t) \end{bmatrix} = \Delta \begin{bmatrix} \tilde{k}(t) \\ \tilde{x}(t) \end{bmatrix} - \begin{bmatrix} \gamma_K(t) \\ \gamma_X(t) \end{bmatrix}, \tag{A.121}$$

where Δ is the Jacobian matrix (with typical element δ_{ij}) and $\gamma_K(t)$ and $\gamma_X(t)$ are (potentially time-varying) forcing terms. By taking the Laplace transform of (A.121) and noting that the capital stock is predetermined ($\tilde{k}(0) = 0$) we obtain:

$$\Lambda(s) \begin{bmatrix} \mathcal{L}\{\tilde{k}, s\} \\ \mathcal{L}\{\tilde{x}, s\} \end{bmatrix} = \begin{bmatrix} -\mathcal{L}\{\gamma_K, s\} \\ \tilde{x}(0) - \mathcal{L}\{\gamma_X, s\} \end{bmatrix}, \tag{A.122}$$

where $\Lambda(s) \equiv s\mathrm{I} - \Delta$. The characteristic roots of Δ are denoted in general terms by $-\lambda_1 < 0$ and $\lambda_2 > 0$. By pre-multiplying both sides of (A.122) by $\Lambda(s)^{-1} \equiv \operatorname{adj}\Lambda(s)/[(s+\lambda_1)(s-\lambda_2)]$ we obtain the following expression in Laplace transforms:

$$(s+\lambda_1) \begin{bmatrix} \mathcal{L}\{\tilde{k}, s\} \\ \mathcal{L}\{\tilde{x}, s\} \end{bmatrix} = \frac{\operatorname{adj}\Lambda(s) \begin{bmatrix} -\mathcal{L}\{\gamma_K, s\} \\ \tilde{x}(0) - \mathcal{L}\{\gamma_X, s\} \end{bmatrix}}{(s-\lambda_2)}, \tag{A.123}$$

where $\operatorname{adj}\Lambda(s)$ is the adjoint matrix of $\Lambda(s)$. In order to eliminate the instability originating from the positive (unstable) characteristic root ($\lambda_2 > 0$) the jump in full consumption at impact ($\tilde{x}(0)$) must be such as to render the numerator of (A.123) equal to zero for $s = \lambda_2$ (for which value of s the denominator is also zero) (see Judd, 1998, pp. 459-460). This furnishes the general expression for $\tilde{x}(0)$:

$$\operatorname{adj}\Lambda(\lambda_2) \begin{bmatrix} -\mathcal{L}\{\gamma_K, \lambda_2\} \\ \tilde{x}(0) - \mathcal{L}\{\gamma_X, \lambda_2\} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{A.124}$$

where it should be noted that $\operatorname{adj}\Lambda(\lambda_2)$ has rank 1 because λ_2 is a distinct eigenvalue of $\Lambda(s)$. Hence, either row of (A.124) can be used to compute $\tilde{x}(0)$.

We next note that $\operatorname{adj}\Lambda(s) = \operatorname{adj}\Lambda(\lambda_2) + \mathrm{I}(s-\lambda_2)$ so that (A.123) and (A.124) can be combined to:

$$(s+\lambda_1) \begin{bmatrix} \mathcal{L}\{\tilde{k}, s\} \\ \mathcal{L}\{\tilde{x}, s\} \end{bmatrix} = \begin{bmatrix} -\mathcal{L}\{\gamma_K, s\} \\ \tilde{x}(0) - \mathcal{L}\{\gamma_X, s\} \end{bmatrix} + \operatorname{adj}\Lambda(\lambda_2) \begin{bmatrix} \frac{\mathcal{L}\{\gamma_K,\lambda_2\} - \mathcal{L}\{\gamma_K,s\}}{s-\lambda_2} \\ \frac{\mathcal{L}\{\gamma_X,\lambda_2\} - \mathcal{L}\{\gamma_X,s\}}{s-\lambda_2} \end{bmatrix}. \tag{A.125}$$

Equations (A.124) and (A.125) together constitute the general solution in terms of Laplace transforms. All the shocks studied in the paper can be represented as special cases of a gradually introduced shock, $\gamma_i(t) \equiv \gamma_i(1 - e^{-\xi_i t})$ with $\xi_i > 0$ and for $i = X, K$. For an unanticipated and permanent shock $\xi_i \to \infty$ whilst for an anticipated shock $0 \ll \xi_i < \infty$. The Laplace transform of the general shock term takes the form $\mathcal{L}\{\gamma_i, s\} = \gamma_i[1/s - 1/(s+\xi_i)]$. By using this in (A.125) we obtain:

$$\begin{aligned} \begin{bmatrix} \mathcal{L}\{\tilde{k}, s\} \\ \mathcal{L}\{\tilde{x}, s\} \end{bmatrix} &= \begin{bmatrix} 0 \\ \tilde{x}(0) \end{bmatrix} \frac{1}{s+\lambda_1} + \begin{bmatrix} \tilde{k}(\infty) \\ \tilde{x}(\infty) \end{bmatrix} \frac{\lambda_1}{s(s+\lambda_1)} \\ &\quad + \operatorname{adj}\Lambda(\lambda_2) \begin{bmatrix} \gamma_K/[(\lambda_2+\xi_K)(s+\xi_K)] \\ \gamma_X/[(\lambda_2+\xi_X)(s+\xi_X)] \end{bmatrix} \frac{1}{(s+\lambda_1)}, \end{aligned} \tag{A.126}$$

[10] The Laplace transform of $x(t)$ is denoted by $\mathcal{L}\{x, s\} \equiv \int_0^\infty x(t)e^{-st}dt$. Intuitively $\mathcal{L}\{x, s\}$ represents the present value of $x(t)$ using s as the discount rate.

where the long-run effects are given by:

$$\begin{bmatrix} \tilde{k}(\infty) \\ \tilde{x}(\infty) \end{bmatrix} = \frac{\operatorname{adj}\Lambda(0)}{\lambda_1\lambda_2}\begin{bmatrix} \gamma_K \\ \gamma_X \end{bmatrix} = \frac{\operatorname{adj}\Delta}{-\lambda_1\lambda_2}\begin{bmatrix} \gamma_K \\ \gamma_X \end{bmatrix}. \tag{A.127}$$

By inverting (A.126) and noting that $\mathcal{L}\{e^{-at}, s\} = 1/(s+a)$ we obtain the expression for the transition paths of $\tilde{k}(t)$ and $\tilde{x}(t)$:

$$\begin{aligned} \begin{bmatrix} \tilde{k}(t) \\ \tilde{x}(t) \end{bmatrix} &= \begin{bmatrix} 0 \\ \tilde{x}(0) \end{bmatrix} e^{-\lambda_1 t} + \begin{bmatrix} \tilde{k}(\infty) \\ \tilde{x}(\infty) \end{bmatrix} \left(1 - e^{-\lambda_1 t}\right) \\ &\quad + \operatorname{adj}\Lambda(\lambda_2) \begin{bmatrix} \gamma_K \mathrm{T}(\xi_K, \lambda_1, t)/(\lambda_2 + \xi_K) \\ \gamma_X \mathrm{T}(\xi_X, \lambda_1, t)/(\lambda_2 + \xi_X) \end{bmatrix}, \end{aligned} \tag{A.128}$$

where $\mathrm{T}(\xi_i, \lambda_1, t)$ is a temporary transition term with properties covered by the following Lemma.

Lemma 1 *Let* $\mathrm{T}(\alpha_1, \alpha_2, t)$ *be a single transition function of the form:*

$$\mathrm{T}(\alpha_1, \alpha_2, t) \equiv \begin{cases} \frac{e^{-\alpha_2 t} - e^{-\alpha_1 t}}{\alpha_1 - \alpha_2} & \text{for } \alpha_1 \neq \alpha_2 \\ t e^{-\alpha_1 t} & \text{for } \alpha_1 = \alpha_2, \end{cases}$$

with $\alpha_1 > 0$ *and* $\alpha_2 > 0$*. Then* $\mathrm{T}(\alpha_1, \alpha_2, t)$ *has the following properties: (i) (positive)* $\mathrm{T}(\alpha_1, \alpha_2, t) > 0$ $t \in (0, \infty)$*, (ii)* $\mathrm{T}(\alpha_1, \alpha_2, t) = 0$ *for* $t = 0$ *and in the limit as* $t \to \infty$*, (iii) (single-peaked)* $d\mathrm{T}(\alpha_1, \alpha_2, t)/dt > 0$ *for* $t \in (0, \hat{t})$*,* $d\mathrm{T}(\alpha_1, \alpha_2, t)/dt < 0$ *for* $t \in (\hat{t}, \infty)$*,* $d\mathrm{T}(\alpha_1, \alpha_2, t)/dt = 0$ *for* $t = \hat{t}$ *and in the limit as* $t \to \infty$*, and* $d\mathrm{T}(\alpha_1, \alpha_2, 0)/dt = 1$*, (iv)* $\hat{t} \equiv \ln(\alpha_1/\alpha_2)/(\alpha_1 - \alpha_2)$ *if* $\alpha_1 \neq \alpha_2$ *and* $\hat{t} \equiv 1/\alpha_1$ *if* $\alpha_1 = \alpha_2$*; (v) (point of inflexion)* $d^2\mathrm{T}(\alpha_1, \alpha_2, t)/dt^2 = 0$ *for* $t^* = 2\hat{t}$*; (vi) if* $\alpha_i \to \infty$ *then* $\mathrm{T}(\alpha_1, \alpha_2, t) = 0$ *for all* $t \geq 0$.

Proof: Property (i) follows by examining the three possible cases. The result is obvious if $\alpha_1 = \alpha_2$. If $\alpha_1 < (>) \alpha_2$, then $\alpha_2 - \alpha_1 > (<) 0$ and $e^{-\alpha_1 t} > (<)\ e^{-\alpha_2 t}$ for all $t \in (0, \infty)$, and $\mathrm{T}(\alpha_1, \alpha_2, 0) > 0$. Property (ii) follows by direct substitution. Property (iii) follows by examining $d\mathrm{T}(\alpha_1, \alpha_2, t)/dt$:

$$\frac{d\mathrm{T}(\alpha_1, \alpha_2, t)}{dt} \equiv \begin{cases} \frac{\alpha_1 e^{-\alpha_1 t} - \alpha_2 e^{-\alpha_2 t}}{\alpha_1 - \alpha_2} & \text{for } \alpha_1 \neq \alpha_2 \\ [1 - \alpha_1 t]\, e^{-\alpha_1 t} & \text{for } \alpha_1 = \alpha_2. \end{cases}$$

Property (iv) is obtained by examining $d\mathrm{T}^2(\alpha_1, \alpha_2, t)/dt^2$:

$$\frac{d^2\mathrm{T}(\alpha_1, \alpha_2, t)}{d^2 t} \equiv \begin{cases} \frac{\alpha_1^2 e^{-\alpha_1 t} - \alpha_2^2 e^{-\alpha_2 t}}{\alpha_2 - \alpha_1} & \text{for } \alpha_1 \neq \alpha_2 \\ -\alpha_1 [2 - \alpha_1 t]\, e^{-\alpha_1 t} & \text{for } \alpha_1 = \alpha_2. \end{cases}$$

Hence, $d^2\mathrm{T}(\alpha_1, \alpha_2, 0)/dt^2 = -(\alpha_1 + \alpha_2) < 0$, and $\lim_{t \to \infty} d^2\mathrm{T}(\alpha_1, \alpha_2, t)/dt^2 = 0$. The inflexion point is found by finding the value of $t = t^*$ where $d^2\mathrm{T}(\alpha_1, \alpha_2, t)/dt^2 = 0$. □

The single transition term is thus bell-shaped, zero at impact and in the long run, and positive during the transition. The terms involving the temporary transition terms $\mathrm{T}(\xi_i, \lambda_1, t)$ in (A.128) describe the modifications to the transition path due to the gradual–rather than abrupt–introduction of a shock $(0 < \xi_i \ll \infty)$.

In order to solve the various cases discussed in the text all that needs to be done is to identify the form of the Jacobian matrix Δ and the shock terms $\gamma_K(t)$ and $\gamma_X(t)$ and to prove saddle-point stability. In each case there is a demographic shock which has an effect on output and thus on the government budget constraint. Depending on the fiscal closure rule, there may be further effects on the macro-economy.

A.6.3 Lump-sum tax financing

With lump-sum taxes, both the output tax and public debt are held constant ($\tilde{t}_Y(t) = \tilde{b}(t) = 0$) and (A.115) and (T5.5) can be combined with (T5.1)-(T5.2) to derive the relevant Jacobian matrix, Δ:

$$\Delta \equiv \begin{bmatrix} \left(\frac{y}{k}\right)_0 [\phi\varepsilon_K - \omega_I] & -\left(\frac{y}{k}\right)_0 (\omega_C + \phi - 1) \\ -(r_0 - \alpha) - (r_0 + \delta)(1 - \phi\varepsilon_K) & (r_0 - \alpha) - (r_0 + \delta)(\phi - 1) \end{bmatrix}, \tag{A.129}$$

and the shock term:

$$\begin{bmatrix} \gamma_K(t) \\ \gamma_X(t) \end{bmatrix} \equiv \begin{bmatrix} \left(\frac{y}{k}\right)_0 \omega_G \tilde{g} + d\eta - d\beta \\ (r_0 - \alpha)\left(\frac{d\eta}{\eta_0} - \frac{d\beta}{\alpha + \beta_0}\right) \end{bmatrix}. \tag{A.130}$$

A.6.3.1 Stability

Saddle-point stability holds provided the determinant of Δ in (A.129) is negative. After some manipulation we find:

$$|\Delta| = -(r_0 + \delta)\left(\frac{y}{k}\right)_0 \left[\omega_G(\phi - 1) + \omega_C\phi(1 - \varepsilon_K) + [\phi(1 - \varepsilon_K) - \omega_G]\left(\frac{r_0 - \alpha}{r_0 + \delta}\right)\right], \tag{A.131}$$

Proposition A.1 summarizes some properties of the model that will prove useful in the discussion of policy shocks.

Proposition 1 *The model satisfies the following properties: (i)* ***Stability:*** *the model is saddle-point stable. (ii)* ***Roots:*** *the characteristic roots are* $\lambda_2 > 0$ *and* $-\lambda_1 < 0$*. The unstable root satisfies the inequality* $\lambda_2 > r_0 - \alpha + \omega_C(r_0 + \delta)$*.*

Proof. Part (i) is confirmed graphically by the configuration of arrows in Figure 1 in the paper. Part (ii) is proved as follows. Since $|\Delta| = -\lambda_1\lambda_2 < 0$, Δ has distinct roots $-\lambda_1 < 0$ and $\lambda_2 > 0$. To prove the inequality for the unstable root we must show that $\Lambda(\bar{\lambda}) < 0$, where $\bar{\lambda} \equiv r_0 - \alpha + \omega_C(r_0 + \delta) > 0$. After some manipulations we get:

$$\Lambda(\bar{\lambda}) = -\left(\frac{y}{k}\right)_0 (\phi + \omega_C - 1)\left[\omega_G(r_0 + \delta) + \bar{\lambda}\,[1 - (1 - t_Y)\varepsilon_K]\right] < 0,$$

which proves part (ii) of Proposition A.1. The root inequality is used in the paper to establish the sign of $\tilde{x}(0)$.

A.6.3.2 Shock in public consumption

Since the system is saddle-point stable the solution method of section A.6.2 is applicable. The shock consists of an unanticipated and permanent increase in government consumption which is financed with a lump-sum tax. The shock terms in (A.130) thus reduce to:

$$\begin{bmatrix} \gamma_K \\ \gamma_X \end{bmatrix} \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left(\frac{y}{k}\right)_0 \omega_G \tilde{g}. \tag{A.132}$$

By using (A.132) in (A.127) we find the long–run effects on the capital stock and full consumption:

$$\frac{\tilde{k}(\infty)}{\tilde{g}} = \frac{\left(\frac{y}{k}\right)_0 \omega_G [(r_0 + \delta)(\phi - 1) - (r_0 - \alpha)]}{\lambda_1\lambda_2} \gtreqless 0, \tag{A.133}$$

$$\frac{\tilde{x}(\infty)}{\tilde{g}} = \frac{\tilde{c}(\infty)}{\tilde{g}} = -\frac{\left(\frac{y}{k}\right)_0 \omega_G [(r_0 + \delta)(1 - \phi\varepsilon_K) + (r_0 - \alpha)]}{\lambda_1\lambda_2} < 0. \tag{A.134}$$

By using (A.133)-(A.134) in (A.115)-(A.118) the long-run results for the remaining variables are obtained:

$$\begin{aligned}
\frac{\tilde{y}(\infty)}{\tilde{g}} &= \phi\varepsilon_K \frac{\tilde{k}(\infty)}{\tilde{g}} - (\phi-1)\frac{\tilde{x}(\infty)}{\tilde{g}} \\
&= \frac{\left(\frac{y}{k}\right)_0 \omega_G \left[(r_0+\delta)(\phi-1) + (r_0-\alpha)\left[\phi(1-\varepsilon_K)-1\right]\right]}{\lambda_1\lambda_2}, && \text{(A.135)}\\
\frac{\tilde{l}(\infty)}{\tilde{g}} &= \left(\frac{\phi-1}{1-\varepsilon_K}\right)\left[\varepsilon_K \frac{\tilde{k}(\infty)}{\tilde{g}} - \frac{\tilde{x}(\infty)}{\tilde{g}}\right] \\
&= \frac{(\phi-1)\left(\frac{y}{k}\right)_0 \omega_G \left[r_0+\delta+r_0-\alpha\right]}{\lambda_1\lambda_2} > 0, && \text{(A.136)}\\
(1-\varepsilon_K)\frac{\tilde{w}(\infty)}{\tilde{g}} &= -\varepsilon_K\left[\frac{\tilde{y}(\infty)}{\tilde{g}} - \frac{\tilde{k}(\infty)}{\tilde{g}}\right] \\
&= -\frac{\varepsilon_K \left(\frac{y}{k}\right)_0 \omega_G (r_0-\alpha)\phi(1-\varepsilon_K)}{\lambda_1\lambda_2} < 0, && \text{(A.137)}\\
\left(\frac{r_0}{r_0+\delta}\right)\frac{\tilde{r}(\infty)}{\tilde{g}} &= \frac{\tilde{y}(\infty)}{\tilde{g}} - \frac{\tilde{k}(\infty)}{\tilde{g}} \\
&= \frac{\left(\frac{y}{k}\right)_0 \omega_G (r_0-\alpha)\phi(1-\varepsilon_K)}{\lambda_1\lambda_2} > 0. && \text{(A.138)}
\end{aligned}$$

The impact effect on full consumption is computed by using (A.124) and noting that:

$$\mathcal{L}\{\gamma_K,\lambda_2\} = \left(\frac{y}{k}\right)_0 \frac{\omega_G \tilde{g}}{\lambda_2}, \qquad \mathcal{L}\{\gamma_X,\lambda_2\} = 0. \tag{A.139}$$

After some manipulation we find:

$$\frac{\tilde{x}(0)}{\tilde{g}} = \frac{\tilde{c}(0)}{\tilde{g}} = -\frac{\omega_G\left[\lambda_2 - (r_0-\alpha) + (r_0+\delta)(\phi-1)\right]}{\lambda_2(\omega_C+\phi-1)} < 0, \tag{A.140}$$

where the sign follows from the fact that $\lambda_2 > r_0 - \alpha$. By using (A.140) in (A.115)-(A.118) and noting that $\tilde{k}(0) = 0$ the impact results for the remaining variables are obtained:

$$\frac{\tilde{y}(0)}{\tilde{g}} = -(\phi-1)\frac{\tilde{x}(0)}{\tilde{g}} = \frac{(\phi-1)\omega_G\left[\lambda_2 - (r_0-\alpha) + (r_0+\delta)(\phi-1)\right]}{\lambda_2(\omega_C+\phi-1)} > 0, \tag{A.141}$$

$$\begin{aligned}
\frac{\tilde{l}(0)}{\tilde{g}} &= \left(\frac{1}{1-\varepsilon_K}\right)\frac{\tilde{y}(0)}{\tilde{g}} > 0, && \text{(A.142)}\\
\frac{\tilde{w}(0)}{\tilde{g}} &= -\left(\frac{\varepsilon_K}{1-\varepsilon_K}\right)\frac{\tilde{y}(0)}{\tilde{g}} < 0, && \text{(A.143)}\\
\left(\frac{r_0}{r_0+\delta}\right)\frac{\tilde{r}(0)}{\tilde{g}} &= \frac{\tilde{y}(0)}{\tilde{g}} > 0. && \text{(A.144)}
\end{aligned}$$

A.6.3.3 Equal shock in birth and death rate

The next shock we consider consists of an equal (unanticipated and permanent) increase in the birth and death rates, i.e. we assume that $d\eta = d\beta > 0$. This shock leaves the population growth rate unchanged ($dn_N = d\eta - d\beta = 0$) and thus does not affect the CSE line. The shock terms in

(A.130) thus reduce to:

$$\begin{bmatrix} \gamma_K \\ \gamma_X \end{bmatrix} \equiv \begin{bmatrix} 0 \\ \zeta_1 \end{bmatrix} d\eta, \tag{A.145}$$

$$\zeta_1 \equiv \frac{(r_0 - \alpha)(\alpha - n_N)_0}{\eta_0 (\alpha + \beta_0)}. \tag{A.146}$$

In the text we assume that the population growth rate is *moderate*, in the sense that $(\alpha - n_N)_0 > 0$. Since $r_0 > \alpha$ this means that ζ_1 is positive. By using (A.145) in (A.127) we find the long–run effects on the capital stock and full consumption:

$$\left(\frac{\tilde{k}(\infty)}{d\eta}\right)_{d\eta=d\beta} = -\frac{\zeta_1 \left(\frac{y}{k}\right)_0 (\omega_C + \phi - 1)}{\lambda_1 \lambda_2} < 0, \tag{A.147}$$

$$\left(\frac{\tilde{x}(\infty)}{d\eta}\right)_{d\eta=d\beta} = \left(\frac{\tilde{c}(\infty)}{d\eta}\right)_{d\eta=d\beta} = -\frac{\zeta_1 \left(\frac{y}{k}\right)_0 (\phi\varepsilon_K - \omega_I)}{\lambda_1 \lambda_2} < 0. \tag{A.148}$$

By using (A.147)-(A.148) in (A.115)-(A.118) the long-run results for the remaining variables are obtained:

$$\begin{aligned} \left(\frac{\tilde{y}(\infty)}{d\eta}\right)_{d\eta=d\beta} &= \phi\varepsilon_K \left(\frac{\tilde{k}(\infty)}{d\eta}\right)_{d\eta=d\beta} - (\phi - 1)\left(\frac{\tilde{x}(\infty)}{d\eta}\right)_{d\eta=d\beta} \\ &= -\frac{\zeta_1 \left(\frac{y}{k}\right)_0 [\phi\varepsilon_K\omega_C + (\phi - 1)\omega_I]}{\lambda_1\lambda_2} < 0, \end{aligned} \tag{A.149}$$

$$\begin{aligned} \left(\frac{\tilde{l}(\infty)}{d\eta}\right)_{d\eta=d\beta} &= \left(\frac{\phi - 1}{1 - \varepsilon_K}\right)\left[\varepsilon_K \left(\frac{\tilde{k}(\infty)}{d\eta}\right)_{d\eta=d\beta} - \left(\frac{\tilde{x}(\infty)}{d\eta}\right)_{d\eta=d\beta}\right] \\ &= \frac{(\phi - 1)\zeta_1 \left(\frac{y}{k}\right)_0 [\varepsilon_K (1 - \omega_C) - \omega_I]}{\lambda_1\lambda_2} \lesseqgtr 0, \end{aligned} \tag{A.150}$$

$$\begin{aligned} \left(\frac{\tilde{w}(\infty)}{d\eta}\right)_{d\eta=d\beta} &= \frac{\varepsilon_K}{1 - \varepsilon_K}\left[\left(\frac{\tilde{y}(\infty)}{d\eta}\right)_{d\eta=d\beta} - \left(\frac{\tilde{k}(\infty)}{d\eta}\right)_{d\eta=d\beta}\right] \\ &= -\frac{\varepsilon_K\zeta_1 \left(\frac{y}{k}\right)_0 [(1 - \phi\varepsilon_K)\omega_C + (\phi - 1)(1 - \omega_I)]}{(1 - \varepsilon_K)\lambda_1\lambda_2} < 0, \end{aligned} \tag{A.151}$$

$$\begin{aligned} \left(\frac{r_0}{r_0 + \delta}\right)\left(\frac{\tilde{r}(\infty)}{d\eta}\right)_{d\eta=d\beta} &= \left(\frac{\tilde{y}(\infty)}{d\eta}\right)_{d\eta=d\beta} - \left(\frac{\tilde{k}(\infty)}{d\eta}\right)_{d\eta=d\beta} \\ &= \frac{\zeta_1 \left(\frac{y}{k}\right)_0 [(1 - \phi\varepsilon_K)\omega_C + (\phi - 1)(1 - \omega_I)]}{\lambda_1\lambda_2} > 0. \end{aligned} \tag{A.152}$$

The impact effect on full consumption is computed by using (A.124) and noting that:

$$\mathcal{L}\{\gamma_K, \lambda_2\} = 0, \qquad \mathcal{L}\{\gamma_X, \lambda_2\} = \frac{\zeta_1 d\eta}{\lambda_2}. \tag{A.153}$$

After some manipulation we find:

$$\left(\frac{\tilde{x}(0)}{d\eta}\right)_{d\eta=d\beta} = \left(\frac{\tilde{c}(0)}{d\eta}\right)_{d\eta=d\beta} = \frac{\zeta_1 d\eta}{\lambda_2} > 0. \tag{A.154}$$

By using (A.154) in (A.115)-(A.118) and noting that $\tilde{k}(0) = 0$ the impact results for the remaining variables are obtained:

$$\left(\frac{\tilde{y}(0)}{d\eta}\right)_{d\eta=d\beta} = -(\phi-1)\left(\frac{\tilde{x}(0)}{d\eta}\right)_{d\eta=d\beta} = -\frac{(\phi-1)\zeta_1}{\lambda_2} < 0, \tag{A.155}$$

$$\left(\frac{\tilde{l}(0)}{d\eta}\right)_{d\eta=d\beta} = \left(\frac{1}{1-\varepsilon_K}\right)\left(\frac{\tilde{y}(0)}{d\eta}\right)_{d\eta=d\beta} < 0, \tag{A.156}$$

$$\left(\frac{\tilde{w}(0)}{d\eta}\right)_{d\eta=d\beta} = -\left(\frac{\varepsilon_K}{1-\varepsilon_K}\right)\left(\frac{\tilde{y}(0)}{d\eta}\right)_{d\eta=d\beta} > 0, \tag{A.157}$$

$$\left(\frac{r_0}{r_0+\delta}\right)\left(\frac{\tilde{r}(0)}{d\eta}\right)_{d\eta=d\beta} = \left(\frac{\tilde{y}(0)}{d\eta}\right)_{d\eta=d\beta} < 0. \tag{A.158}$$

A.6.3.4 Combined shock in birth and death rate: No generational turnover effect

The next shock we consider is a knife-edge case for which both β and η change, though in such a way that the generational turnover effect is neutralized, i.e. we ensure that $\gamma_X = 0$. In view of (A.130) we set $d\eta/\eta_0 = d\beta/(\alpha+\beta_0)$ so that the shock vector reduces to:

$$\begin{bmatrix} \gamma_K(t) \\ \gamma_X(t) \end{bmatrix} \equiv \begin{bmatrix} -\zeta_2 \\ 0 \end{bmatrix} d\eta, \tag{A.159}$$

$$\zeta_2 \equiv \left(\frac{\alpha - n_N}{\eta}\right) > 0, \tag{A.160}$$

where we assume that $\alpha > n_N$ as before. In this case, both β and η increase but the former by more than the latter so that the population growth rate falls. By using (A.159) in (A.127) we find the long–run effects on the capital stock and full consumption:

$$\left(\frac{\tilde{k}(\infty)}{d\eta}\right)_{\gamma_X=0} = \frac{\zeta_2\left[r_0 - \alpha - (r_0+\delta)(\phi-1)\right]}{\lambda_1\lambda_2} \gtreqless 0, \tag{A.161}$$

$$\left(\frac{\tilde{x}(\infty)}{d\eta}\right)_{\gamma_X=0} = \left(\frac{\tilde{c}(\infty)}{d\eta}\right)_{\gamma_X=0} = \frac{\zeta_2\left[r_0 - \alpha + (r_0+\delta)(1-\phi\varepsilon_K)\right]}{\lambda_1\lambda_2} > 0. \tag{A.162}$$

By using (A.161)-(A.162) in (A.115)-(A.118) the long-run results for the remaining variables are obtained:

$$\begin{aligned}
\left(\frac{\tilde{y}(\infty)}{d\eta}\right)_{\gamma_X=0} &= \phi\varepsilon_K \left(\frac{\tilde{k}(\infty)}{d\eta}\right)_{\gamma_X=0} - (\phi-1)\left(\frac{\tilde{x}(\infty)}{d\eta}\right)_{\gamma_X=0} \\
&= -\frac{\zeta_2\left[(r_0+\delta)(\phi-1)+(r_0-\alpha)\left[\phi(1-\varepsilon_K)-1\right]\right]}{\lambda_1\lambda_2} \gtreqless 0, \qquad \text{(A.163)} \\
\left(\frac{\tilde{l}(\infty)}{d\eta}\right)_{\gamma_X=0} &= \left(\frac{\phi-1}{1-\varepsilon_K}\right)\left[\varepsilon_K\left(\frac{\tilde{k}(\infty)}{d\eta}\right)_{\gamma_X=0} - \left(\frac{\tilde{x}(\infty)}{d\eta}\right)_{\gamma_X=0}\right] \\
&= -\frac{(\phi-1)\zeta_2\left[r_0-\alpha+r_0+\delta\right]}{\lambda_1\lambda_2} < 0, \qquad \text{(A.164)} \\
\left(\frac{\tilde{w}(\infty)}{d\eta}\right)_{\gamma_X=0} &= \frac{\varepsilon_K}{1-\varepsilon_K}\left[\left(\frac{\tilde{y}(\infty)}{d\eta}\right)_{\gamma_X=0} - \left(\frac{\tilde{k}(\infty)}{d\eta}\right)_{\gamma_X=0}\right] \\
&= \frac{\varepsilon_K\zeta_2\phi(r_0-\alpha)}{\lambda_1\lambda_2} > 0, \qquad \text{(A.165)} \\
\left(\frac{r_0}{r_0+\delta}\right)\left(\frac{\tilde{r}(\infty)}{d\eta}\right)_{\gamma_X=0} &= \left(\frac{\tilde{y}(\infty)}{d\eta}\right)_{\gamma_X=0} - \left(\frac{\tilde{k}(\infty)}{d\eta}\right)_{\gamma_X=0} \\
&= -\frac{\varepsilon_K\zeta_2\phi(r_0-\alpha)}{\lambda_1\lambda_2} < 0. \qquad \text{(A.166)}
\end{aligned}$$

The impact effect on full consumption is computed by using (A.124) and noting that:

$$\mathcal{L}\{\gamma_K,\lambda_2\} = -\frac{\zeta_2 d\eta}{\lambda_2}, \qquad \mathcal{L}\{\gamma_X,\lambda_2\} = 0. \qquad \text{(A.167)}$$

After some manipulation we find:

$$\left(\frac{\tilde{x}(0)}{d\eta}\right)_{\gamma_X=0} = \left(\frac{\tilde{c}(0)}{d\eta}\right)_{\gamma_X=0} = \frac{\zeta_2\left[\lambda_2-(r_0-\alpha)+(r_0+\delta)(\phi-1)\right]}{\lambda_2\left(\frac{y}{k}\right)_0(\omega_C+\phi-1)} > 0. \qquad \text{(A.168)}$$

By using (A.154) in (A.115)-(A.118) and noting that $\tilde{k}(0)=0$ the impact results for the remaining variables are obtained:

$$\begin{aligned}
\left(\frac{\tilde{y}(0)}{d\eta}\right)_{\gamma_X=0} &= -(\phi-1)\left(\frac{\tilde{x}(0)}{d\eta}\right)_{\gamma_X=0} \\
&= -\frac{\zeta_2(\phi-1)\left[\lambda_2-(r_0-\alpha)+(r_0+\delta)(\phi-1)\right]}{\lambda_2\left(\frac{y}{k}\right)_0(\omega_C+\phi-1)} < 0, \qquad \text{(A.169)} \\
\left(\frac{\tilde{l}(0)}{d\eta}\right)_{\gamma_X=0} &= \left(\frac{1}{1-\varepsilon_K}\right)\left(\frac{\tilde{y}(0)}{d\eta}\right)_{\gamma_X=0} < 0, \qquad \text{(A.170)} \\
\left(\frac{\tilde{w}(0)}{d\eta}\right)_{\gamma_X=0} &= -\left(\frac{\varepsilon_K}{1-\varepsilon_K}\right)\left(\frac{\tilde{y}(0)}{d\eta}\right)_{\gamma_X=0} > 0, \qquad \text{(A.171)} \\
\left(\frac{r_0}{r_0+\delta}\right)\left(\frac{\tilde{r}(0)}{d\eta}\right)_{\gamma_X=0} &= \left(\frac{\tilde{y}(0)}{d\eta}\right)_{\gamma_X=0} < 0. \qquad \text{(A.172)}
\end{aligned}$$

A.6.3.5 Shock in birth rate

??? Shock is $d\eta > 0$ (both CSE and FCE lines affected). Results are ambiguous.

A.6.3.6 Shock in death rate

??? Shock is $d\beta > 0$ (both CSE and FCE lines affected). Results are ambiguous.

A.7 Discrete-time model

In this section we present the discrete-time version of the model which is used to simulate the non-linear version of the model with the aid of the DYNARE MATLAB package of Juillard (1996, 2003). Variable names are the same as for the continuous time version.

A.7.1 Individual households

Lifetime utility of the household of vintage v at time t is:

$$\Lambda_{v,t} \equiv \sum_{\tau=t}^{\infty} \left[\frac{\bar{u}_{v,\tau}^{1-1/\sigma_X} - 1}{1 - 1/\sigma_X} \right] \left(\frac{1-\beta}{1+\alpha} \right)^{t-\tau}, \tag{A.173}$$

where α is the rate of time preference, β is the probability of death, and $v < t$.[11] Sub-utility is:

$$\bar{u}_{v,\tau} \equiv \left[\varepsilon_C \left(\bar{c}_{v,\tau} \right)^{(\sigma_C - 1)/\sigma_C} + (1-\varepsilon_C) \left[1 - \bar{l}_{v,\tau} \right]^{(\sigma_C-1)/\sigma_C} \right]^{\sigma_C/(\sigma_C-1)}. \tag{A.174}$$

The household budget identity is:

$$\bar{a}_{v,\tau} = \left(\frac{1+r_\tau}{1-\beta} \right) \bar{a}_{v,\tau-1} + W_\tau - \bar{z}_\tau - P_{U,\tau} \bar{u}_{v,\tau}, \tag{A.175}$$

$$P_{U,\tau} \bar{u}_{v,\tau} = x_{v,\tau}. \tag{A.176}$$

where $\bar{a}_{v,\tau-1}$ is the stock of financial wealth at the *end* of period $\tau - 1$. Financial wealth is defined as:

$$\bar{a}_{v,\tau} \equiv \bar{k}_{v,\tau} + \bar{b}_{v,\tau}. \tag{A.177}$$

Full consumption definition:

$$\bar{x}_{v,\tau} \equiv \bar{c}_{v,\tau} + W_\tau \left[1 - \bar{l}_{v,\tau} \right]. \tag{A.178}$$

Discount factor:

$$R_{t,\tau} \equiv \begin{cases} 1 & \text{for } \tau = t \\ \prod_{\mu=t+1}^{\tau} \frac{1-\beta}{1+r_\mu} & \text{for } \tau > t \end{cases}. \tag{A.179}$$

Useful relationships:

$$\frac{R_{t,\tau}}{R_{t-1,\tau}} = \frac{1}{R_{t-1,t}} = \frac{1+r_t}{1-\beta}, \qquad \frac{R_{t,\tau+1}}{R_{t,\tau}} = \frac{1-\beta}{1+r_{\tau+1}}. \tag{A.180}$$

Intertemporal budget restriction:

$$\sum_{\tau=t}^{\infty} \bar{x}_{v,\tau} R_{t,\tau} = TW_{v,t}, \tag{A.181}$$

$$TW_{v,t} \equiv \left(\frac{1+r_t}{1-\beta} \right) \left[\bar{a}_{v,t-1} + \bar{h}_{t-1} \right]. \tag{A.182}$$

[11] A newly born household starts to consume one period after it is born, i.e. $\Lambda_{t-1,t}$ is its lifetime utility.

Lagrangian expression:

$$Z^H_{v,t} = \sum_{\tau=t}^{\infty} \left(\frac{1-\beta}{1+\alpha}\right)^{\tau-t} \left\{ \frac{\bar{u}_{v,\tau}^{1-1/\sigma_X} - 1}{1-1/\sigma_X} + \lambda^H_{v,\tau} \left[\left(\frac{1+r_\tau}{1-\beta}\right) \bar{a}_{v,\tau-1} + W_\tau - \bar{z}_\tau - P_{U,\tau}\bar{u}_{v,\tau} - \bar{a}_{v,\tau} \right] \right\}. \tag{A.183}$$

The (interesting) first-order conditions:

$$\bar{u}_{v,\tau}^{-1/\sigma_X} = \lambda^H_{v,\tau} P_{U,\tau}, \tag{A.184}$$

$$\lambda^H_{v,\tau} = \lambda^H_{v,\tau+1} \frac{1+r_{\tau+1}}{1+\alpha}. \tag{A.185}$$

Euler equation for $\bar{u}_{v,\tau}$:

$$\frac{\bar{u}_{v,\tau+1}}{\bar{u}_{v,\tau}} = \left(\frac{P_{U,\tau}}{P_{U,\tau+1}}\right)^{\sigma_X} \left(\frac{1+r_{\tau+1}}{1+\alpha}\right)^{\sigma_X}. \tag{A.186}$$

Euler equation for full consumption $\bar{x}_{v,\tau}$:

$$\frac{\bar{x}_{v,\tau+1}}{\bar{x}_{v,\tau}} = \left(\frac{P_{U,\tau+1}}{P_{U,\tau}}\right)^{1-\sigma_X} \left(\frac{1-\beta}{1+\alpha}\frac{1+r_{\tau+1}}{1-\beta}\right)^{\sigma_X}. \tag{A.187}$$

The conditional solution for the Euler equation is:

$$R_{t,\tau}\bar{x}_{v,\tau} = \left(\frac{P_{U,\tau}}{P_{U,t}}\right)^{1-\sigma_X} \left(\frac{1-\beta}{1+\alpha}\right)^{\sigma_X(\tau-t)} R_{t,\tau}^{1-\sigma_X}\bar{x}_{v,t}. \tag{A.188}$$

By using (A.188) in (A.181) we obtain the solution for full consumption:

$$\bar{x}_{v,t} = TW_{v,t}/\Delta_t, \tag{A.189}$$

$$\Delta_t \equiv \sum_{\tau=t}^{\infty} \left(\frac{P_{U,\tau}R_{t,\tau}}{P_{U,t}}\right)^{1-\sigma_X} \left(\frac{1-\beta}{1+\alpha}\right)^{\sigma_X(\tau-t)}. \tag{A.190}$$

We note various aspects of (A.190):

- If $\sigma_X = 1$ then:

$$\frac{1}{\Delta_t} \equiv \left[\sum_{\tau=t}^{\infty} \left(\frac{1-\beta}{1+\alpha}\right)^{\tau-t}\right]^{-1} = \frac{\beta+\alpha}{1+\alpha}. \tag{A.191}$$

- In general, the difference equation for Δ_t is:

$$\Delta_t = 1 + \left(\frac{P_{U,t+1}}{P_{U,t}}\frac{1-\beta}{1+r_{t+1}}\right)^{1-\sigma_X} \left(\frac{1-\beta}{1+\alpha}\right)^{\sigma_X} \Delta_{t+1} \tag{A.192}$$

Human wealth (*excluding* current non-interest income) is defined as:

$$\bar{h}_t \equiv \sum_{\tau=t+1}^{\infty} \left[W_\tau - \bar{z}_\tau\right] R_{t,\tau}. \tag{A.193}$$

The difference equation for human wealth is:

$$\bar{h}_t = \left(\frac{1+r_t}{1-\beta}\right) \bar{h}_{t-1} - \left[W_t - \bar{z}_t\right] \tag{A.194}$$

Consumption of goods and leisure:

$$W_t \left[1 - \bar{l}_{v,t}\right] = \left[1 - \omega_{CX,t}\right] \bar{x}_{v,t}, \tag{A.195}$$

$$\bar{c}_{v,t} = \omega_{CX,t} \bar{x}_{v,t}, \tag{A.196}$$

$$\omega_{CX,t} \equiv \frac{\varepsilon_C^{\sigma_C}}{\varepsilon_C^{\sigma_C} + (1-\varepsilon_C)^{\sigma_C} W_t^{1-\sigma_C}}. \tag{A.197}$$

Definition $P_{U,t}$:

$$P_{U,t} \equiv \begin{cases} \left[(\varepsilon_C)^{\sigma_C} + (1-\varepsilon_C)^{\sigma_C} W_t^{1-\sigma_C}\right]^{1/(1-\sigma_C)} & \text{for } \sigma_C \neq 1 \\ \left(\frac{1}{\varepsilon_C}\right)^{\varepsilon_C} \left(\frac{W_t}{1-\varepsilon_C}\right)^{1-\varepsilon_C} & \text{for } \sigma_C = 1 \end{cases} \tag{A.198}$$

A.7.2 Population dynamics

Total population stock at the end of period $t-1$ (beginning of period t) is N_{t-1}. Definition:

$$N_t \equiv \sum_{a=1}^{\infty} N_{t-a,t-1} \tag{A.199}$$

where $N_{t-1,t-1}$ is the new generation born at end of period $t-1$ ($a = 1$ for newborn). Population growth rate:

$$N_t \equiv (1 + n_N) N_{t-1}. \tag{A.200}$$

Or:

$$N_t = (1 + n_N)^t \tag{A.201}$$

Digression on timing convention:

- stock at beginning of period 2 is $N_{2-1} = N_1$
- then early during period 2 [winter-time] death occurs, i.e. population dwindles to $(1-\beta) N_1$
- then later during period 2 [spring-time] births occur, i.e. population expands to $(1+\eta)(1-\beta) N_1$
- hence, at end of period 2 we have $N_2 = (1+n_N) N_1$ so $1 + n_N \equiv (1+\eta)(1-\beta)$

Hence:

$$1 + n_N = (1-\beta)(1+\eta). \tag{A.202}$$

Newborn generation:

$$N_{v,v} = \eta (1-\beta) N_{v-1}. \tag{A.203}$$

Size of existing generations:

$$N_{v,t-1} = (1-\beta)^{t-v-1} N_{v,v}, \qquad t-1 \geq v. \tag{A.204}$$

Or in terms of age, $a \equiv t - v$:

$$N_{t-a,t-1} = (1-\beta)^{a-1} N_{t-a,t-a}, \qquad a \geq 1. \tag{A.205}$$

Cohort size (for $a \geq 1$):

$$N_{t-a,t-1} = \eta \left(\frac{1-\beta}{1+n_N} \right)^a N_{t-1} = \eta \left(1+\eta\right)^{-a} N_{t-1}, \tag{A.206}$$

where we have used the fact that $N_{t-1} = (1+n_N)^a N_{t-a-1}$. Cohort size in terms of v and t (for $v \leq t-1$):

$$N_{v,t-1} = \eta \left(1+\eta\right)^{v-1} \left(1-\beta\right)^{t-1}. \tag{A.207}$$

A.7.3 Aggregate household sector

A.7.3.1 Total human wealth

Total human wealth is defined as follows:

$$H_t \equiv \sum_{v=t}^{-\infty} \bar{h}_t N_{v,t} = N_t \bar{h}_t. \tag{A.208}$$

Note that it includes the human wealth of newborns.

A.7.3.2 Difference equation aggregate human wealth

By using (A.194) in (A.208) we find:

$$\begin{aligned} H_t &= N_t \left[\left(\frac{1+r_t}{1-\beta} \right) \bar{h}_{t-1} - \left[W_t - z_t\right] \right] \\ &= \left(1+\eta\right)\left(1-\beta\right) N_{t-1} \left(\frac{1+r_t}{1-\beta} \right) \bar{h}_{t-1} - N_t \left[W_t - \bar{z}_t\right] \\ &= \left(1+r_t\right)\left(1+\eta\right) H_{t-1} - N_t \left[W_t - \bar{z}_t\right], \end{aligned} \tag{A.209}$$

where we have used (A.200), (A.202), and noted that $H_{t-1} = N_{t-1}\bar{h}_{t-1}$.

A.7.3.3 Aggregate full consumption

Aggregate full consumption is defined as follows:

$$X_t \equiv \sum_{v=t-1}^{-\infty} \bar{x}_{v,t} N_{v,t-1}. \tag{A.210}$$

The thing to note is that newborns start to consume when they are 1 period old. We recall from above that:

$$\bar{x}_{v,t} = \frac{1}{\Delta_t} TW_{v,t} \tag{A.211}$$

$$TW_{v,t} \equiv \frac{1+r_t}{1-\beta} \left[\bar{a}_{v,t-1} + \bar{h}_{t-1}\right] \tag{A.212}$$

These equations are found in (A.189) and (A.182) respectively. After some manipulation we find that aggregate total wealth is:

$$\begin{aligned} TW_t &\equiv \sum_{v=t-1}^{-\infty} TW_{v,t} N_{v,t-1} \\ &= \frac{1+r_t}{1-\beta} \sum_{v=t-1}^{-\infty} \left[\bar{a}_{v,t-1} + \bar{h}_{t-1}\right] N_{v,t-1} \\ &= \frac{1+r_t}{1-\beta} \left[\left(1-\beta\right) A_{t-1} + H_{t-1}\right], \end{aligned} \tag{A.213}$$

where we have used:

$$\begin{aligned}\sum_{v=t-1}^{-\infty} \bar{a}_{v,t-1} N_{v,t-1} &= \underbrace{\bar{a}_{t-1,t-1}}_{=0} N_{t-1,t-1} + \sum_{v=t-2}^{-\infty} \bar{a}_{v,t-1} N_{v,t-1} \\ &= (1-\beta) \sum_{v=t-2}^{-\infty} \bar{a}_{v,t-1} N_{v,t-2} \\ &= (1-\beta) A_{t-1} \qquad (\text{as } N_{v,t-1} = (1-\beta) N_{v,t-2}).\end{aligned} \tag{A.214}$$

Recall that newborns are born without financial wealth, i.e. $\bar{a}_{t-1,t-1} = 0$.

It follows from (A.210)-(A.211) that aggregate full consumption can be written as:

$$\begin{aligned}X_t &= \frac{1}{\Delta_t} TW_t \\ &= \frac{1}{\Delta_t} \frac{1+r_t}{1-\beta} \left[(1-\beta) A_{t-1} + H_{t-1}\right].\end{aligned} \tag{A.215}$$

We know that by definition:

$$\begin{aligned}X_{t+1} &\equiv \sum_{v=t}^{-\infty} \bar{x}_{v,t+1} N_{v,t} \\ &= N_{t,t} \bar{x}_{t,t+1} + \sum_{v=t-1}^{-\infty} \bar{x}_{v,t+1} N_{v,t} \\ &= \eta (1-\beta) N_{t-1} \bar{x}_{t,t+1} + (1-\beta) \Gamma_t X_t,\end{aligned} \tag{A.216}$$

where we have used:

$$\Gamma_t \equiv \left(\frac{P_{U,t+1}}{P_{U,t}}\right)^{1-\sigma_X} \left(\frac{1+r_{t+1}}{1+\alpha}\right)^{\sigma_X}, \tag{A.217}$$

$$\sum_{v=t-1}^{-\infty} \bar{x}_{v,t+1} N_{v,t} = \sum_{v=t-1}^{-\infty} \Gamma_t \bar{x}_{v,t} (1-\beta) N_{v,t-1} = (1-\beta) \Gamma_t X_t. \tag{A.218}$$

A.7.3.4 Aggregate Euler equation

To derive the aggregate Euler equation for full consumption we first write:

$$\bar{x}_{t,t+1} = \frac{1}{\Delta_{t+1}} TW_{t,t+1}, \tag{A.219}$$

$$TW_{t,t+1} \equiv \left(\frac{1+r_{t+1}}{1-\beta}\right) \left[\underbrace{\bar{a}_{t,t}}_{=0} + \bar{h}_t\right] = \left(\frac{1+r_{t+1}}{1-\beta}\right) \bar{h}_t. \tag{A.220}$$

It follows that:

$$\begin{aligned}\eta (1-\beta) N_{t-1} \bar{x}_{t,t+1} &= \frac{1+r_{t+1}}{\Delta_{t+1}} \eta N_{t-1} \bar{h}_t \\ &= \frac{1+r_{t+1}}{\Delta_{t+1}} \frac{\eta H_t}{(1-\beta)(1+\eta)},\end{aligned} \tag{A.221}$$

where we have used (A.208) and noted that $N_t = (1-\beta)(1+\eta) N_{t-1}$. By using (A.221) in (A.216) we find:

$$\begin{aligned} \Delta_{t+1} X_{t+1} &= \frac{1+r_{t+1}}{1-\beta} \frac{\eta H_t}{1+\eta} + (1-\beta) \Gamma_t \Delta_{t+1} X_t \\ &= \frac{1+r_{t+1}}{1-\beta} \left[\frac{\eta H_t}{1+\eta} + (1-\beta)(\Delta_t - 1) X_t \right], \end{aligned} \tag{A.222}$$

where we have used:

$$\Gamma_t \Delta_{t+1} = \frac{1+r_{t+1}}{1-\beta} (\Delta_t - 1). \tag{A.223}$$

(Note that (A.223) follows by definition from (A.192) and (A.217).) We derive from (A.215) that:

$$\begin{aligned} \Delta_{t+1} X_{t+1} &= \frac{1+r_{t+1}}{1-\beta} \left[(1-\beta) A_t + H_t\right] \qquad \Longleftrightarrow \\ H_t &= \frac{1-\beta}{1+r_{t+1}} \Delta_{t+1} X_{t+1} - (1-\beta) A_t. \end{aligned} \tag{A.224}$$

Using (A.224) in (A.222) we find after some manipulations:

$$\begin{aligned} \Delta_{t+1} X_{t+1} &= \frac{\eta}{1+\eta} \frac{1+r_{t+1}}{1-\beta} \left[\frac{1-\beta}{1+r_{t+1}} \Delta_{t+1} X_{t+1} - (1-\beta) A_t \right] \\ &\quad + (1+r_{t+1})(\Delta_t - 1) X_t \\ &= (1+r_{t+1}) \Big[(1+\eta)(\Delta_t - 1) X_t - \eta A_t \Big]. \end{aligned} \tag{A.225}$$

Equation (A.225) is the most general expression for the aggregate Euler equation for full consumption. Several special cases are worth noting:

- For the unit-elastic case (for which $\sigma_X = 1$) we find that $\Delta_t = \Delta_{t+1} = \frac{1+\alpha}{\beta+\alpha}$ so that (A.225) reduces to:

$$X_{t+1} = \frac{1+r_{t+1}}{1+\alpha} \Bigg[(1+n_N) X_t - \eta(\beta+\alpha) A_t \Bigg]. \tag{A.226}$$

- The standard representative-agent case is obtained by setting $\beta = 0$, $\eta = 0$ (so that $n_N = 0$) in (A.225):

$$X_{t+1} = \Gamma_t X_t, \tag{A.227}$$

where Γ_t is defined in (A.217) above. Note that (A.227) is identical to (A.187).

A.7.3.5 Aggregate financial wealth

Aggregate financial wealth is defined as:

$$A_t \equiv \sum_{a=1}^{\infty} N_{t-a,t-1} \bar{a}_{t-a,t}, \tag{A.228}$$

where A_t is financial wealth at the end of period t. By using (A.175)-(A.176) and (A.178) (and noting $v \equiv t-a$) we find:

$$\begin{aligned} \sum_{a=1}^{\infty} N_{t-a,t-1} \bar{a}_{t-a,t} = \sum_{a=1}^{\infty} N_{t-a,t-1} \left[\left(\frac{1+r_t}{1-\beta} \right) \bar{a}_{t-a,t-1} + [W_t - \bar{z}_t] - \bar{x}_{t-a,t} \right] \quad \Leftrightarrow \\ A_t = (1+r_t) A_{t-1} + N_{t-1} [W_t - \bar{z}_t] - X_t. \end{aligned} \tag{A.229}$$

In deriving (A.229) we have also used:

$$\sum_{a=1}^{\infty} N_{t-a,t-1}\bar{a}_{t-a,t-1} = N_{t-1,t-1}\bar{a}_{t-1,t-1} + \sum_{a=2}^{\infty} N_{t-a,t-1}\bar{a}_{t-a,t-1} = (1-\beta)\,A_{t-1}, \tag{A.230}$$

as $\bar{a}_{t-1,t-1} = 0$ (newborns have no financial assets).

A.7.3.6 Aggregate goods consumption and labour supply

Aggregate goods consumption is defined as follows:

$$C_t \equiv \sum_{v=t-1}^{-\infty} \bar{c}_{v,t} N_{v,t-1}. \tag{A.231}$$

By using (A.196) in (A.231) and noting (A.210) we find:

$$C_t \equiv \sum_{v=t-1}^{-\infty} \omega_{CX,t}\bar{x}_{v,t}N_{v,t-1} = \omega_{CX,t}X_t. \tag{A.232}$$

In a similar fashion, aggregate leisure demand is given by:

$$\sum_{v=t-1}^{-\infty} W_t \left[1-\bar{l}_{v,t}\right] N_{v,t-1}. \tag{A.233}$$

By using (A.195) in (A.233) and noting (A.210) we obtain:

$$\sum_{v=t-1}^{-\infty} W_t \left[1-\bar{l}_{v,t}\right] N_{v,t-1} = W_t\left[N_{t-1}-L_t\right] = \left[1-\omega_{CX,t}\right]X_t, \tag{A.234}$$

where L_t is aggregate labour supply in period t:

$$L_t \equiv \sum_{v=t-1}^{-\infty} \bar{l}_{v,t}N_{v,t-1}. \tag{A.235}$$

A.7.4 Firms

Output is given by:

$$Y_t = F\left[K_{t-1}, L_t\right] \tag{A.236}$$

$$\equiv \Psi_Y\left[\varepsilon_K K_{t-1}^{(\sigma_K-1)/\sigma_K} + (1-\varepsilon_K)L_t^{(\sigma_K-1)/\sigma_K}\right]^{\sigma_K/(\sigma_K-1)}. \tag{A.237}$$

Real profit is defined as:

$$\Pi_t \equiv (1-t_{Y,t})\,Y_t - W_t^K K_{t-1}^N - W_t L_t. \tag{A.238}$$

The first-order conditions are:

$$(1-t_{Y,t})\,\frac{\partial F\left[K_{t-1},L_t\right]}{\partial K_{t-1}} = W_t^K, \tag{A.239}$$

$$(1-t_{Y,t})\,\frac{\partial F\left[K_{t-1},L_t\right]}{\partial L_t} = W_t. \tag{A.240}$$

A.7.5 Portfolio investments

A.7.5.1 Capital accumulation:

By purchasing goods for investment purposes the household creates capital which is rented out to the firms at a rental rate W_τ^K. The objective function associated with the portfolio investment problem is:

$$\bar{V}_t = \sum_{\tau=t}^{\infty} \left[W_\tau^K K_{\tau-1} - I_\tau \right] D_{t,\tau} \tag{A.241}$$

$$= W_t^K K_{t-1} - I_t + \underbrace{\sum_{\tau=t+1}^{\infty} \left[W_\tau^K K_{\tau-1} - I_\tau \right] D_{t,\tau}}_{V_t}, \tag{A.242}$$

$$D_{t,\tau} \equiv \begin{cases} 1 & \text{for } \tau = t \\ \prod_{\mu=t+1}^{\tau} \frac{1}{1+r_\mu} & \text{for } \tau > t \end{cases}, \tag{A.243}$$

where r_μ is the real interest rate. The capital accumulation identity is:

$$K_\tau = I_\tau + (1-\delta) K_{\tau-1}. \tag{A.244}$$

The investor chooses paths for gross investment and the capital stock in order to maximize (A.241) subject to (A.244) and taking as given the path of the rental rate (W_τ^K) and the initial capital stock (K_{t-1}). The first-order conditions are (for $\tau = t, t+1, ...$):

$$\frac{\partial \bar{V}_t}{\partial I_\tau} = -D_{t,\tau} + \lambda_\tau = 0, \tag{A.245}$$

$$\frac{\partial \bar{V}_t}{\partial K_\tau} = W_{\tau+1}^K D_{t,\tau+1} - \lambda_\tau + (1-\delta) \lambda_{\tau+1} = 0, \tag{A.246}$$

where λ_τ is the Lagrange multiplier associated with (A.244). Combining these first-order conditions yields:

$$W_{\tau+1}^K + (1-\delta) = \frac{D_{t,\tau}}{D_{t,\tau+1}} \equiv 1 + r_{\tau+1}. \tag{A.247}$$

By simplifying (A.247) we obtain the usual rental rate expression:

$$W_{\tau+1}^K = r_{\tau+1} + \delta \qquad (\text{for } \tau = t, t+1, ...). \tag{A.248}$$

Note that V_t satisfies the following arbitrage relationship:

$$(1 + r_t) V_{t-1} = W_t^K K_{t-1} - I_t + V_t. \tag{A.249}$$

It is not difficult to show that:

$$V_t = K_t. \tag{A.250}$$

The proof proceeds as follows:

$$\begin{aligned}
V_t &\equiv \sum_{\tau=t+1}^{\infty} \left[W_\tau^K K_{\tau-1} - I_\tau \right] D_{t,\tau} \\
&= \sum_{\tau=t+1}^{\infty} \left[W_\tau^K K_{\tau-1} - K_\tau + (1-\delta) K_{\tau-1} \right] D_{t,\tau} \\
&= \left[\left(W_{t+1}^K + (1-\delta) \right) K_t - K_{t+1} \right] D_{t,t+1} \\
&\quad + \left[\left(W_{t+2}^K + (1-\delta) \right) K_{t+1} - K_{t+2} \right] D_{t,t+2} \\
&\quad + \left[\left(W_{t+3}^K + (1-\delta) \right) K_{t+2} - K_{t+3} \right] D_{t,t+3} + ...
\end{aligned} \tag{A.251}$$

Gathering terms we find:

$$\begin{aligned}
V_t &= \left[W_{t+1}^K + (1-\delta) \right] K_t D_{t,t+1} \\
&\quad + \left[\left(W_{t+2}^K + (1-\delta) \right) D_{t,t+2} - D_{t,t+1} \right] K_{t+1} \\
&\quad + \left[\left(W_{t+3}^K + (1-\delta) \right) D_{t,t+3} - D_{t,t+2}^N \right] K_{t+2} \\
&\quad + ...
\end{aligned} \tag{A.252}$$

Using the first-order condition, $\left[W_{\tau+1}^K + (1-\delta) \right] D_{t,\tau+1} = D_{t,\tau}$ (and noting that $D_{t,t} = 1$) we can simplify (A.252) to obtain (A.250).

A.7.6 Loose ends

The government budget identity is given by:

$$B_t = (1+r_t) B_{t-1} + G_t - N_{t-1} \bar{z}_t - t_{Y,t} Y_t \tag{A.253}$$

where r_t denotes the real rate of interest on government bonds, B is aggregate government debt, and G is government consumption. Dividing both sides of (A.253) by $N_t \equiv (1+n_N) N_{t-1}$ we find:

$$b_t = \frac{(1+r_t) b_{t-1} + g_t - \bar{z}_t - t_{Y,t} y_t}{1+n_N}, \tag{A.254}$$

where $b_t \equiv B_t/N_t$, $g_t \equiv G_t/N_{t-1}$, and $y_t \equiv Y_t/N_{t-1}$.

The goods market clearing condition is:

$$Y_t = C_t + I_t + G_t. \tag{A.255}$$

The discrete-time model is given in Table A.9. We note the following conventions. Flow variables are expressed in terms of the population stock at the beginning of the period (i.e. at the end of the previous period):

$$\begin{array}{lll}
c_t \equiv \frac{C_t}{N_{t-1}} & g_t \equiv \frac{G_t}{N_{t-1}} & l_t \equiv \frac{L_t}{N_{t-1}} \\
y_t \equiv \frac{Y_t}{N_{t-1}} & x_t \equiv \frac{X_t}{N_{t-1}} &
\end{array} \tag{A.256}$$

Stock variables are expressed as follows:

$$\begin{array}{ll}
b_t \equiv \frac{B_t}{N_t} & a_t \equiv \frac{A_t}{N_t} \\
k_t \equiv \frac{K_t}{N_t} &
\end{array} \tag{A.257}$$

Table A.9: Discrete-time version of the model(♯)

(a) Dynamic equations:

$$k_t - k_{t-1} = \frac{y_t - c_t - g_t - (\delta + n_N)\, k_{t-1}}{1 + n_N} \tag{T9.1}$$

$$\Delta_{t+1} x_{t+1} = (1 + r_{t+1}) \left[\frac{(\Delta_t - 1)\, x_t}{1 - \beta} - \eta \left[k_t + b_t \right] \right] \tag{T9.2}$$

$$\Delta_t = 1 + \left(\frac{P_{U,t+1}}{P_{U,t}} \frac{1-\beta}{1 + r_{t+1}} \right)^{1-\sigma_X} \left(\frac{1-\beta}{1+\alpha} \right)^{\sigma_X} \Delta_{t+1} \tag{T9.3}$$

(b) Static equations:

$$y_t = \Psi_Y \left[\varepsilon_K k_{t-1}^{(\sigma_K - 1)/\sigma_K} + (1 - \varepsilon_K) l_t^{(\sigma_K - 1)/\sigma_K} \right]^{\sigma_K/(\sigma_K - 1)} \tag{T9.4}$$

$$\frac{W_t}{1 - t_{Y,t}} = (1 - \varepsilon_K)\, \Psi_Y^{1-1/\sigma_K} \left(\frac{y_t}{l_t} \right)^{1/\sigma_K} \tag{T9.5}$$

$$\frac{r_t + \delta}{1 - t_{Y,t}} = \varepsilon_K \Psi_Y^{1-1/\sigma_K} \left(\frac{y_t}{k_{t-1}} \right)^{1/\sigma_K} \tag{T9.6}$$

$$c_t = \omega_{CX,t} x_t \tag{T9.7}$$

$$W_t \left[1 - l_t \right] = \left[1 - \omega_{CX,t} \right] x_t \tag{T9.8}$$

(c) Miscellaneous equations:

$$\omega_{CX,t} \equiv \frac{\varepsilon_C^{\sigma_C}}{(\varepsilon_C)^{\sigma_C} + (1 - \varepsilon_C)^{\sigma_C} W_t^{1-\sigma_C}} \tag{T9.9}$$

$$P_{U,t} \equiv \begin{cases} \left[(\varepsilon_C)^{\sigma_C} + (1 - \varepsilon_C)^{\sigma_C} W_t^{1-\sigma_C} \right]^{1/(1-\sigma_C)} & \text{for } \sigma_C \neq 1 \\ \left(\frac{1}{\varepsilon_C} \right)^{\varepsilon_C} \left(\frac{W_t}{1-\varepsilon_C} \right)^{1-\varepsilon_C} & \text{for } \sigma_C = 1 \end{cases} \tag{T9.10}$$

$$b_t - b_{t-1} = \frac{(r_t - n_N)\, b_{t-1} + g_t - \bar{z}_t - t_{Y,t} y_t}{1 + n_N} \tag{T9.11}$$

(♯) See the list of variables for the definitions.

A.7.7 Numerical simulations

In order to illustrate the quantitative significance of returns to scale and the mode of financing, this section presents a calibrated example of the model. Since we wish to study the effects of the intertemporal substitution elasticity in labour supply (ω_{LL}), the pre-existing output tax (t_Y), and the diversity parameter (η) on the various output multipliers, the model is calibrated in such a way that these parameters can be freely varied. In terms of Tables A.5 - A.8, the parameters that are held fixed throughout the simulations are the rate of pure time preference ($\alpha = 0.03$), the rate of depreciation of the capital stock ($\delta = 0.07$), the share of labour income ($1 - \varepsilon_K = 0.7$), and the share of government spending ($\omega_G = 0.2$). In Table A.5, $t_Y = 0$ and ω_{LL} is varied and in Table A.6, $\omega_{LL} = 2$ and t_Y is varied. Table A.5-A.6 refer to the case of infinitely-lived agents ($\beta = 0$). Once these coefficients are set, all other information regarding shares can be obtained by using the information at the bottom of Table A.2.[12]

[12]Indeed, $\beta = 0$ implies that $r = \alpha$ and $y = (\alpha + \delta)/[\varepsilon_K(1 - t_Y)]$. It follows that $\omega_I = \delta/y$, $\omega_C = 1 - \omega_I - \omega_G$, $\omega_T = \omega_G - t_Y$, and $\omega_A = (\varepsilon_K(1 - t_Y) - \omega_I$.

References

Auerbach, A. J. and Kotlikoff, L. J. (1987). *Dynamic Fiscal Policy*. Cambridge University Press, Cambridge.

Blanchard, O.-J. (1985). Debts, deficits, and finite horizons. *Journal of Political Economy*, 93:223–247.

Blinder, A. S. and Solow, R. M. (1973). Does fiscal policy matter? *Journal of Public Economics*, 2:319–337.

Bovenberg, A. L. (1993). Investment promoting policies in open economies: The importance of intergenerational and international distributional effects. *Journal of Public Economics*, 51:3–54.

Bovenberg, A. L. (1994). Capital taxation in the world economy. In van der Ploeg, F., editor, *Handbook of International Macroeconomics*. Basil Blackwell, Oxford.

Bovenberg, A. L. and Heijdra, B. J. (1998). Environmental tax policy and intergenerational distribution. *Journal of Public Economics*, 67:1–24.

Bovenberg, A. L. and Heijdra, B. J. (2002). Environmental abatement and intergenerational distribution. *Environmental and Resource Economics*, 23:45–84.

Buiter, W. H. (1988). Death, birth, productivity growth and debt neutrality. *Economic Journal*, 98:279–293.

Diamond, P. A. (1965). National debt in a neoclassical growth model. *American Economic Review*, 55:1126–1150.

Heijdra, B. J. (2003). Macroeconomic effects of demographic shocks: Mathematical appendix. Mimeo, University of Groningen. Download from: `http://www.eco.rug.nl/medewerk/heijdra/download.htm`.

Heijdra, B. J. and Ligthart, J. E. (2000). The dynamic macroeconomic effects of tax policy in an overlapping generations model. *Oxford Economic Papers*, 52:677–701.

Heijdra, B. J. and Ligthart, J. E. (2002). Tax policy, the macroeconomy, and intergenerational distribution. *IMF Staff Papers*, 49:106–127.

Judd, K. L. (1982). An alternative to steady-state comparisons in perfect foresight models. *Economics Letters*, 10:55–59.

Judd, K. L. (1998). *Numerical Methods in Economics*. MIT Press, Cambridge, MA.

Marini, G. and Van der Ploeg, F. (1988). Monetary and fiscal policy in an optimising model with capital accumulation and finite lives. *Economic Journal*, 98:772–786.

Samuelson, P. A. (1958). An exact consumption-loan model of interest with or without the social contrivance of money. *Journal of Political Economy*, 66:467–482.

Weil, P. (1989). Overlapping families of infinite-lived agents. *Journal of Public Economics*, 38:183–198.

Yaari, M. E. (1965). Uncertain lifetime, life insurance, and the theory of the consumer. *Review of Economic Studies*, 32:137–150.